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Metallic Ratios and Ramanujan Summation

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Abstract - *Metallic Ratios are class of numbers which are irrationals. The well known Golden Ratio and Silver Ratio are special cases of sequences of Metallic Ratios. In this paper, after introducing Metallic Ratios formally, I have proved some interesting inequalities for powers of metallic ratios whose lower and upper bounds will be connected to Ramanujan Summation method leading to very interesting and new results. In this paper, I had extended the concept of Ramanujan Summation technique to the bounds of powers of metallic ratios. In Ramanujan Summation Method, Ramanujan showed that Ramanujan Sum of all even powers of positive integers is always zero. Similar to this, I had proved that Ramanaujan Summation of all even powers of lower bounds of Metallic ratios of order k is always* $-\frac{1}{2}$. *The result regarding computation of upper bounds of Metallic ratios through two previous lower bounds has been established. This result enables us to compute the Ramanujan Summation of Upper bounds of Metallic Ratios in terms of the corresponding Lower bounds. Further, I had shown that the Ramanujan Summation of Upper bounds of Metallic ratios of order 2r + 1 are equal to Upper bounds of Metallic ratios of order 2r + 2. In the final section, the computation of Ramanujan Summation values of lower and upper bounds for first eight powers of metallic powers were carried out. These values verify the theorems proved in this paper. The whole idea of assigning Ramanujan Summation to bounds of Metallic Ratios is very new and so the results obtained in this paper provide great insights and opens great scope towards understanding the behavior of metallic ratios.*

Keywords - *Recurrence Relation, Powers of Metallic Ratios, Lower and Upper Bounds, Mathematical Induction, Ramanujan Summation.*

I. Introduction

The sequence of metallic ratios occurs as irrational real numbers through a specified recurrence relation. In this paper, using the closed expression for *k*th term of the sequence of metallic ratios, I have obtained lower and upper bounds for *r*th powers of metallic ratios. Interestingly, such bounds have direct relationship with entries of Pascal's triangle and Fibonacci numbers. After defining Ramanujan Summation method through a definite integral, I had proved new results connecting the bounds of powers of metallic ratios with that of Ramanujan Summation method.

II. Definitions And Notations

2.1 The *k*th term of sequence of metallic ratios denoted by M_k is defined as the positive root of the equation $x^2 - kx - 1 = 0 \quad (2.1)$. Thus $M_k = \frac{k+\sqrt{k^2+4}}{2} \quad (2.2)$ and $-\frac{1}{M_k} = \frac{k-\sqrt{k^2+4}}{2} \quad (2.3)$.

2.2 The lower and upper bounds for *r*th powers of metallic ratios M_k^r are denoted by $L_r(k)$ and $U_r(k)$ respectively. In this paper, I will determine the bounds $L_r(k)$ and $U_r(k)$ such that $L_r(k) \le M_k^r \le U_r(k) \quad (2.4)$.

To know more about metallic ratios and their properties see **[4 – 10]**.

III. Bounds for Powers of Metallic Ratios

3.1 Theorem 1

For any natural number *k*, if M_k is the *k*th Metallic Ratio, then we have

$1 \le M_k^0 \le 1$ (3.1)
$k \le M_k^1 \le k+1$ (3.2)
$k^2+1 \le M_k^2 \le k^2+k+1$ (3.3)
$k^3+2k \le M_k^3 \le k^3+k^2+2k+1$ (3.4)
$k^4+3k^2+1 \le M_k^4 \le k^4+k^3+3k^2+2k+1$ (3.5)
$k^5+4k^3+3k \le M_k^5 \le k^5+k^4+4k^3+3k^2+3k+1$ (3.6)
$k^6+5k^4+6k^2+1 \le M_k^6 \le k^6+k^5+5k^4+4k^3+6k^2+3k+1$ (3.7)
$k^7+6k^5+10k^3+4k \le M_k^7 \le k^7+k^6+6k^5+5k^4+10k^3+6k^2+4k+1$ (3.8)
$k^8+7k^6+15k^4+10k^2+1 \le M_k^8 \le k^8+k^7+7k^6+6k^5+15k^4+10k^3+10k^2+4k+1$ (3.9)

Proof: By (2.2), since M_k is non-zero, it follows that $M_k^0 = 1$ and so (3.1) is satisfied.

Now, for any natural number *k*, from (2.2), $M_k = \frac{k+\sqrt{k^2+4}}{2} \ge \frac{k+k}{2} = k$ and

$$M_k = \frac{k+\sqrt{k^2+4}}{2} \le \frac{k+\sqrt{k^2+4k+4}}{2} = \frac{k+(k+2)}{2} = k+1.$$

Thus, $k \le M_k^1 \le k+1$. This proves (3.2).

From (2.2), $M_k^2 = \frac{1}{4}\left(k^2+k^2+4+2k\sqrt{k^2+4}\right) = k\left(\frac{k+\sqrt{k^2+4}}{2}\right)+1 = kM_k+1$

Thus, $M_k^2 = kM_k + 1$ (3.10).

For any non-negative integer *r* let us assume that

$L_r(k) \le M_k^r \le U_r(k)$ (3.11)
$L_{r+1}(k) \le M_k^{r+1} \le U_{r+1}(k)$ (3.12)

Then using (3.10) we get $M_k^{r+2} = M_k^r \times M_k^2 = M_k^r \times \left(kM_k+1\right) = kM_k^{r+1} + M_k^r$ (3.13)

Now from (3.11) and (3.12), we get $kL_{r+1}(k) + L_r(k) \le kM_k^{r+1} + M_k^r \le kU_{r+1}(k) + U_r(k)$ (3.14)

Thus from (3.13) and (3.14), we get $kL_{r+1}(k) + L_r(k) \le M_k^{r+2} \le kU_{r+1}(k) + U_r(k)$

Hence, we get $L_{r+2}(k) \le M_k^{r+2} \le U_{r+2}(k)$ where

$L_{r+2}(k) = kL_{r+1}(k) + L_r(k),\ U_{r+2}(k) = kU_{r+1}(k) + U_r(k)$ (3.15).

Now using (3.11), (3.12) and (3.15) we can easily prove the inequalities from (3.3) to (3.9).

In particular, if $r = 0$ then from (3.1) and (3.2) we have

$L_0(k) = 1, L_1(k) = k, U_0(k) = 1, U_1(k) = k+1$. Now from (3.15), we get

$L_2(k) = kL_1(k) + L_0(k) = k^2 + 1$ and $U_2(k) = kU_1(k) + U_0(k) = k^2 + k + 1$.

Thus from $L_2(k) \le M_k^2 \le U_2(k)$ we obtain $k^2 + 1 \le M_k^2 \le k^2 + k + 1$. This proves (3.3).

Similarly if $r = 1$, from (3.15), we see that

$L_3(k) = kL_2(k) + L_1(k) = k^3 + 2k$ and $U_3(k) = kU_2(k) + U_1(k) = k^3 + k^2 + 2k + 1$.

Thus from $L_3(k) \le M_k^3 \le U_3(k)$ we obtain $k^3 + 2k \le M_k^3 \le k^3 + k^2 + 2k + 1$. This proves (3.4).

In similar fashion, using (3.15) recursively, we can obtain equations (3.5) to (3.9).

This completes the proof.

IV. Ramanujan Summation

The great Indian mathematician Srinivasa Ramanujan introduced a novel way of assigning a sum to divergent series like Cesaro Summation method. Today such results provided by Ramanujan were known as Ramanujan Summation Method.

I present two Ramanujan Summation identities. For details of these results, see **[1 – 3].**

For any positive integer r, we have

$$(RS)\ \sum_{k=1}^{\infty} k^{2r} = 1^{2r} + 2^{2r} + 3^{2r} + \cdots = 0 \quad (4.1)$$

$$(RS)\ \sum_{k=1}^{\infty} k^{2r-1} = 1^{2r-1} + 2^{2r-1} + 3^{2r-1} + \cdots = -\frac{B_{2r}}{2r} \quad (4.2)$$

where B_r is the rth Bernoulli number and $B_0 = 1$. The prefix *RS* mentioned before equations (4.1) and (4.2) indicate the term ‘Ramanujan Summation’.

In **[1]** by the corresponding author, it has been shown that

$$(RS)\ \sum_{k=1}^{\infty} k^{r} = \int_{-1}^{0} \left(\sum_{k=1}^{n} k^{r} \right) dn \quad (4.3)$$

In (4.3) the term $\sum_{k=1}^{\infty} k^r$ denotes the sum of *r*th powers of positive integers. Thus according to (4.1) and (4.2), we notice that the Ramanujan summation of even powers of positive integers is zero and that of odd powers of positive integers is connected to the Bernoulli Numbers. In this paper, I will determine Ramanujan Summation for the bounds of powers of metallic ratios.

V. Ramanujan Summation and Bounds of Powers of Metallic Ratios

Using the lower and upper bounds namely $L_r(k), U_r(k)$ for *r*th powers of metallic ratios M_k^r obtained in theorem 1, we can obtain the Ramanujan Summation for them and see if we could get some pattern.

5.1 Definitions

In view of (4.3), the Ramanujan Summation for the lower and upper bounds $L_r(k), U_r(k)$ is defined through the following expressions.

$$(RS)\ \sum_{k=1}^{\infty} L_r(k) = \int_{-1}^{0} \left(\sum_{k=1}^{n} L_r(k) \right) dn \quad (5.1)$$

$$(RS)\ \sum_{k=1}^{\infty} U_r(k) = \int_{-1}^{0} \left(\sum_{k=1}^{n} U_r(k) \right) dn \quad (5.2)$$

5.2 Theorem 2

$$(RS)\ \sum_{k=1}^{\infty} L_0(k) = (RS)\ \sum_{k=1}^{\infty} U_0(k) = -\frac{1}{2} \quad (5.3)$$

Proof: From (3.1), we know that $L_0(k) = U_0(k) = 1$. Hence, by definitions (5.1) and (5.2) we have $(RS)\ \sum_{k=1}^{\infty} L_0(k) = (RS)\ \sum_{k=1}^{\infty} U_0(k) = \int_{-1}^{0} \left(\sum_{k=1}^{n} (1) \right) dn = \int_{-1}^{0} n\, dn = -\frac{1}{2}$.

This completes the proof.

5.3 Theorem 3

For any positive integer *r*, we have

$$(RS)\ \sum_{k=1}^{\infty} L_{2r}(k) = -\frac{1}{2} \quad (5.4)$$

Proof: First, we observe that

$$L_{2r}(k) = \binom{2r}{0}k^{2r} + \binom{2r-1}{1}k^{2r-2} + \binom{2r-2}{2}k^{2r-4} + \cdots + \binom{r+1}{r-1}k^2 + \binom{r}{r}k^0$$
$$= \sum_{m=0}^{r}\binom{2r-m}{m}k^{2r-2m}$$

Using (4.1), (4.3) and (5.3) we have

$$(RS)\sum_{k=1}^{\infty} L_{2r}(k) = \int_{-1}^{0}\left(\sum_{k=1}^{n} L_{2r}(k)\right)dn = \int_{-1}^{0}\left(\sum_{k=1}^{n}\sum_{m=0}^{r}\binom{2r-m}{m}k^{2r-2m}\right)dn$$
$$= \binom{2r}{0}\int_{-1}^{0}\left(\sum_{k=1}^{n}k^{2r}\right)dn + \binom{2r-1}{1}\int_{-1}^{0}\left(\sum_{k=1}^{n}k^{2r-2}\right)dn$$
$$+\cdots+\binom{r+1}{r-1}\int_{-1}^{0}\left(\sum_{k=1}^{n}k^{2}\right)dn + \binom{r}{r}\int_{-1}^{0}\left(\sum_{k=1}^{n}(k^{0})\right)dn$$
$$= \binom{2r}{0}(RS)\left(\sum_{k=1}^{\infty}k^{2r}\right) + \binom{2r-1}{1}(RS)\left(\sum_{k=1}^{\infty}k^{2r-2}\right)$$
$$+\cdots+\binom{r+1}{r-1}(RS)\left(\sum_{k=1}^{\infty}k^{2}\right) + \int_{-1}^{0}\left(\sum_{k=1}^{n}L_0(k)\right)dn$$
$$= 0+0+\cdots+0+(RS)\sum_{k=1}^{\infty}L_0(k) = -\frac{1}{2}$$

This completes the proof.

5.4 Theorem 4

If v is any positive integer and if $L_v(k), U_v(k)$ are lower and upper bounds of M_k^v then

$$U_v(k) = L_v(k) + L_{v-1}(k) \quad (5.5)$$

Proof: We will prove (5.5) by Mathematical Induction. For v = 1, from (3.1) and (3.2), we have $U_1(k) = k+1$ and $L_1(k) + L_0(k) = k+1$. Hence the result is true for v = 1.

Now by Induction Hypothesis, assume that (5.5) is true for all values of v up to r + 1, i.e. for all $v = 1, 2, 3, \ldots, r-1, r, r+1$. We will prove for $v = r + 2$.

Using (3.15), and the Induction Hypothesis, we get

$$U_{r+2}(k) = kU_{r+1}(k) + U_r(k) = k\left[L_{r+1}(k) + L_r(k)\right] + \left[L_r(k) + L_{r-1}(k)\right]$$
$$= \left[kL_{r+1}(k) + L_r(k)\right] + \left[kL_r(k) + L_{r-1}(k)\right] = L_{r+2}(k) + L_{r+1}(k)$$

Hence (5.5) remain true for $v = r + 2$ also. Thus by Principle of Mathematical Induction, (5.5) is true for all positive integers v. This completes the proof.

5.5 Theorem 5
If r is any non-negative integer, then

$$(RS)\ \sum_{k=1}^{\infty} U_{2r+1}(k) = (RS)\ \sum_{k=1}^{\infty} L_{2r+1}(k) - \frac{1}{2} \qquad (5.6)$$

$$(RS)\ \sum_{k=1}^{\infty} U_{2r+2}(k) = (RS)\ \sum_{k=1}^{\infty} L_{2r+1}(k) - \frac{1}{2} \qquad (5.7)$$

$$(RS)\ \sum_{k=1}^{\infty} U_{2r+1}(k) = (RS)\ \sum_{k=1}^{\infty} U_{2r+2}(k) \qquad (5.8)$$

Proof: Using (5.1), (5.4), (5.5) and additive property of integrals, we have

$$(RS)\ \sum_{k=1}^{\infty} U_{2r+1}(k) = (RS)\sum_{k=1}^{\infty}\left[L_{2r+1}(k) + L_{2r}(k)\right] = \int_{-1}^{0}\left(\sum_{k=1}^{n}\left[L_{2r+1}(k) + L_{2r}(k)\right]\right) dn$$
$$= \int_{-1}^{0}\left(\sum_{k=1}^{n} L_{2r+1}(k)\right) dn + \int_{-1}^{0}\left(\sum_{k=1}^{n} L_{2r}(k)\right) dn$$
$$= (RS)\ \sum_{k=1}^{\infty} L_{2r+1}(k) + (RS)\ \sum_{k=1}^{\infty} L_{2r}(k) = (RS)\ \sum_{k=1}^{\infty} L_{2r+1}(k) - \frac{1}{2}$$

This proves (5.6). Similarly, we have

$$(RS)\ \sum_{k=1}^{\infty} U_{2r+2}(k) = (RS)\sum_{k=1}^{\infty}\left[L_{2r+2}(k) + L_{2r+1}(k)\right] = \int_{-1}^{0}\left(\sum_{k=1}^{n}\left[L_{2r+1}(k) + L_{2(r+1)}(k)\right]\right) dn$$
$$= \int_{-1}^{0}\left(\sum_{k=1}^{n} L_{2r+1}(k)\right) dn + \int_{-1}^{0}\left(\sum_{k=1}^{n} L_{2(r+1)}(k)\right) dn$$
$$= (RS)\ \sum_{k=1}^{\infty} L_{2r+1}(k) + (RS)\ \sum_{k=1}^{\infty} L_{2(r+1)}(k) = (RS)\ \sum_{k=1}^{\infty} L_{2r+1}(k) - \frac{1}{2}$$

This proves (5.7). Identity (5.8) follows directly upon comparing (5.6) and (5.7).

This completes the proof.

VI. Computation

Using the theorems established in section 5, we can calculate the Ramanujan Summation of lower and upper bounds of *r*th power of metallic ratios. First, let us compute the Ramanujan Summation of sum of $L_1(k) = k$. Using (5.1), we have

$$(RS)\sum_{k=1}^{\infty} L_1(k) = \int_{-1}^{0}\left(\sum_{k=1}^{n} k\right) dn = \int_{-1}^{0}\left(\frac{n(n+1)}{2}\right) dn = -\frac{1}{12} \quad (6.1)$$

Now using (5.6), we get $(RS)\sum_{k=1}^{\infty} U_1(k) = (RS)\sum_{k=1}^{\infty} L_1(k) - \frac{1}{2} = -\frac{1}{12} - \frac{1}{2} = -\frac{7}{12}$ (6.2)

From (5.4), we see that $(RS)\sum_{k=1}^{\infty} L_2(k) = -\frac{1}{2}$ (6.3)

Using (5.7), we get $(RS)\sum_{k=1}^{\infty} U_2(k) = (RS)\sum_{k=1}^{\infty} L_1(k) - \frac{1}{2} = -\frac{1}{12} - \frac{1}{2} = -\frac{7}{12}$ (6.4)

Again using (5.1), we get

$$(RS)\sum_{k=1}^{\infty} L_3(k) = \int_{-1}^{0}\left(\sum_{k=1}^{n} (k^3 + 2k)\right) dn = \int_{-1}^{0}\left[\left(\frac{n(n+1)}{2}\right)^2 + n(n+1)\right] dn = -\frac{19}{120} \quad (6.5)$$

From (5.6), we have $(RS)\sum_{k=1}^{\infty} U_3(k) = (RS)\sum_{k=1}^{\infty} L_3(k) - \frac{1}{2} = -\frac{19}{120} - \frac{1}{2} = -\frac{79}{120}$ (6.6)

From (5.4), we see that $(RS)\sum_{k=1}^{\infty} L_4(k) = -\frac{1}{2}$ (6.7)

Using (5.7), we get $(RS)\sum_{k=1}^{\infty} U_4(k) = (RS)\sum_{k=1}^{\infty} L_3(k) - \frac{1}{2} = -\frac{19}{120} - \frac{1}{2} = -\frac{79}{120}$ (6.8)

$$(RS)\sum_{k=1}^{\infty} L_5(k) = \int_{-1}^{0}\left(\sum_{k=1}^{n} (k^5 + 4k^3 + 3k)\right) dn = -\frac{1}{252} + \frac{1}{30} - \frac{1}{4} = -\frac{139}{630} \quad (6.9)$$

From (5.6), we have $(RS)\sum_{k=1}^{\infty} U_5(k) = (RS)\sum_{k=1}^{\infty} L_5(k) - \frac{1}{2} = -\frac{139}{630} - \frac{1}{2} = -\frac{227}{315}$ (6.10)

From (5.4), we see that $(RS)\sum_{k=1}^{\infty} L_6(k) = -\frac{1}{2}$ (6.11)

Using (5.7), we get $(RS)\sum_{k=1}^{\infty}U_6(k)=(RS)\sum_{k=1}^{\infty}L_5(k)-\frac{1}{2}=-\frac{139}{630}-\frac{1}{2}=-\frac{227}{315}$ (6.12)

$$(RS)\sum_{k=1}^{\infty}L_7(k)=\int_{-1}^{0}\left(\sum_{k=1}^{n}(k^7+6k^5+10k^3+4k)\right)dn=\frac{1}{240}-\frac{1}{42}+\frac{1}{12}-\frac{1}{3}=-\frac{151}{560} \quad (6.13)$$

From (5.6), we have $(RS)\sum_{k=1}^{\infty}U_7(k)=(RS)\sum_{k=1}^{\infty}L_7(k)-\frac{1}{2}=-\frac{151}{560}-\frac{1}{2}=-\frac{431}{560}$ (6.14)

From (5.4), we see that $(RS)\sum_{k=1}^{\infty}L_8(k)=-\frac{1}{2}$ (6.15)

Using (5.7), we get $(RS)\sum_{k=1}^{\infty}U_8(k)=(RS)\sum_{k=1}^{\infty}L_7(k)-\frac{1}{2}=-\frac{151}{560}-\frac{1}{2}=-\frac{431}{560}$ (6.16)

VII. Conclusion

This paper mainly focus about determining Ramanujan Summation values for lower and upper bounds for powers of metallic ratios. Ramanujan devised a novel method of assigning a real number to the divergent series, which today is named after him as Ramanujan Summation. As mentioned in (4.1), (4.2), Ramanujan proved that the Ramanujan Summation of even powers of natural numbers is 0 and that of odd powers is connected to the Bernoulli Numbers. In this paper, I had extended the concept of Ramanujan Summation technique to the bounds of powers of metallic ratios obtained in Theorem 1 from equations (3.1) to (3.9).

In performing such modification, I had shown that the Ramanujan Summation of all lower bounds of even powers of metallic ratios namely $L_{2r}(k)$ is $-\frac{1}{2}$ through equation (5.4) of theorem 3. This result resembles the constant value of 0 obtained by Ramanujan for sum of even powers of natural numbers in equation (4.1). In theorem 5, equations (5.6) and (5.7) provide the way of computing the Ramanujan Summation of upper bounds of odd and even order respectively knowing the Ramanujan Summation of lower bound. These two results will subsequently prove that the Ramanujan Summation of upper bounds of consecutive orders namely $U_{2r+1}(k), U_{2r+2}(k)$ of powers of metallic ratios are equal as shown in equation (5.8).

In section 6, through equations (6.1) to (6.16), I had computed the Ramanujan Summation values for the first eight lower and upper bounds using the results established in theorems 3 and 5. These values resemble the Ramanujan Summation method for sum of powers of natural numbers though we get different values in this case. It is well known in Analytic Number Theory that Ramanujan Summation method presented in equations (4.1) and (4.2) is connected to the values of Riemann Zeta Function through analytic continuation. In similar way, the concept of Ramanujan Summation of bounds of powers of metallic ratios discussed in this paper can be thought of analytic continuation of powers of metallic ratios in to extended complex plane. In this sense, the results proved in this paper, paves more scope for further investigations.

Finally, we see that the lower and upper bounds for powers of metallic ratios evaluated at $k = 1$ provides consecutive Fibonacci numbers. For example, from $L_r(k) \le M_k^r \le U_r(k)$ we notice through equations (3.1) to (3.9) of theorem 1 that $L_r(1)=F_{r+1}, U_r(1)=F_{r+2}$ where F_{r+1}, F_{r+2} are (r+1)th, (r+2)th Fibonacci numbers respectively. This paper thus contains plenty of new results as well as provides scope for further extension regarding the connection of powers of metallic ratios with Ramanujan Summation method.

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