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Stability Analysis of Low Carbon Product Price Game Model with Double Delay

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Abstract — *This paper, bifurcation theory is used to study the price game model of low-carbon products with double time delays. Firstly, the delay decision of the retailer and the manufacturer is selected as the delay parameter, and the stability point and the critical value of maintaining the stability of the system are obtained by using the eigenvalue method, and the condition of Hopf bifurcation is discussed. Finally, the validity of the conclusion is verified by numerical simulation with mathematical software.*

Keywords — *Hopf bifurcation, stability, price game model, numerical simulation*

I. MODEL BUILDING

In recent years, the social economy continues to grow rapidly, but with the increase of economic wealth, more and more resources are consumed, resulting in frequent incidents of resource waste and environmental damage. Therefore, ecological environment and economic development are the two cornerstones of the existence of human society, and neither is indispensable. Low-carbon products are an effective way to place equal emphasis on both green and economic development[1-3]. Therefore, scholars have carried out a lot of studies on green supply chain. In this paper, Sun Zhenjie[4] explored the role of market-oriented green supply chain integration on the growth benefits of retail enterprises, and analyzed the adjustment effect of competitive intensity to guide China's retail enterprises to pay full attention to green economy. Literature [5] gives the optimal strategy of manufacturer and retailer under decentralized decision-making, and builds a green supply chain model for manufacturer and retailer, so as to realize revenue sharing through coordination mechanism.

Wang Jing, Si Fengshan et al. [6] adopted nonlinear dynamics theory and system complexity theory to establish a delay continuous differential price game model, analyzed the dynamic behavior and stability of the price game model, and further adopted variable feedback control method to control the stability of the unstable system. The price game system is as follows:

$$\begin{cases} \dfrac{dp(t)}{dt} = v_R p(a\beta - 2bp\beta + bw\beta + \alpha\theta\beta), \\ \dfrac{dw(t)}{dt} = v_M w(a\beta - bp\beta + \alpha\theta\beta), \end{cases}$$

where p and w represent the selling price of the product and the wholesale price of the product provided by the manufacturer respectively, v_R and v_M represent the adjustment speed of retail price and wholesale price respectively, a represents the maximum potential market demand, β represents revenue sharing coefficient; Both α and b represent sensitivity coefficients, θ represents the low-carbon degree of the product, that is, the green level of the product. In addition, the paper also considers the price game model in which retailers adopt delay strategy:

$$\begin{cases} \dfrac{dp(t)}{dt} = v_R p(a\beta - 2bp(t-\tau)\beta + bw\beta + \alpha\theta\beta), \\ \dfrac{dw(t)}{dt} = v_M w(a\beta - bp(t-\tau)\beta + \alpha\theta\beta), \end{cases}$$

Typically, manufacturers and retailers in price decisions require reference to the current profit margins, also need to follow a certain moment ago profit margins, but the reference information of time lag between manufacturers and retailers will differ, so in this paper, on the basis of the above price game model, with two time delays is given price game model of low carbon products:

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$$\begin{cases} \dfrac{dp(t)}{dt} = v_R p(a\beta - 2bp(t-\tau_1)\beta + bw(t-\tau_2)\beta + \alpha\theta\beta), \\ \dfrac{dw(t)}{dt} = v_M w(a\beta - bp(t-\tau_1)\beta + \alpha\theta\beta), \end{cases} \tag{1}$$

where, τ_1 represents the delay parameter of the retailer in making price decisions, τ_2 represents the delay parameter when the manufacturer makes the price decision.

II. MODEL BUILDING

We first assume that the equilibrium point of system (1) is (x_0, y_0), then it satisfies the following equation:

$$\begin{cases} v_R p(a\beta - 2bp(t-\tau_1)\beta + bw(t-\tau_2)\beta + \alpha\theta\beta) = 0, \\ v_M w(a\beta - bp(t-\tau_1)\beta + \alpha\theta\beta) = 0, \end{cases}$$

the equilibrium point $E_1(0,0)$, $E_2(\frac{a+\alpha\theta}{2b},0)$ and $E_3(\frac{a+\alpha\theta}{b},\frac{a+\alpha\theta}{b})$ can be obtained by solving the above equation. Combined with practical problems, it is meaningful only when the sales price and wholesale price are positive. Therefore, we only analyze the dynamics analysis of the system by the time-delay parameters at the equilibrium point $E_3(\frac{a+\alpha\theta}{b},\frac{a+\alpha\theta}{b})$, where $p_0 = w_0 = \frac{a+\alpha\theta}{b}$.

Let $x(t) = p(t) - p_0$ and $y(t) = w(t) - w_0$. After linearizing the system (1) at the equilibrium point $E_3(x_0, y_0)$, the linearization equation is

$$\begin{cases} \dfrac{dx(t)}{dt} = a_1 x(t) + a_2 x(t-\tau_1) + a_3 y(t-\tau_2), \\ \dfrac{dy(t)}{dt} = b_1 y(t) + b_2 x(t-\tau_1), \end{cases} \tag{2}$$

among them

$a_1 = v_R(a\beta - 2bx_0\beta + by_0\beta + \alpha\theta\beta)$, $a_2 = -2bv_R\beta x_0$, $a_3 = bv_R\beta x_0$, $b_1 = v_M(a\beta - bx_0\beta + \alpha\theta\beta)$, $b_2 = -bv_M\beta y_0$.

The characteristic equation of the system (2) is

$$\lambda^2 - a_2\lambda e^{-\lambda\tau_1} - a_3 b_2 e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{3}$$

In order to study the stability of equilibrium point E_3 of the system and the generation of *Hopf* branches, we need to discuss the distribution of roots of equation (3). Since there are two delays τ_1 and τ_2 in equation (3), we discuss the stability of positive equilibrium point for system (1) in the following cases.

Case 1: $\tau_1 = \tau_2 = 0$

Theorem 1. For system (1), if $\tau_1 = \tau_2 = 0$, when $(H_1): a_3b_2 < 0, a_2 < 0, a_2a_3b_2 > 0$, the equilibrium point E_3 is locally asymptotically stable, Otherwise the equilibrium point E_3 is unstable.

Proof. When $\tau_1 = \tau_2 = 0$, the characteristic equation of system (3) becomes

$$\lambda^2 - a_2\lambda - a_3b_2 = 0, \tag{4}$$

Then, according to the characteristic equation (4) and the Raws-Hurwitz criterion, when $(H_1): a_3b_2 < 0, a_2 < 0, a_2a_3b_2 > 0$, both roots of equation (4) have negative real parts, then for $\tau_1 = \tau_2 = 0$, the equilibrium point E_3 of the system is locally asymptotically stable. Otherwise, the equilibrium point E_3 of the system is unstable. To prove to complete.

Case 2: $\tau_1 = 0, \tau_2 > 0$

Lemma 1. For system (1), if $\tau_1 = 0, \tau_2 > 0$ is satisfied, and let's say that (H_1) is true. In this case, the characteristic equation has a pair of pure imaginary roots $\pm i\omega_2(\omega_2 > 0)$, which are single roots, where

$$\omega_2 = \sqrt{\frac{-a_2{}^2 + \sqrt{a_2{}^4 + 4a_3{}^2b_2{}^2}}{2}}, \quad \tau_{2k} = \frac{1}{\omega_2}(\arcsin\frac{a_2\omega_2}{a_3b_2} + 2k\pi), k = 0,1,2,\cdots,$$

Proof. When $\tau_1 = 0, \tau_2 > 0$, the characteristic equation of system (1) becomes

$$\lambda^2 - a_2\lambda - a_3b_2e^{-\lambda\tau_2} = 0, \tag{5}$$

First, we assume that $\lambda = i\omega_2(\omega_2 > 0)$ is a root of the characteristic equation (5), then it satisfies the following equation

$$-\omega_2{}^2 - ia_2\omega_2 - a_3b_2\cos\omega_2\tau_2 + ia_3b_2\sin\omega_2\tau_2 = 0, \tag{6}$$

the separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega_2^{\ 2} - a_3 b_2 \cos\omega_2\tau_2 = 0, \\ -a_2\omega_2 + a_3 b_2 \cos\omega_2\tau_2 = 0, \end{cases} \tag{7}$$

from (7) we obtain

$$\omega_2^{\ 4} + a_2^{\ 2}\omega_2^{\ 2} - a_3^{\ 2}b_2^{\ 2} = 0, \tag{8}$$

calculated

$$\omega_2^{\ 2} = \frac{-a_2^{\ 2} \pm \sqrt{a_2^{\ 4} + 4a_3^{\ 2}b_2^{\ 2}}}{2}, \tag{9}$$

Due to $\omega_2 > 0$, so

$$\omega_2 = \sqrt{\frac{-a_2^{\ 2} + \sqrt{a_2^{\ 4} + 4a_3^{\ 2}b_2^{\ 2}}}{2}},$$

Therefore, the characteristic equation (7) has a pair of pure imaginary roots $\pm i\omega_2$, which can be calculated according to Equation (7)

$$\tau_{2k} = \frac{1}{\omega_2}(\arcsin\frac{a_2\omega_2}{a_3b_2} + 2k\pi), k = 0,1,2,\cdots.$$

To prove to complete.

Lemma 2. Let $\tau_{20} = min\{\tau_{2k} | k = 0,1,2,\cdots\}$, and the corresponding value of ω_2 be ω_{20}, and let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be the root of the characteristic equation (5), and the conditions $\alpha(\tau_{20}) = 0$ and $\alpha(\tau_{20}) = 0$ are satisfied, then the cross-sectional condition $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_{20}} > 0$ is true.

Proof. Substitute $\lambda(\tau_2)$ into the characteristic equation (5) and take the derivative of both ends with respect to τ_2 at the same time, using the implicit function theorem:

$$\frac{d\lambda}{d\tau_2}|_{\tau_2=\tau_{20}} = -\frac{a_3b_2\lambda e^{-\lambda\tau_2}}{2\lambda - a_2 + a_3b_2\tau_2e^{-\lambda\tau_2}}|_{\tau_2=\tau_{20}},$$

so

$$[\frac{d\lambda}{d\tau_2}]^{-1}|_{\tau_2=\tau_{20}} = -\frac{2\lambda - a_2 + a_3b_2\tau_2e^{-\lambda\tau_2}}{a_3b_2\lambda e^{-\lambda\tau_2}}|_{\tau_2=\tau_{20}} = -\frac{2\lambda - a_2}{a_3b_2\lambda e^{-\lambda\tau_2}} - \frac{\tau_2}{\lambda}|_{\tau_2=\tau_{20}}$$

$$= -\frac{2i\omega_2 - a_2}{a_3b_2i\omega_2e^{-i\omega_2\tau_2}} - \frac{\tau_2}{i\omega_2} = -\frac{2i\omega_2 - a_2}{a_3b_2i\omega_2(\cos\omega_2\tau_2 - i\sin\omega_2\tau_2)} - \frac{\tau_2}{i\omega_2}$$

$$\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1} = \frac{a_3b_2\omega_2(a_2\sin\omega_2\tau_2 - 2\cos\omega_2\tau_2)}{(a_3b_2\omega_2\cos\omega_2\tau_2)^2 + (a_3b_2\omega_2\sin\omega_2\tau_2)^2},$$

according to Equation (7), it can be known

$$\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1} = \frac{a_2^{\ 2}\omega_2^{\ 2} + 2\omega_2^{\ 3}}{(a_3b_2\omega_2\cos\omega_2\tau_2)^2 + (a_3b_2\omega_2\sin\omega_2\tau_2)^2} > 0,$$

then the cross-sectional condition is true. The proof is completed.

According to the above calculation and analysis and the *Hopf* branching theory in literature [7-8], the following theorems are obtained.

Theorem 2. For system (1), when $\tau_1 = 0, \tau_2 > 0$, and satisfies condition (H_1) ,then when $\tau_2 \in [0, \tau_{20})$, the positive equilibrium point E_3 is locally asymptotically stable, When $\tau_2 > \tau_{20}$, the positive equilibrium E_3 is unstable; When $\tau_2 = \tau_{20}$, the *Hopf* branch appears at the equilibrium point E_3.

Case 3: $\tau_1 = \tau_2 = \tau$

Lemma 3. For system (1), if $\tau_1 = \tau_2 = \tau$ is satisfied, and assuming that (H_2) is true.

(1) If $2a_3b_2 + a_2^{\ 2} \le 0$, that is, equation (11) has no positive root, then the characteristic equation (11) has no pure imaginary root,

(2) When $2a_3b_2 + a_2^{\ 2} > 0$ is true and $\tau = \tau_k$, the characteristic equation (11) has positive roots, then equation (10) has a pair of pure imaginary roots $\pm i\omega$, where:

$$\omega=\sqrt{\frac{2a_3b_2+a_2{}^2+\sqrt{(2a_3b_2+a_2{}^2)^2-4a_3{}^2b_2{}^2}}{2}}\ ,\quad \tau_k=\frac{1}{\omega}[\arcsin(-\frac{a_2\omega}{\omega^2-a_3b_2})+2k\pi],k=0,1,2,\cdots.$$

Proof. When $\tau_1=\tau_2=\tau$,the characteristic equation of system (3) becomes

$$\lambda^2-a_2\lambda e^{-\lambda\tau}-a_3b_2e^{-2\lambda\tau}=0\,, \tag{10}$$

multiply both sides of equation (10) by $e^{\lambda\tau}$ and get

$$\lambda^2e^{\lambda\tau}-a_2\lambda-a_3b_2e^{-\lambda\tau}=0\,, \tag{11}$$

We assume that $\lambda=i\omega(\omega>0)$ is a solution to the characteristic equation (11), which satisfies the following equation:

$$-\omega^2(\cos\omega\tau+i\sin\omega\tau)-ia_2\omega-a_3b_2(\cos\omega\tau-i\sin\omega\tau)=0\,,$$

by separating the real and imaginary parts, we get:

$$\begin{cases}(\omega^2+a_3b_2)\cos\omega\tau=0\\(\omega^2-a_3b_2)\sin\omega\tau=-a_2\omega\end{cases} \tag{12}$$

If condition $(H_2):\omega^2+a_3b_2\neq0$ holds, equation (12) can be simplified as:

$$\begin{cases}\cos\omega\tau=0\\(\omega^2-a_3b_2)\sin\omega\tau=-a_2\omega\end{cases} \tag{13}$$

it can be obtained from equation (13)

$$\omega^4-(2a_3b_2+a_2{}^2)\omega^2+a_3{}^2b_2{}^2=0\,, \tag{14}$$

have a type

$$\omega^2=\frac{2a_3b_2+a_2{}^2\pm\sqrt{(2a_3b_2+a_2{}^2)^2-4a_3{}^2b_2{}^2}}{2}\,,$$

$(2a_3b_2+a_2{}^2)^2-4a_3{}^2b_2{}^2>0$ is known from calculation. Therefore, when $2a_3b_2+a_2{}^2\leq0$ is appropriate, the characteristic equation (11) has no positive roots. When $2a_3b_2+a_2{}^2>0$, ω is the real root, then the characteristic equation (11) has a positive root, and the calculation has

$$\omega=\sqrt{\frac{2a_3b_2+a_2{}^2+\sqrt{(2a_3b_2+a_2{}^2)^2-4a_3{}^2b_2{}^2}}{2}}\,,$$

therefore, the characteristic equation (11) has a pair of pure imaginary roots $\pm i\omega$, which can be calculated according to Equation (13):

$$\tau_k=\frac{1}{\omega}[\arcsin(-\frac{a_2\omega}{\omega^2-a_3b_2})+2k\pi],k=0,1,2,\cdots.$$

The proof is completed.

Lemma 4. Let $\tau_0=min\{\tau_k\,|\,k=0,1,2,\cdots\}$ and say the corresponding value of ω is ω_0, let $\lambda(\tau)=\alpha(\tau)+i\omega(\tau)$ is the root of the characteristic equation (11), and satisfy the conditions $\alpha(\tau_0)=0$ and $\omega(\tau_0)=\omega_0$, and assume that (H_1) is true, then the cross-sectional condition $\mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}|_{\tau=\tau_0}>0$ is true.

Proof. Substitute $\lambda(\tau)$ into the characteristic equation (10) and take the derivative of both ends with respect to τ at the same time, using the implicit function theorem:

$$\frac{d\lambda}{d\tau}|_{\tau=\tau_0}=\frac{-a_2\lambda^2e^{-\lambda\tau}-2a_3b_2\lambda e^{-2\lambda\tau}}{2\lambda-a_2e^{-\lambda\tau}+a_2\lambda\tau e^{-\lambda\tau}+2a_3b_2\tau e^{-2\lambda\tau}}|_{\tau=\tau_0}\,,$$

so

$$\begin{aligned}[\frac{d\lambda}{d\tau}]^{-1}\Big|_{\tau=\tau_0}&=\frac{2\lambda e^{\lambda\tau}-a_2}{-a_2\lambda^2-2\lambda a_3b_2e^{-\lambda\tau}}-\frac{\tau}{\lambda}\Big|_{\tau=\tau_0}\\&=\frac{2i\omega_0(\cos\omega_0\tau_0+i\sin\omega_0\tau_0)-a_2}{a_2\omega_0{}^2-2i\omega_0a_3b_2(\cos\omega_0\tau_0-i\sin\omega_0\tau_0)}-\frac{\tau_0}{i\omega_0},\end{aligned}$$

$$\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau=\tau_2}=\frac{-4\omega_0{}^2a_3b_2\cos^2\omega_0\tau_0+(-2\omega_0\sin\omega_0\tau_0-a_2)(a_2\omega_0{}^2-2\omega_0a_3b_2\cos\omega_0\tau_0)}{(a_2\omega_0{}^2-2\omega_0a_3b_2\sin\omega_0\tau_0)^2+(2\omega_0a_3b_2\cos\omega_0\tau_0)^2}$$
$$=\frac{-4\omega_0{}^2a_3b_2\cos^2\omega_0\tau_0}{(a_2\omega_0{}^2-2\omega_0a_3b_2\sin\omega_0\tau_0)^2+(2\omega_0a_3b_2\cos\omega_0\tau_0)^2},$$

if we assume (H_1) is true, then $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau=\tau_2}>0$, the cross-sectional condition is true.The proof is completed.

According to the above calculation and analysis and the *Hopf* branching theory in literature [7-8], the following theorems are obtained.

Theorem 3. For system (1), when satisfy $\tau_1=\tau_2=\tau$, assume that (H_1), (H_2) are true. The following conclusions are true: when $\tau\in[0,\tau_0)$, the equilibrium point of system (1) is locally asymptotically stable, when $\tau=\tau_0$, it has *Hopf* branch at the equilibrium point, When $\tau>\tau_0$, the equilibrium of system (1) is unstable.

Case 4: $\tau_1>0,\tau_2\in[0,\tau_{20})$ and $\tau_1\neq\tau_2$

Lemma 5. For system (1), let $G(\omega_1)=\omega_1{}^4-\omega_1{}^2a_2{}^2-a_3{}^2b_2{}^2+2a_3b_2a_2\omega_1\sin\omega_1\tau_1$, then $G(\omega_1)$ has positive roots $\{\omega_{11},\omega_{12},\omega_{13},\omega_{14}\}$, then when $\tau_1=\tau_{1i}^k$, and satisfied $\omega_{10}\tau_2<\frac{\pi}{2}$, equation (15) has a pure imaginary root $\pm i\omega_{1i}$, where

$$\tau_{1i}^{(k)}=\frac{1}{\omega_{1i}}[\arccos(-\frac{a_3b_2\cos\omega_{1i}\tau_2}{\omega_{1i}{}^2})+2k\pi],k=0,1,2,\cdots,i=1,2,3,4.$$

Proof. When $\tau_1>0,\tau_2\in[0,\tau_{20})$, multiply both sides of equation (3) by $e^{\lambda\tau}$ and get

$$\lambda^2e^{\lambda\tau_1}-a_2\lambda-a_3b_2e^{-\lambda\tau_2}=0, \tag{15}$$

assuming that $\lambda=i\omega_1(\omega_1>0)$ is a solution to the characteristic equation (15), it satisfies the following equation

$$-\omega_1{}^2\cos\omega_1\tau_1-\omega_1{}^2\sin\omega_1\tau_1-a_2i\omega_1-a_3b_2\cos\omega_1\tau_2+ia_3b_2\sin\omega_1\tau_2=0,$$

by separating the real and imaginary parts, we get

$$\begin{cases}\omega_1{}^2\cos\omega_1\tau_1+a_3b_2\cos\omega_1\tau_2=0\\ \omega_1{}^2\sin\omega_1\tau_1-a_3b_2\sin\omega_1\tau_2=-a_2\omega_1\end{cases} \tag{16}$$

to solve the above equations

$$\cos\omega_1\tau_1=-\frac{a_3b_2\cos\omega_1\tau_2}{\omega_1{}^2},\quad \sin\omega_1\tau_1=\frac{a_3b_2\sin\omega_1\tau_2-a_2\omega_1}{\omega_1{}^2},$$

the $\cos\omega_1\tau_1$ and the $\sin\omega_1\tau_1$ cancel out

$$\omega_1{}^4-\omega_1{}^2a_2{}^2-a_3{}^2b_2{}^2+2a_3b_2a_2\omega_1\sin\omega_1\tau_2=0. \tag{17}$$

Let $G(\omega_1)=\omega_1{}^4-\omega_1{}^2a_2{}^2-a_3{}^2b_2{}^2+2a_3b_2a_2\omega_1\sin\omega_1\tau_2$,then $G(0)=-a_3{}^2b_2{}^2<0$, and because of $\lim\limits_{\omega_1\to+\infty}G(\omega_1)\to+\infty$, then equation (15) has at least a finite number of positive roots, denoted as ω_{11}, ω_{12}, ω_{13}, ω_{14}, then $\{\tau_{1i}^{(k)}|i=1,2,3,4,k=1,2,\cdots\}$ corresponding to each positive root can be calculated by equation (16), and we know by calculation

$$\tau_{1i}^{(k)}=\frac{1}{\omega_{1i}}[\arccos(-\frac{a_3b_2\cos\omega_{1i}\tau_2}{\omega_{1i}{}^2})+2k\pi],k=0,1,2,\cdots,i=1,2,3,4.$$

Therefore, when $\tau_1=\tau_{1i}^k$, equation (15) has pure imaginary roots $\pm i\omega_1$.

Lemma 6. Let $\tau_{10}=min\{\tau_{1i}^{(0)}i=1,2,3,4\}$ and say the corresponding value of ω_1 is ω_{10}, let $\lambda(\tau_1)=\alpha(\tau_1)+i\omega(\tau_1)$ is the root of the characteristic equation (15) at $\tau_1=\tau_{10}$, and satisfy the conditions $\alpha(\tau_{10})=0$ and $\omega(\tau_{10})=\omega_{10}$, and assume that (H_1) is true, then the cross-sectional condition $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau_1=\tau_{10}}>0$ is true.

Proof. Substitute $\lambda(\tau_1)$ into the characteristic equation (15) and take the derivative of both sides with respect to τ_1:

$$\frac{d\lambda}{d\tau_1}|_{\tau_1=\tau_{10}}=-\frac{a_2\lambda^2e^{-\lambda\tau_1}+a_3b_2\lambda e^{-\lambda(\tau_1+\tau_2)}}{2\lambda-a_2e^{-\lambda\tau_1}+a_2\tau_1\lambda e^{-\lambda\tau_1}+a_3b_2(\tau_1+\tau_2)e^{-\lambda(\tau_1+\tau_2)}}|_{\tau_1=\tau_{10}}$$
$$=-\frac{a_2\lambda^2+a_3b_2\lambda e^{-\lambda\tau_2}}{2\lambda e^{\lambda\tau_1}-a_2+a_2\tau_1\lambda+a_3b_2(\tau_1+\tau_2)e^{-\lambda\tau_2}}$$

$$[\frac{d\lambda}{d\tau_1}]^{-1}\Big|_{\tau_1=\tau_{10}} = -\frac{2}{\lambda^2} - \frac{a_3 b_2 \tau_2 e^{-\lambda\tau_2} - a_2}{a_2\lambda^2 + a_3 b_2 \lambda e^{-\lambda\tau_2}} - \frac{\tau_1}{\lambda}\Big|_{\tau_1=\tau_{10}}$$

$$\begin{aligned}\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1} &= \frac{2}{\omega_{10}{}^2} + \frac{a_2 - a_3 b_2 \tau_2(\cos\omega_{10}\tau_2 - i\sin\omega_{10}\tau_2)}{a_3 b_2 i\omega_{10}(\cos\omega_{10}\tau_2 - i\sin\omega_{10}\tau_2)} \\ &= \frac{2}{\omega_{10}{}^2} + \frac{a_2 a_3 b_2 \omega_{10}\sin\omega_{10}\tau_2}{(a_3 b_2 \omega_{10}\cos\omega_{10}\tau_2)^2 + (a_3 b_2 \omega_{10}\sin\omega_{10}\tau_2)^2}\end{aligned}$$

it can be seen that when (H_1) is satisfied, $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau=\tau_1} > 0$ is true, that is, the cross-sectional condition is true. The proof is completed.

According to the above calculation and analysis and the *Hopf* branching theory in literature [7-8], the following theorems are obtained.

Theorem 4. For system (1), when $\tau_1 > 0, \tau_2 \in [0, \tau_{20})$, assume that (H_1) are true. The following conclusions are true: when $\tau_1 \in [0, \ \tau_{10})$, the equilibrium point of system (1) is locally asymptotically stable, when $\tau_1 = \tau_{10}$, it has *Hopf* branch at the equilibrium point, When $\tau_1 > \tau_{10}$, the equilibrium of system (1) is unstable.

III. NUMERICAL SIMULATION

In this section, we will verify the above results by using mathematical software to draw diagrams.

When $\tau_1 = 0, \tau_2 > 0$, we first select the same parameter as in literature [6] : $a = 1.2$, $b = 0.7$, $v_R = v_M = 0.5$, $\beta = 0.5$, $\alpha = 0.6$, $\theta = 0.8$. $p_0 = w_0 = 2.4, \omega_{20} = 0.2041, \tau_{20} = 6.53206$ is obtained through calculation, and these coefficients satisfy (H_1). When $\tau_2 = 5.5 < \tau_{20}$ is taken, system (1) is locally asymptotically stable at the equilibrium point, as shown in Figure 1. Take $\tau_2 = 6.53206 = \tau_{20}$, system (1) generates *Hopf* bifurcation at the equilibrium point, as shown in Figure 2. When $\tau_2 = 7 > \tau_{20}$ is taken, system (1) is unstable at the equilibrium point, as shown in Figure 3.

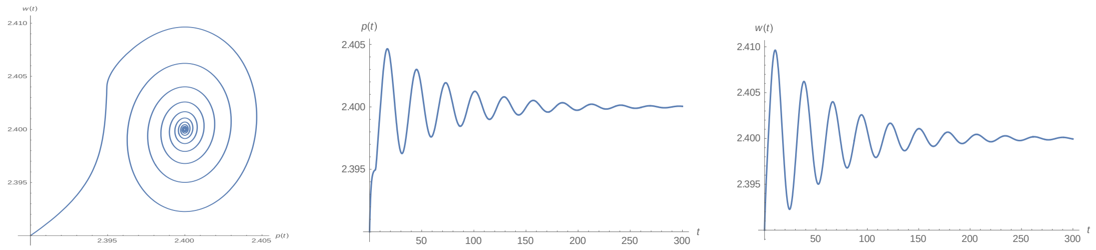

Fig. 1 Phase diagram and time history diagram of $\tau_2 = 5.5 < \tau_{20}$.

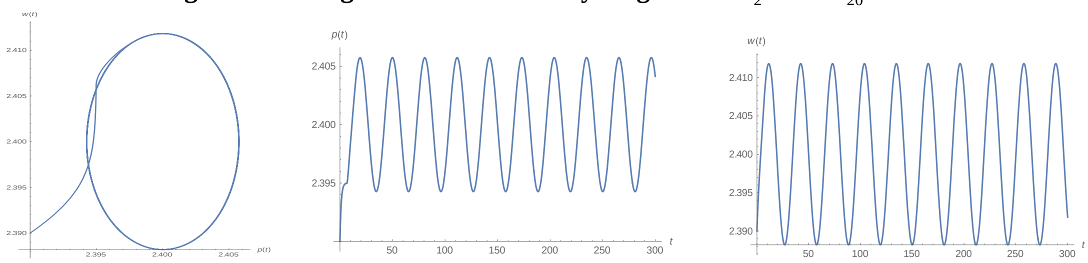

Fig. 2 Phase diagram and time history diagram of $\tau_2 = 6.53206 = \tau_{20}$.

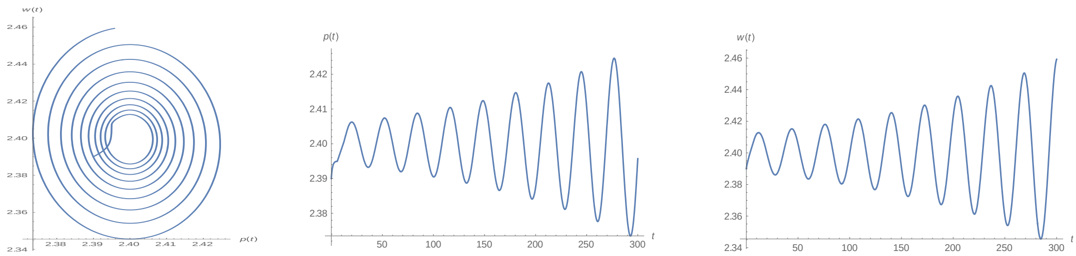

Fig. 3 Phase diagram and time history diagram of $\tau_2 = 7 > \tau_{20}$.

When $\tau_1 = \tau_2 = \tau$, we first select the same parameter as in literature [6] : $a = 1.2$, $b = 0.7$, $v_R = v_M = 0.5$, $\beta = 0.5$, $\alpha = 0.6$, $\theta = 0.8$. $p_0 = w_0 = 2.4$, $\omega_{20} = 0.42$, $\tau_0 = 3.74$ is obtained through calculation, and these coefficients satisfy (H_1), (H_2). When $\tau = 3 < \tau_0$ is taken, system (1) is locally asymptotically stable at the equilibrium point, as shown in Figure 4. Take $\tau = 3.74 = \tau_0$, system (1) generates *Hopf* bifurcation at the equilibrium point, as shown in Figure 5. When $\tau = 5.42 > \tau_0$ is taken, system (1) is unstable at the equilibrium point, as shown in Figure 6.

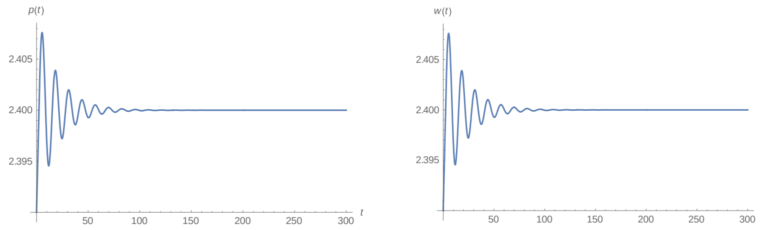

Fig. 4 Phase diagram of $\tau = 3 < \tau_0$.

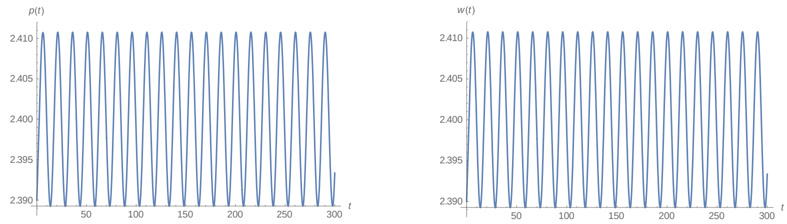

Fig. 5 Phase diagram of $\tau = 3.74 = \tau_0$.

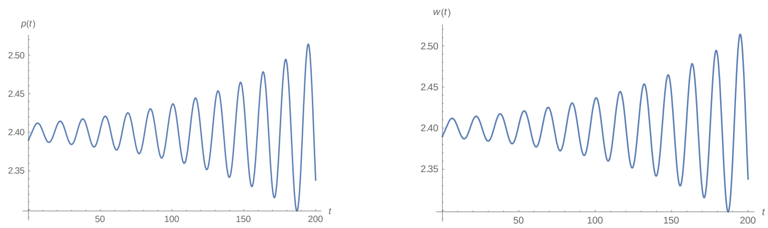

Fig. 6 Phase diagram of $\tau = 4 > \tau_0$.

When $\tau_1 > 0, \tau_2 \in (0, \tau_{20}]$, we first select the same parameter as in literature [6] : $a = 1.2$, $b = 0.7$, $\beta = 0.5$, $v_R = v_M = 0.5$, $\alpha = 0.6$, $\theta = 0.8$. $p_0 = 2.4$, $w_0 = 2.4$, $\tau_{10} = 6.11$ is obtained through calculation, and these coefficients satisfy (H_1). When $\tau_1 = 5.5 < \tau_{10}$ is taken, system (1) is locally asymptotically stable at the equilibrium point, as shown in Figure 7. Take $\tau_1 = 6.11 = \tau_{10}$, system (1) generates *Hopf* bifurcation at the equilibrium point, as shown in Figure 8. When $\tau_1 = 6 > \tau_{10}$ is taken, system (1) is unstable at the equilibrium point, as shown in Figure 9.

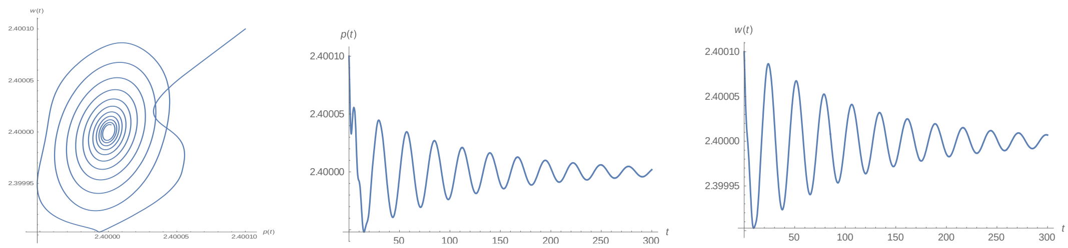

Fig. 7 Phase diagram and time history diagram of $\tau_1 = 5.5 < \tau_{10}$.

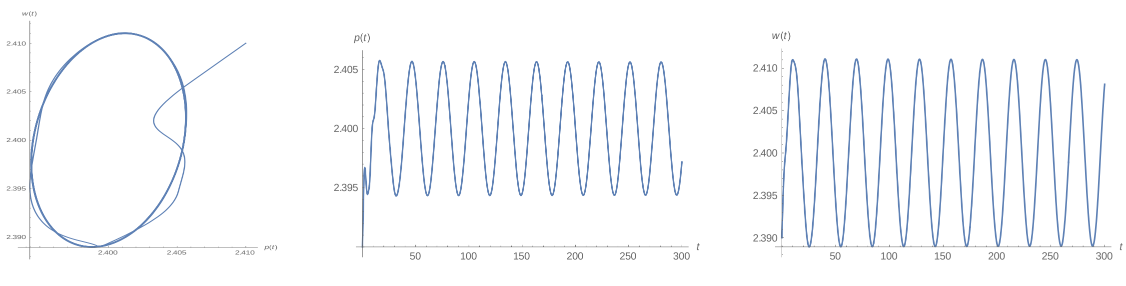

Fig. 8 Phase diagram and time history diagram of $\tau_1 = 5.7061 = \tau_{10}$.

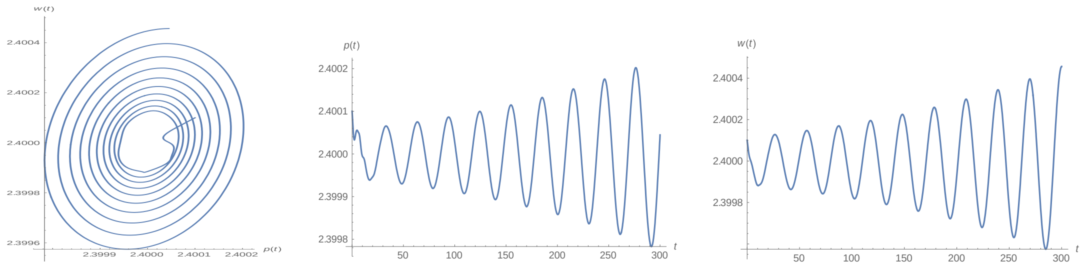

Fig. 9 Phase diagram and time history diagram of $\tau_1 = 6 > \tau_{10}$ **.**

V. CONCLUSIONS

This paper studies the double time delay price game model of low carbon products. On the basis of the original price game model, considering that both manufacturers and retailers may adopt delayed decision, so that the system is more realistic and can better solve the related price game model problems. In this paper, some conclusions about the stability of equilibrium point and the critical value of Hopf bifurcation are studied. Finally, the correctness of the results is verified by mathematical software. The results show that the delay parameter has a very important influence on the decision-making of retailers and manufacturers. When the delay value is lower than the critical value, the price game process remains stable. Once the delay value exceeds the critical value, the system will lose stability and Hopf branch will produce periodic solutions, which will cause the price fluctuation. Therefore, when adjusting prices, manufacturers and retailers should not refer to the historical price too far back, and should choose the recent price.

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