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Inverse Perfect Restrained Domination in Graphs

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Abstract - *Let $G = (V(G), E(G))$ be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$. A dominating set S is called a restrained dominating set if for each $u \in V(G) \setminus S$ there exist $v \in S$ and $z \in V(G) \setminus S$ $(z \neq u)$ such that u is adjacent to v and z. A restrained dominating set S is called a perfect restrained dominating set of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. Further, if D is a minimum perfect restrained dominating set of G, then a perfect restrained dominating set $S \subseteq V(G) \setminus D$ is called an inverse perfect restrained dominating set of G with respect to D. In this paper, we investigate the concept and give some important results.*

Keywords - *dominating set, restrained dominating set, perfect restrained dominating set, inverse perfect restrained dominating set*

I. INTRODUCTION

Suppose that $G = (V(G), E(G))$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$. In simple graph, we mean, finite and undirected graph with neither loops nor multiple edges. For the general graph theoretic terminology, the readers may refer to [1].

A vertex v is said to dominate a vertex u if uv is an edge of G or $v = u$. A set of vertices $S \subseteq V(G)$ is called a dominating set of G if every vertex not in S is dominated by at least one member of S. The size of a set of least cardinality among all dominating sets for G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called γ-set of G. Domination in a graph has been a huge area of research in graph theory. It was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [2]. Domination in graphs has been studied in [3-10].

A dominating set S is called a restrained dominating set of G if for each $u \in V(G) \setminus S$ there exists $v \in S$ such that $uv \in E(G)$ and there exists $z \in V(G) \setminus (S \cup \{u\})$ such that $uz \in E(G)$. The restrained domination number of G, is the minimum cardinality of a restrained dominating set of G and is denoted by $\gamma_r(G)$. A restrained dominating set of cardinality $\gamma_r(G)$ is called γ_r-set of G. Restrained domination has been studied in [11-17].

A restrained dominating set S is called a perfect restrained dominating set of G if each $u \in V(G) \setminus S$ is dominated by exactly one element of S. The perfect restrained domination number of G, is the minimum cardinality of a perfect restrained dominating set of G and is denoted by $\gamma_{pr}(G)$. A perfect restrained dominating set of cardinality $\gamma_{pr}(G)$ is called γ_{pr}-set of G. Perfect domination has been studied in [18-21].

Motivated by [18] and the idea of inverse domination in graphs [22-28], we initiate the study of an inverse perfect restrained dominating set. Let D be a minimum perfect restrained dominating set of G. A perfect restrained dominating set $S \subseteq V(G) \setminus D$ is called an inverse perfect restrained dominating set of G with respect to D. The inverse perfect restrained domination number of G, is the minimum cardinality of an inverse perfect restrained dominating set of G and is denoted by $\gamma_{pr}^{-1}(G)$. An inverse perfect restrained dominating set of cardinality $\gamma_{pr}^{-1}(G)$ is called γ_{pr}^{-1}-set of G.

In this paper, we investigate the concept and give some important results. We further give the characterization of an inverse perfect restrained

II. RESULTS

Remark 2.1 The set $S = V(G)$ is a restrained dominating set and a perfect dominating set.

Proof: If $S = V(G)$, then every vertex in $V(G) \setminus S = \emptyset$ vacuously satisfies the definitions of a restrained dominating set and a perfect dominating set. ■

Remark 2.2 Every graph G has a restrained dominating set and a perfect dominating set.

Proof: By Remark 2.1, $S = V(G)$ is a restrained dominating set and a perfect dominating set. ∎

From the definitions of inverse perfect secure dominating set and Remark 2.2 the following is immediate.

Remark 2.3 Let G be a nontrivial graph. Then $1 \leq \gamma(G) \leq \gamma_{pr}^{-1}(G) \leq \frac{n}{2}$.

For a nontrivial connected graph G, the following result says that $\gamma_{pr}^{-1}(G)$ ranges over all integers from 1 to $\frac{n}{2}$.

Theorem 2.4 Given positive integers k, m and n such that $1 \leq k \leq m \leq \frac{n}{2}$, where $n \geq 3$ there exists a connected graph G with $|V(G)| = n$, $\gamma_r(G) = k$, and $\gamma_{pr}(G)^{-1} = m$.

Proof: Consider the following cases.

Case 1. Suppose that $1 = k = m \leq \frac{n}{2}$.

Let $G = (V, E)$ with $E(G) = \{x_1x_2, x_1x_3, x_1x_4, x_1x_n, x_2x_3, x_2x_4, x_2x_n\}$ and $V(G) = \{x_1, x_2, \ldots, x_n\}$ (see Figure 1).

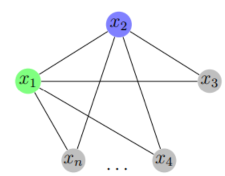

Figure 1: A graph G with $1 = k = m \leq \frac{n}{2}$.

The set $D = \{x_1\}$ is a γ_r-set of G, the set $S = \{x_2\}$ is a γ_{pr}^{-1}-set of G. Thus $\gamma_r(G) = 1 = k$, $\gamma_{pr}^{-1}(G) = 1 = m$ and $|V(G)| = |V(K_n)| = n$.

Case 2. Suppose that $1 \leq k = m < \frac{n}{2}$.

Let $G = P_k \circ (K_1 + \overline{K}_r)$ where $k \geq 1$ and $r \geq 1$ (see Figure 2).

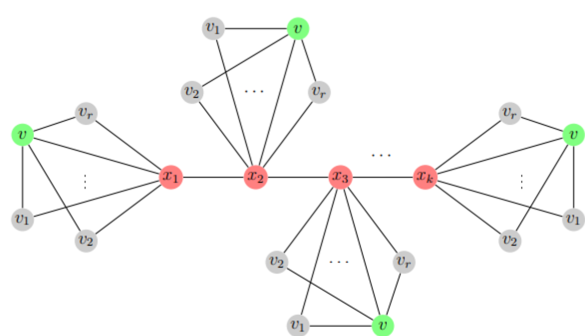

Figure 2: A graph G with $1 \leq k = m < \frac{n}{2}$.

Let $n = k(r + 1)$. The set $D = V(P_k)$ is a γ_r-set and a γ_{pr}-set of G, and the set $S = \bigcup_{x \in V(P_k)} S_x$ where $S_x = \{v\}$ for all $x \in V(P_k)$ is a γ_{pr}^{-1}-set of G with repect to D. Thus, $\gamma_r(G) = |V(P_k)| = k$, $\gamma_{pr}^{-1}(G) = |S| = |\bigcup_{x \in V(P_k)} S_x| = k \cdot 1 = k = m$, and $|V(G)| = |V(P_k \circ K_r)| = k + kr = k(1 + r) = n$.

Case 3. Suppose that $1 < k < m = \frac{n}{2}$.

Let $G = (V, E)$ where $V = \bigcup_{x \in V(P_r)} S_x$, $S_x = V(P_4)$ for all $x \in V(P_r')$, $r \neq 1$ and $r \equiv 1(mod 3)$ and $E = [\bigcup_{x \in V(P_r)} E(\langle S_x \rangle)] \cup [E(P_r') \cup E(P_r'')]$, $P_r' = [x_1, x_2, \dots, x_r]$, and $P_r'' = [w_1, w_2, \dots, w_r]$ (see Figure 3).

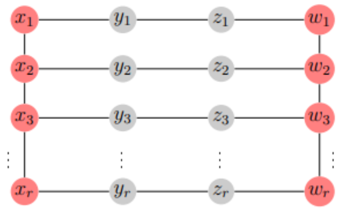

Figure 3: A graph G with $1 < k < m = \frac{n}{2}$.

Let $n = 4r$, $k = \frac{4r+2}{3}$ and $m = 2r$. Set $A = \left\{z_{3i-1}, z_{3i}: i = 1,2, \dots, \frac{r-1}{3}\right\} \cup \left\{x_{3i-2}, w_{3i-2}: i = 1,2, \dots, \frac{r-1}{3}\right\} \cup \{x_r, w_r\}$ is a γ_r -set of G, the set $D = V(P_r') \cup V(P_r'')$ is a γ_{pr}-set of G, and $S = \{y_i: i = 1,2, \dots, r\} \cup \{z_i: i = 1,2, \dots, r\}$ is a γ_{pr}^{-1}-set of G. Thus,

$$\gamma_r(G) = |A| = 2 \cdot \frac{r-1}{3} + 2 \cdot \frac{r-1}{3} + 2 = \frac{4r+2}{3} = k,$$

$$\gamma_{pr}^{-1}(G) = |S| = r + r = m, \text{ and}$$

$$|V(G)| = |\bigcup_{x \in V(P_r)} S_x| = |V(P\_r)| \cdot |S_x| = r|V(P_4)| = r(4) = n.$$

■

Corollary 2.5 The difference between $\gamma_{pr}^{-1}(G) - \gamma_r(G)$ can be made arbitrarily large.

Proof: Let n be an integer. By Theorem 2.4, there exists a connected graph G such that $\gamma_r(G) = 2n + 2$ and $\gamma_{pr}^{-1}(G) = 3n + 2$. Then $\gamma_{pr}^{-1}(G) - \gamma_r(G) = (3n + 2) - (2n + 2) = n$. Hence, the difference between $\gamma_{pr}^{-1}(G) - \gamma_r(G)$ can be made arbitrarily large. ■

Let $C_n = [v_1, v_2, \dots, v_n]$ such that $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and $E(C_n) = \backslash\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. The next result shows the inverse perfect restrained domination number of a cycle graph C_n.

Theorem 2.6 Let $G = C_n$ of order $n \geq 3$. Then

$$\gamma_{pr}^{-1}(G) = \begin{cases} \frac{n}{3}, & if\ n \equiv 0(mod\ 3) \\ \frac{n+2}{3}, & if\ n \equiv 1(mod\ 3) \\ \frac{n+4}{3}, & if\ n \equiv 2(mod\ 3), n \neq 5 \end{cases}$$

Proof: Case 1: Suppose that $n \equiv 0(mod\ 3)$. Consider the graph C_n where $n \neq 5$ (see Figure 4).

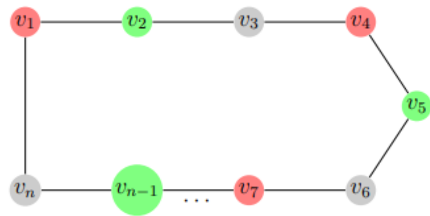

Figure 4: A graph with $\gamma_{pr}^{-1}(G) = \frac{n}{3}$.

The set $D = \left\{v_{3i-2}: i = 1,2,\ldots,\frac{n}{3}\right\}$ is a γ_{pr}-set of G, the set $S = \left\{v_{3i-1}: i = 1,2,\ldots,\frac{n}{3}\right\}$ is a γ_{pr}^{-1}-set of G with respect to D. Thus, $\gamma_{pr}^{-1}(G) = \frac{n}{3}$.

Case 2: Suppose that $n \equiv 1(\mathit{mod}\ 3)$. Consider the graph G (see Figure 5).

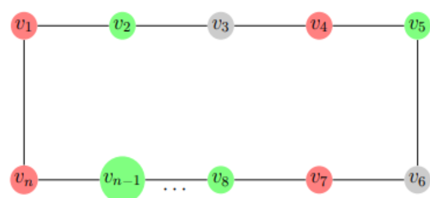

Figure 5: A graph with $\gamma_{pr}^{-1}(G) = \frac{n+2}{3}$.

The set $D = \left\{v_{3i-2}: i = 1,2,\ldots,\frac{n-1}{3}\right\} \cup \{v_n\}$ is a γ_{pr}-set of G, the set $S = \left\{v_{3i-1}: i = 1,2,\ldots,\frac{n-1}{3}\right\} \cup \{v_{n-1}\}$ is a γ_{pr}^{-1}-set of G. Thus, $\gamma_{pr}(G) = |S| = \frac{n-1}{3} + 1 = \frac{n+2}{3}$.

Case 3: Suppose that $n \equiv 2(\mathit{mod}\ 3)$ and $n \neq 5$. Consider the graph G (see Figure 6).

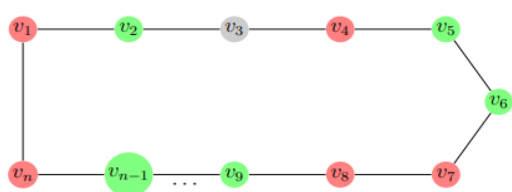

Figure 6: A graph with $\gamma_{pr}^{-1}(G) = \frac{n+4}{3}$.

The set $D = \left\{v_{3i-2}: i = 1,2,\ldots,\frac{n-2}{3}\right\} \cup \{v_{n-3}, v_n\}$ is a γ_{pr}-set of G, the set $S = \left\{v_{3i-1}: i = 1,2,\ldots,\frac{n-5}{3}\right\} \cup \{v_{n-1}, v_{n-2}, v_{n-5}\}$ is a γ_{pr}^{-1}-set of G. Thus, $\gamma_{pr}(G) = |S| = \frac{n-5}{3} + 3 = \frac{n+4}{3}$, $n \neq 5$. ■

Theorem 2.7 Let $G = K_1 + H$ and H is a nontrivial connected graph. Then $\gamma_{pr}^{-1}(G) = 1$ if and only if $\gamma(H) = 1$.

Proof: Suppose that $\gamma_{pr}^{-1}(G) = 1$. Let $S = \{x\}$ be an inverse perfect restrained dominating set of G. Since H is a nontrivial connected graph, $D = V(K_1) = \{v\}$ is a restrained dominating set of G. Let $u \in V(G) \setminus D$. Then every u is dominated by one vertex v. Hence, D is a perfect dominating set of G, that is, D is a perfect restrained dominating set of G. Since $S \subseteq V(G) \setminus D = V(H)$, it follows that $\gamma(H) = 1$.

For the converse, suppose that $\gamma(H) = 1$. Let $S = \{x\}$ be a dominating set of H and hence a dominating set in $G = K_1 + H$. Since H is a nontrivial connected graph, $S = \{x\}$ is a restrained dominating set of G. Let $u \in V(G) \setminus S$. Then every u is dominated by one vertex x. Hence, S is a perfect dominating set of G, that is, S is a perfect restrained dominating set of G. Similarly, $D = V(K_1)$ is a perfect restrained dominating set of G and a γ_{pr}-set of G. Since $S \subseteq V(H) = V(G) \setminus D$, it follows that S is an inverse perfect restrained dominating set of G with respect to D, that is, $\gamma_{pr}^{-1}(G) = 1$. ∎

A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The following result is an immediate consequence of Theorem 2.7.

Corollary 2.8 Let $G = K_n$ of order $n \geq 3$. Then $\gamma_{pr}^{-1}(G) = 1$.

Proof: Let $G = K_n$. Then $G = K_1 + K_{n-1}$ where $\gamma(K_{n-1}) = 1$. By Theorem 2.7, $\gamma_{pr}^{-1}(G) = 1$. ∎

The complete graph $G = K_n$ of order $n \geq 3$ is shown below (see Figure 7).

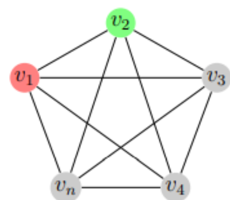

Figure 7: A graph with $\gamma_{pr}^{-1}(G) = 1$.

The set $D = \{v_1\}$ is a γ_{pr}-set of G and the set $S = \{v_2\}$ is γ_{pr}^{-1}-set of G with respect to D.

Theorem 2.9 Let $G = P_2 \circ H$ where H is a nontrivial connected graph. Then $\gamma_{pr}^{-1}(G) = 2$ if and only if $\gamma(H) = 1$.

Proof: Suppose that $\gamma_{pr}^{-1}(G) = 2$. Then there exists a γ_{pr}^{-1}-set $S = \{v_1, v_2\}$ such such that $S \subseteq V(G) \setminus D$ were D is a γ_{pr}-set of G. Let $P_2 = [x_1, x_2]$ and let $v_1 \in V(x_1 + H^{x_1})$, $v_2 \in V(x_2 + H^{x_2})$. Since $v_1 \in S$, v_1 must be a dominating set of H^{x_1}. Similarly, v_2 must be a dominating set of H^{x_2}. Hence, $\gamma(H) = 1$.

For the converse, suppose that $\gamma(H) = 1$. Let $S_v = \{v\}$ be a dominating set of H. Clearly, S_v^x is a perfect restrained dominating set of $x + H^x$ where $x \in V(P_2)$. Thus $S = \bigcup_{x \in V(P_2)} S_v^x = \{v_1, v_2\}$ is a perfect restrained dominating set of G. Similarly, $D = V(P_2)$ is a perfect restrained dominating set of G. Since $S \subseteq V(G) \setminus D$, it follows that S is an inverse perfect restrained dominating set of G with respect to D. Hence, $\gamma_{pr}^{-1}(G) = |S| = 2$. ∎

The next result follows immediately from Theorem 2.9.

Corollary 2.10 Let $G = P_2 \circ K_n$ of order $n \geq 2$. Then $\gamma_{pr}^{-1}(G) = 2$.

Proof: Let $G = P_2 \circ K_n$. Since $n \geq 2$ and $\gamma(K_n) = 1$, by Theorem 2.9, $\gamma_{pr}^{-1}(G) = 2$. ∎

III. Conclusion

In this paper, we introduced the concept of inverse perfect restrained domination in graphs and prove that given positive integers k, m and n such that $1 \leq k \leq m \leq \frac{n}{2}$, where $n \geq 3$ there exists a connected graph G with $|V(G)| = n, \gamma_r(G) = k$, and $\gamma_{pr}(G)^{-1} = m$. Further, we prove the inverse perfect restrained domination number of a cycle C_n graph. Prove the characterization of the inverse perfect restrained domination number of a graph $G = K_1 + H$ and $G = K_2 \circ H$ with $\gamma(H) = 1$. Some related problems on inverse perfect restrained domination in graphs are still open for research.

1. Characterize the inverse perfect restrained dominating sets of the join, corona, Cartesian product, and lexicographic product of two graphs.
2. Find the inverse perfect restrained domination number of the join, corona, Cartesian product, and lexicographic product of two graphs.

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