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Cantor Set: Far-Reaching Implications

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Abstract - *This work is hinged on revealing some far-reaching implications of the Cantor Set which has some remarkable properties. Description of the construction and subsequently computation of the Cantor Ternary set was charted. It was concluded by establishing some far-reaching implications of the Cantor set in the area of music.*

Keywords - *Cantor set, Ternary set, Cardinality, Cantor's function, isolated points*

I. Introduction

Georg Cantor, first discovered the Cantor set. He coined the term everywhere dense, which is currently used today. In his papers, Cantor later introduced the idea of perfect sets. He made the ternary Cantor set famous in this same paper when attempting to prove that it was possible to have a set that is closed and nowhere dense. He noted that this was a perfect, infinite set that is nowhere dense in any interval, regardless of the size (Stettin, 2017). The Cantor set is created by starting with the closed interval [0,1]. The next step is to remove a middle portion of the interval. This initial removal will then leave two intervals, namely $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. This process is continued an infinite number of times, with the final set of numbers, after the infinite iterations, being the Cantor set itself. This set is the intersection of all iterations, denoted as, where T_i represents the i^{th} iteration (Xu, 2020). There are different proportions that can be removed with this set, the most well-known being $\frac{1}{3}$. This set, referred to as the Cantor Ternary Set, can also be referred to as the classic set.

We will describe the construction of the ternary Cantor set, T, in a similar manner. Begin with the closed interval $[0,1]$ and remove a dense open set, G. The remaining set, $T = [0,1] \setminus G$ will also be closed and nowhere dense in $[0,1]$. Based on our construction of G, T will have no isolated points.

$$T = \cap_{i=1}^{\infty} \mathrm{T}_i$$

In mathematics, the Cantor Set is a set of points lying on a single line segment that has a number of remarkable and deep properties (wikipedia, 2021). Through the consideration of this set, Cantor and others helped lay the foundations of modern point set topology. It is on the closed interval $C_0 = [0,1]$. One then has to divide it into three equal sections and remove the middle-third which is an open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. The process is based on the nth set given by $c_n = \frac{c_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{c_{n-1}}{3}\right)$ for $n \geq 1$, and $c_0 = [0,1]$.

II. Preliminaries

A. Cantor Set

This is done with our clue from the Authors of the document captioned *The Cantor Set and the Cantor Function, TMA 4225- Foundation of Analysis:*

Definition 2.1.1: Cantor’s set is the set C left after the procedure below of deleting the open middle third interval is performed infinitely many times.

i. Step 0: begin with the interval [0, 1]
ii. Step 1: divide [0, 1] into 3 subintervals and delete the open middle subinterval.
iii. Step 2: divide each of the 2 resulting intervals above into 3 subintervals and delete the open middle subintervals.

This procedure is continued indefinitely. At each step, we delete the open middle third subinterval of each interval obtained in the previous step (Fleron, 1994).

B. Ternary Representation of Cantor’s Set

Every real number can be represented by an infinite sequence of digits:

$$\frac{1}{3} = 0.33333K$$

$$\text{golden ratio} = 1.6180339887498948482045K$$

$$\frac{1}{10} = 0.10000K = 0.09999K$$

We can represent real numbers in any base. We will use the *ternary* (base 3) representation, because Cantor’s set has a special representation in base 3.

A number is in Cantor’s set if and only if its ternary representation contains only the digits 0 and 2 (in other words, it has no 1’s).

Theorem 2.2.1: The cardinality of the Cantor’s set is the *continuum*. That is, Cantor’s set has the same cardinality as the interval [0,1].

Theorem 2.2.2: Cantor’s set is *negligible*. In other words, its “length”/Lebesgue measure is 0.

C. Topological structure of Cantor’s Set

Theorem *2.3.1*: Cantor's set has no interior points/ it is nowhere dense. In other words, it is just “dust”**.** That's because its length is 0, so it contains no continuous parts (no intervals).

Theorem 2.3.4: Cantor's set has no isolated points.

That is, in any neighborhood of a point in Cantor’s set, there is another point from Cantor's set.

In topology a set which is compact and has no isolated points is called a perfect set (Abbott, 2012).

Theorem 2.3.5: Cantor's set is totally disconnected.

III. Main Results

A. Computing the Cantor Ternary set

Lemma 3.1.1: 2^i intervals are removed from C_i when constructing C_{i+1}.

Theorem 3.1.2: Cantor's set is closed. That's because it is the complement relative to [0, 1] of open intervals, the ones removed in its construction.

Let's now compute the ternary set. The process of removing the middle-thirds is shown as follows

$c_0 = [0,1]$

$c_1 = [0,1] - c_n\left(\frac{1}{3}, \frac{2}{3}\right)$

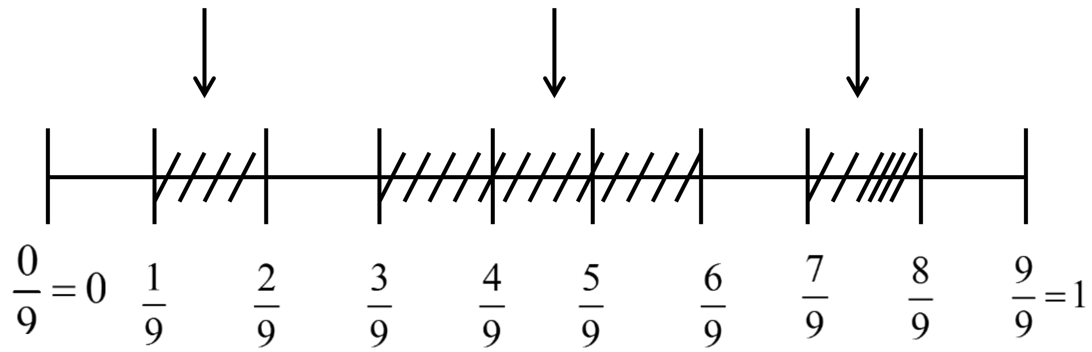
$=[0,1] - \left(\frac{c_0}{3} \cup \frac{2}{3} + \frac{c_0}{3}\right)$

$=[0,1] -$

$=\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{3}{3}\right]$

$for = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

Also $C_2 = [0,1]$ - We have



$c_2 = \left(\left[0, \frac{1}{3}\right] - \left[\frac{1}{9}, \frac{2}{9}\right]\right) \cup \left(\left[\frac{2}{3}, \frac{3}{3}\right] - \left[\frac{7}{9}, \frac{8}{9}\right]\right)$

$= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, \frac{9}{9}\right]$

$= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{6}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

For C_3 we have

$c_3 = \frac{c_2}{3} \cup \left(\frac{2}{3}, \frac{c_2}{2}\right)$

Now using the formular the *n*th set gives

$c_1 = \frac{c_0}{3} \cup \left(\frac{2}{3}, \frac{c_0}{3}\right)$

$$c1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Also

$$c_2 = \frac{c_1}{3} \cup \left(\frac{2}{3}, \frac{c_1}{3}\right) = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

And

$$c_3 = \frac{c_2}{3} \cup \left(\frac{2}{3}, \frac{c_2}{3}\right)$$

Think through the endpoints that remain. We have $\frac{1}{4}$ amongst numbers which are not interval points in the Cantor Set. There are more of such numbers. The Cantor Set is really outstanding (Singh, 2015). The Cantor Set is homeomorphic to the product space and one can speak of the odd or even intervals as well (Lipschutz, 1965).

Application: In music, the Cantor Set is relevant in that it portrays the self-similar facet:

In Key of C, the d or d m s is represented on the piano as CEG, whilst in Key of D, d or d m s is denoted by CF$^{\#}$A on the piano.

Another scenario is in Key of C and Key of D in playing d r m f s l t d', the similarities can be seen in line with the Cantor Set. These are factual in all the keys on the piano and bounded in their domains.

IV. Conclusion

The Cantor Ternary Set was successfully computed and explained related the practical aspects in music. It was reiterated that, the Cantor ternary set is bounded in [0, 1].

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