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Original Article

Iterative Cubic Spline Technique To Approximate Two-Dimensional And Axisymmetric Flow of A Viscous Incompressible Fluid

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***Abstract** - The present paper contains the study of the two-dimensional axisymmetric flow of a viscous incompressible fluid between two parallel plates due to the normal motion of the plates. The flow problem is governed by non-linear fourth order differential equation which is solved using iterative spline collocation method. Solutions are obtained for axisymmetric as well as two dimensional squeezing flow at different values of a squeeze number. Impacts of various physical parameters on the flow field are examined and shown graphically. For validation of the method,the obtained numerical results are compared with Homotopy perturbration method (HPM) graphically. The proposed work also emphasizes the application of spline collocation method to the nonlinear ordinary differential equations.*

Keywords — *Non-linear differential equation, Spline collocation method, Squeezing flow*

I. INTRODUCTION

In recent years, the analytical and experimental study of squeezing flow, together with the inertial term between the rotating cylinders and parallel plate, plays a crucial role in lubrication systems, chemical engineering, dusty fluids and food business. Because of its wide applications, this study has stimulated the interest of many researchers. Some authors have discussed a study of different flows induced by stretching of a sheet. Ariel et.al [20] studied axisymmetric flow over a stretching sheet by homotopy perturbation method. The solutions obtained for flow problems due to moving boundaries using HPM are quite accurate. The problem of boundary layer flows with moving boundaries was analyzed using homotopy perturbation method. Kuzma analyzed fluid inertia effects in squeeze films by corrected theory for the squeeze film and compared the obtained results with experimental results[8]. Qayyum et al. [1] studied unsteady squeezing flow of viscoelastic Jeffery fluid between parallel disks. Series solution was obtained for the governing equation. They analysed velocity profile with respect to changes in porosity and squeezing. Many researchers analysed various types of effects for different type of flows using Homotopy perturbation method, Homotopy analysis method,application of spline functions etc. Blue[16, 18, 23] discussed the applicability of spline functions to nonlinear differential equations. Domairry and Aziz[9] examined the suction and injection effects on the flow of magneto-hydrodynamic MHD squeeze squeezed between two parallel disks using Homotopy perturbation method. Ran et. al.[25] studied an axisymmetric Newtonian fluid squeezed between two parallel plates and applied Homotopy analysis method (HAM) to find analytical solution of the equation. He introduced Homotopy perturbation method (HPM)[11-15]. However, due to few limitations of HAM , some researchers have proposed few modifications in standard HAM [22]. Several researchers [21, 19] have contributed their efforts to make squeezing flow model more understandable. Many authors used the spline collocation technique to establish the accurate and efficient numerical schemes for the solution of partial integro-differential equations, time fractional differential equations, 2D fourth-order reaction-diffusion equation[2, 5, 10, 17, 24]. The governing equations are highly nonlinear differential equations. In the present work, the solutions of governing equations are obtained by iterative spline collocation method.[3, 4]

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II. FLOW DEVELOPMENT AND MATHEMATICAL FORMUALTION

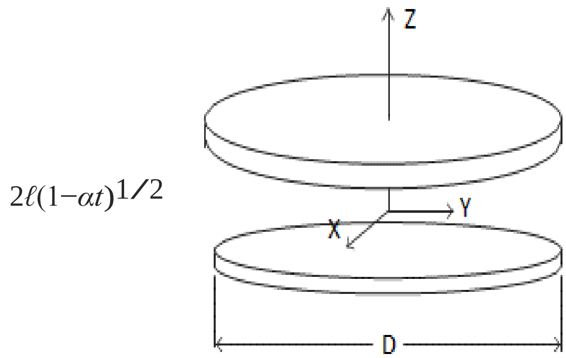

Figure -I: Schamatic Diagram of the Problem

Let the position of two plates be at $z = \pm l\ (1-at)^{1/2}$,where l indicates position at time $t=0$ as shown in figure 1. Suppose that the length l (in the two-dimensional case) or diameter D (in the axisymmetric case) are assumed to be very large in the comparision of the gap width $2z$ at any time such that the end effects can be neglected. The value of α is positive when the two plates are squeezed until they touch at $t = 1/\alpha$,while α will be negative if the two plates are separated. x_1, y_1 and w_1 are the velocity components in the x, y and z directions respectively [7].Wang [6] introduced the following transformation for two-dimensional flow,

$$x_1 = \frac{\alpha x}{[2(1-\alpha t)]} g'(\eta), \qquad z_1 = \frac{-\alpha l}{[2(1-\alpha t)^{1/2}]} g(\eta), \tag{1}$$

where $\eta = \dfrac{z}{[l(1-\alpha t)^{1/2}]}$

Substituting (1) into the unsteady two –dimensional Navier-stokes equations gives following non-linear ordinary differential equation

$$g^{iv} + S\{-\eta g''' - 3g'' - g'g'' + gg'''\} = 0 \tag{2}$$

The flow is characterized by $S = al^2/2v$ (squeeze number).

The boundary conditions are given by

$$\begin{aligned} &g(0)=0, \quad g''(0)=0 \\ &g(1)=1, \quad g'(1)=0 \end{aligned} \tag{3}$$

Similarly, the Wang transforms for axisymmetric flow are [6],

$$x_1 = \frac{\alpha x}{[4(1-\alpha t)]} g'(\eta),$$

$$y_1 = \frac{\alpha y}{[4(1-\alpha t)]} g'(\eta)$$

$$z_1 = \frac{-\alpha l}{[2(1-\alpha t)^{1/2}]} g(\eta)$$

(4)

Using transformation (4), unsteady axisymmetric Navier-Stokes equations reduces to,

$$g^{iv} + S\{-\eta g''' - 3g'' + gg'''\} = 0$$

(5)

Subject to the boundary conditions (3).

Consequently, we should solve the non-linear ordinary differential equation

$$g^{iv} + S\{-\eta g''' - 3g'' - \beta g' g'' + gg'''\} = 0$$

(6)

where,

$\beta = 0$, Axisymmetric

(7)

$= 1$, Two-dimensional

subject to boundary conditions (3) .

III. METHOD INTRODUCTION

To solve a fourth order boundary value problem,

$$y^{iv} = f(x, y, y', y'', y''')$$

(8)

Subject to the boundary conditions,

$$H_1\left[y(a),\ y'(a),\ y''(a),\ y'''(a)\right] = 0$$

$$H_2\left[y(b),\ y'(b),\ y''(b),\ y'''(b)\right] = 0$$

(9)

Since $s(x)$ is a quintic spline interpolating $y(x)$ given by equations (8), with boundary conditions (9) we have $s(x_i) = y(x_i)$ and $s^{iv}(x)$ is a linear function. Let us define $s^{iv}(x)$ in subinterval $\left[x_i, x_{i+1}\right]$of $\left[a,b\right]$ as,

$$s^{iv}(x) = y_{i+1}^{iv} \frac{(x - x_i)}{h_{i+1}} + y_i^{iv} \frac{(x_{i+1} - x)}{h_{i+1}};\ i = 0, 1, 2.... N-1$$

(10)

where $h_{i+1} = x_{i+1} - x_i$ and n is the number of subinterval of $\left[a,b\right]$.

Four successive integrations of equation (10) produce a quintic spline in $\left[x_i, x_{i+1}\right]$, which is of the form,

$$s(x)=y_{i+1}^{iv}\frac{(x-x_i)^5}{120h_{i+1}}+y_i^{iv}\frac{(x_{i+1}-x)^5}{120h_i}+\left(y_i^{"}-\frac{h^2}{6}y_i^{iv}\right)\frac{(x_{i+1}-x)^3}{6h_{i+1}}+\left(y_{i+1}^{"}-\frac{h^2}{6}y_{i+1}^{iv}\right)\frac{(x-x_i)^3}{6h_{i+1}}+\left(y_i-\frac{h^2}{6}y_i^{"}+\frac{7h^4}{360}y_i^{iv}\right)\frac{(x_{i+1}-x)}{h_{i+1}}$$
$$+\left(y_{i+1}-\frac{h^2}{6}y_{i+1}^{"}+\frac{7h^4}{360}y_{i+1}^{iv}\right)\frac{(x-x_i)}{h_{i+1}}$$

(11)

Similarly, the quintic spline in $\left[x_{i-1},\ x_i\right]$ is obtained in the form,

$$s(x)=y_i^{iv}\frac{(x-x_{i-1})^5}{120h_i}+y_{i-1}^{iv}\frac{(x_i-x)^5}{120h_i}+\left(y_{i-1}^{"}-\frac{h^2}{6}y_{i-1}^{iv}\right)\frac{(x_i-x)^3}{6h_i}+\left(y_i^{"}-\frac{h^2}{6}y_i^{iv}\right)\frac{(x-x_{i-1})^3}{6h_i}+\left(y_{i-1}-\frac{h^2}{6}y_{i-1}^{"}+\frac{7h^4}{360}y_{i-1}^{iv}\right)\frac{(x_i-x)}{h_i}$$

(12)

Using the continuity of $s'(x)$ at $x=x_i$, we obtain the recurrence relation

$$y_{i-1}-2y_i+y_{i+1}=\frac{h^2}{6}\left(y_{i-1}^{"}+4y_i^{"}+y_{i+1}^{"}\right)-\frac{h^4}{360}\left(7y_{i-1}^{iv}+16y_i^{vi}+7y_{i+1}^{iv}\right);\ i=1,\ 2,\ 3,\ldots,N$$

(13)

and continuity of $s'''(x)$ at $x=x_i$ gives,

$$y_{i-1}^{"}-2y_i^{"}+y_{i+1}^{"}=\frac{h^2}{6}\left(y_{i-1}^{iv}+4y_i^{iv}+y_{i+1}^{iv}\right);\ i=1,\ 2,\ 3,\ldots,N\quad .$$

(14)

Here the equations (13) and (14) are also used to solve the fourth order boundary value problem given by equation (9). These equations give a tridiagonal system of N equations in $y_i, i=0,1,2,\ldots N$ and this process is iterative.

IV. SOLUTION PROCEDURE

Let us express the equation (6) in the form,

$$g^{iv}=-S\{-\eta g'''-3g''-\beta g'g''+gg'''\}$$

(15)

Subject to boundary conditions (3).

In this case, initial approximations begin with $g_{(0)}(\eta)\ =\ a\eta^3+b\eta^2+c\eta+d$ substituting and satisfying the conditions (3) as an initial guess. This is found to be,

$$g_{(0)}(\eta)\ =\ -\frac{1}{2}\eta_i^3+\frac{3}{2}\eta_i$$

(16)

Initially compute $g_i^{"}$ from the following system,

$$-2g_1^{"}+g_2^{"}=\frac{h^2}{6}\left(g_o^{iv}+4g_1^{iv}+g_2^{iv}\right)-g_0^{"}$$

$$g''_{i-1} - 2g''_i + g''_{i+1} = \frac{h^2}{6}\left(g^{iv}_{i-1} + 4g^{iv}_i + g^{iv}_{i+1}\right), \text{ i=1, 2…n-1}$$

(17)

$$g''_{N-2} - 2g''_{N-1} = \frac{h^2}{6}\left(g^{iv}_{N-2} + 4g^{iv}_{N-1} + g^{iv}_N\right) - g''_N$$

Now the calculation of g_i 's, $i=0,1,2,\ldots, N$ to be carried out through the solution of tridiagonal system of equations given below:

$$-2g_1 + g_2 = \frac{h^2}{6}\left(g''_{i-1} + 4g''_i + g''_{i+1}\right) - \frac{h^4}{360}\left(7g^{iv}_{i-1} + 16g^{vi}_i + 7g^{iv}_{i+1}\right) - g_0$$

$$g_{i-1} - 2g_i + g_{i+1} = \frac{h^2}{6}\left(g''_{i-1} + 4g''_i + g''_{i+1}\right) - \frac{h^4}{360}\left(7g^{iv}_{i-1} + 16g^{vi}_i + 7g^{iv}_{i+1}\right), i = 1, 2, .., N-1$$

$$g_{N-2} - 2g_{N-1} = \frac{h^2}{6}\left(g''_{N-1} + 4g''_N + g''_{N+1}\right) - \frac{h^4}{360}\left(7g^{iv}_{N-1} + 16g^{vi}_N + 7g^{iv}_{N+1}\right) - g_N$$

(18)

where N is the number of sub intervals of mesh points $[0,1]$ for equation (15). The coefficient matrix is a nonsingular matrix of dimension $(N+1)\,\mathrm{x}(\mathrm{N}+1)$ and hence, the solution of this system exists so that the quantities g_i 's , $i = 1, 2, 3, \ldots, N$ can be obtained.

Table I : Solution for $s = 1.5$ of Axisymmetric Flow

η	$i=1$	$i=2$	$i=3$	$i=4$	**HPM**
0	0	0	0	0	0
0.2	0.251614341	0.262413692	0.265657278	0.270193352	0.280859419
0.4	0.494238476	0.511984946	0.517258811	0.524645364	0.544546845
0.6	0.715221002	0.733354169	0.738661415	0.746111435	0.771162858
0.8	0.894415311	0.905853819	0.909151621	0.913791263	0.934940174
1	1	1	1	1	1

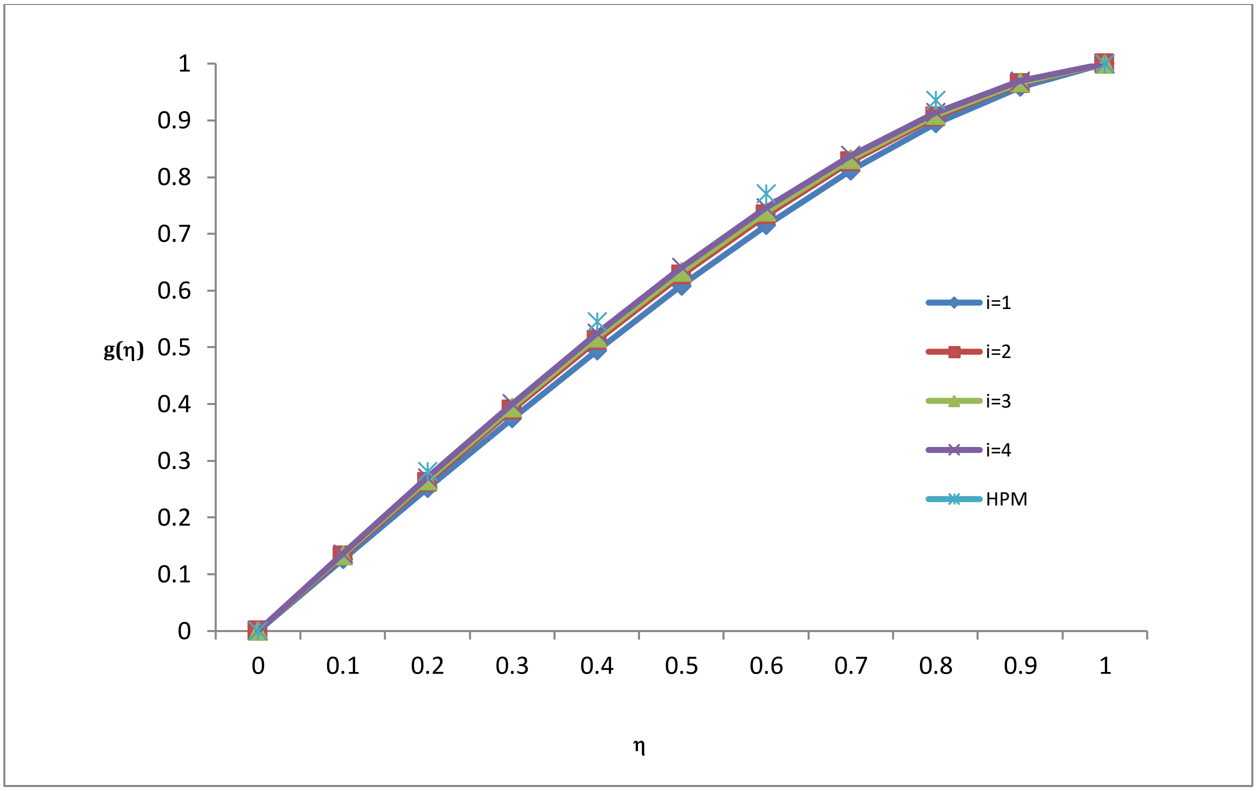

Figure -II:Graph of $s=1.5$ for Axisymmetric Flow

Table II: Different Values of s for Axisymmetric Flow ($\beta=0$)

η	$s=1.5$	$s=0.5$	$s=-0.5$	$s=-1.5$
0	0	0	0	0
0.1	0.136064147	0.142848681	0.158288835	0.172673
0.2	0.270193352	0.283253267	0.31282786	0.340386
0.3	0.400409687	0.418739302	0.459911821	0.498289
0.4	0.524645364	0.546771003	0.595924991	0.641762
0.5	0.640687748	0.664719014	0.717386891	0.766528
0.6	0.746111435	0.769826083	0.820998897	0.868779
0.7	0.838191602	0.859169715	0.903691662	0.945295
0.8	0.913791263	0.929620563	0.962672933	0.986541
0.9	0.969212729	0.977795019	0.993584691	0.993585
1	1	1	1	1

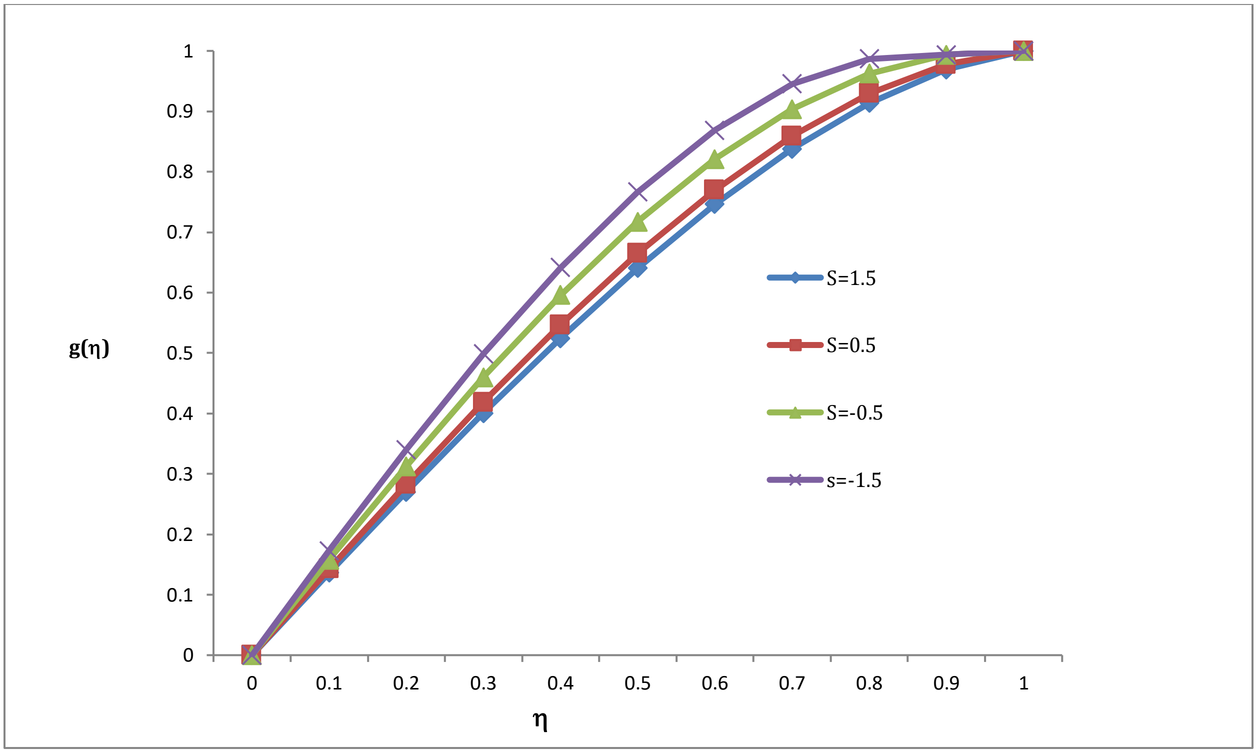

Figure III: Graph of Different Values of s for Axisymmetric Flow ($\beta = 0$)

Table III: Solution of Two-dimensional Flow for $s = 1.5$ ($\beta = 1$)

η	0	0.2	0.4	0.6	0.8	1
$i = 1$	0	0.251614	0.494238	0.715221	0.894415	1
$i = 2$	0	0.250093	0.491828	0.712871	0.892986	1
$i = 3$	0	0.255430	0.500417	0.721384	0.898193	1
$i = 4$	0	0.259270	0.506594	0.727508	0.901942	1
$i = 5$	0	0.262265	0.511413	0.732285	0.904867	1
$i = 6$	0	0.271721	0.526658	0.747366	0.913991	1
HPM	0	0.276432	0.537752	0.765249	0.932471	1

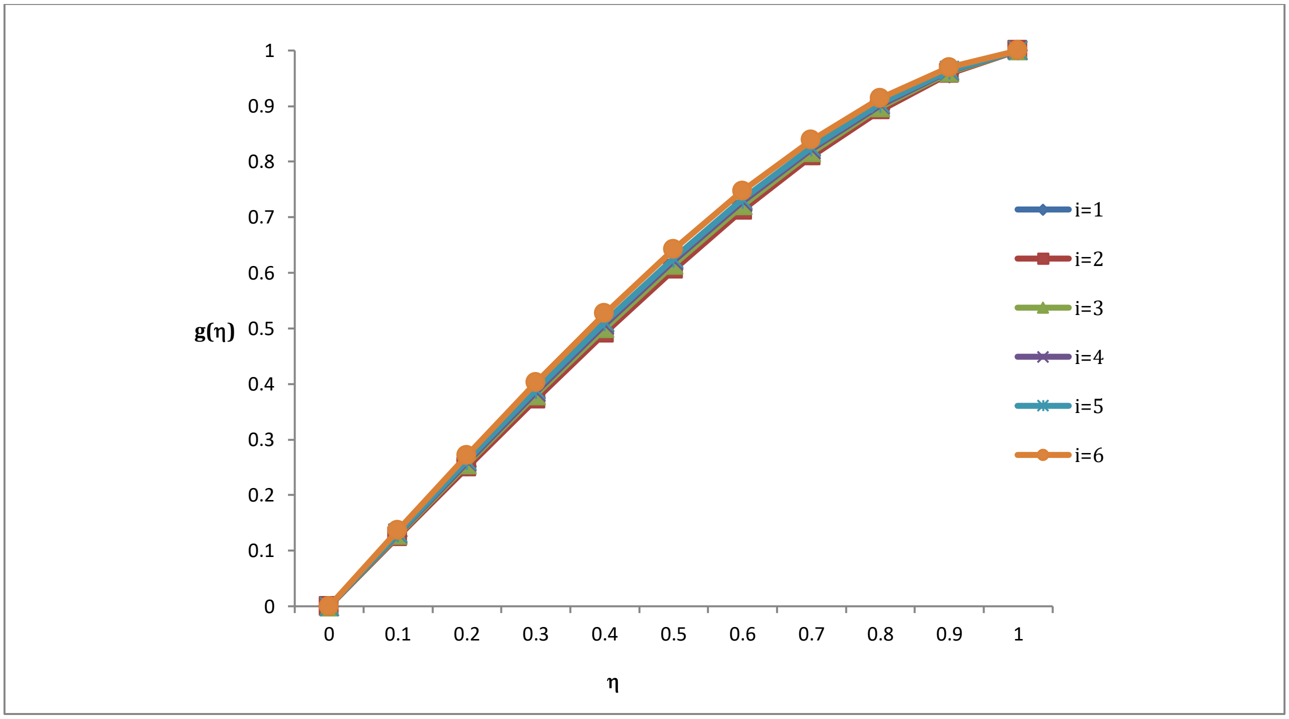

Figure IV: Graph of Two dimensional Flow for $s = 1.5$ ($\beta = 1$)

Table IV: Solution of Different Values of s for Two-dimensional Flow ($\beta = 1$)

η	$s = -1.5$	$s = -0.5$	$s = 0.5$	$s = 1.5$
0	0	0	0	0
0.1	0.18078	0.15845	0.1427	0.136904759
0.2	0.35584	0.31314	0.28297	0.271721774
0.3	0.51962	0.46035	0.41835	0.402350678
0.4	0.66693	0.59644	0.54631	0.526658188
0.5	0.79312	0.71794	0.66423	0.642441699
0.6	0.89419	0.82153	0.76936	0.747366776
0.7	0.93898	0.90414	0.85877	0.838866853
0.8	0.96503	0.963	0.92933	0.913991556
0.9	0.99734	0.99565	0.97764	0.969188296
1	1	1	1	1

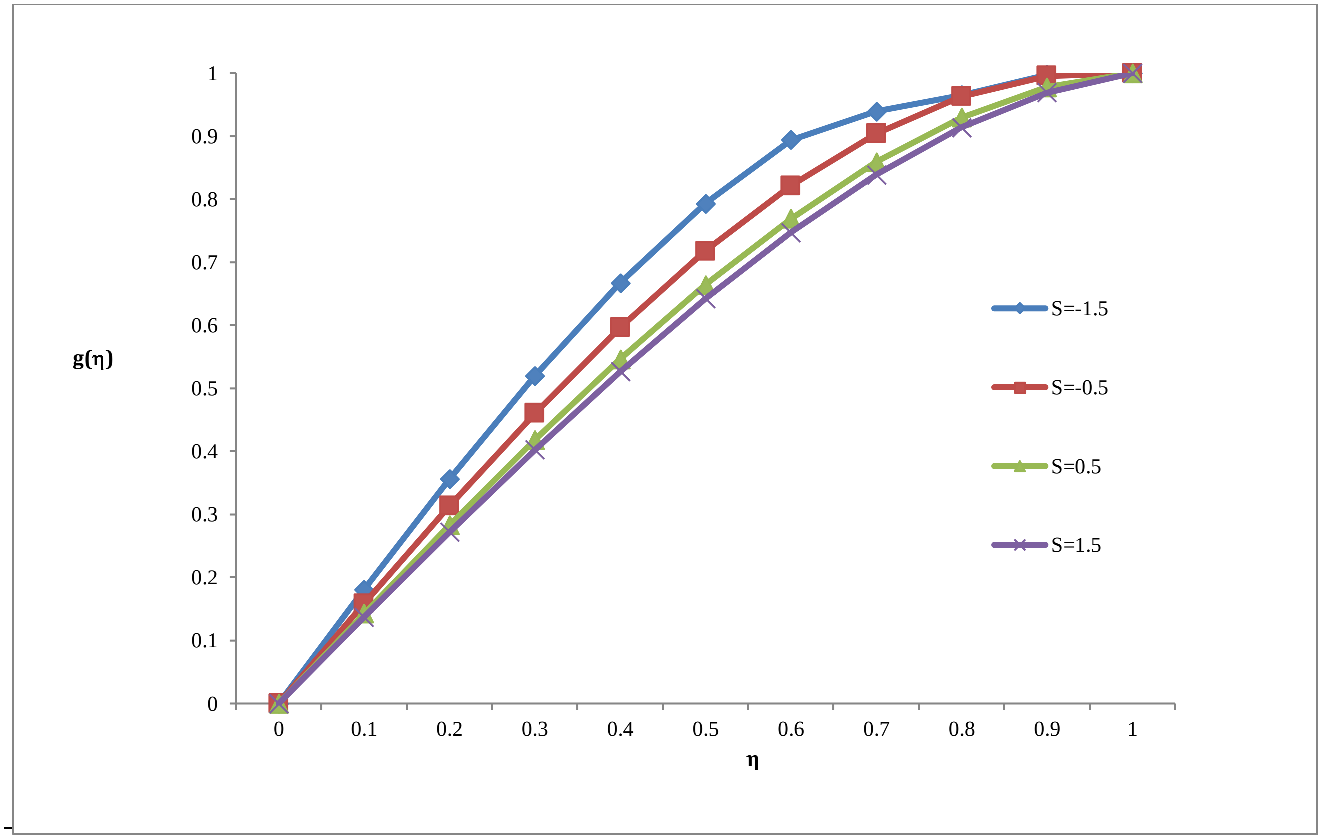

Figure V: Graph of Different Values of s for Two-Dimensional Flow ($\beta = 1$)

V. RESULTS AND DISCUSSION

The fourth-order ordinary differential equation (1), with the boundary conditions (3), is solved numerically using iterative spline method. Our main concern is to obtain various values of $g(\eta)$. The quantity $g(\eta)$ describes the flow behavior. For several values of s, the function $g(\eta)$ is obtained by the different order of approximation for the axisymmetric and two-dimensional cases. For axisymmetric flow, different values of $g(\eta)$ at $s = 1.5$ are obtained by the iterative process upto four consecutive cubic spline approximations as shown in fgure-2. From figure 2, we can see that there is a very good agreement between the purely analytic results of the HAM and numerical results. Also figure 2 indicates the convergent nature of cubic spline approximations, in which the fourth iteration of cubic spline closely approximates the flow profiles obtained by Homotopy perturbation method. Figure 3 shows the influence of negative and positive values of s on $g(\eta)$ for axisymmetric case, which are obtained after fourth iteration. Figure 4 shows the different values of $g(\eta)$ at $s = 1.5$ for two-dimensional case for four consecutive cubic spline approximations. The variations of $g(\eta)$ with the change in the positive values of s as well as for negative values of s, for the two-dimensional case are plotted in figure 5.

VI.CONCLUSION

To study the unsteady axisymmetric and two-dimensional squeezing flows between two parallel plates, governed by a non-linear two-point boundary value problem of fourth order, the differential equation obtained was treated numerically with iterative spline collocation method. It is observed that iterative spline collocation method gives the better approximation of the solution by increasing the number of sub intervals. The results are compared with the HPM. The obtained solution compared with the numerical solutions, demonstrate remarkable accuracy. An ultimate conclusion drawn from this work is that spline collocation requires compact computations. The iterative technique is advantageous in the sense that it involves a more compact coefficient matrix. This reduces the laborwork for obtaining approximations.

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