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A Study On the Recurrence Properties of Generalized Tetranacci Sequence

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Abstract - *In this paper, we investigate the recurrence properties of the generalized Tetranacci sequence and present how the generalized Tetranacci sequence at negative indices can be expressed by the sequence itself at positive indices.*

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Keywords - *Tetranacci numbers, Tetranacci sequence, negative indices, recurrence relations.*

1. Introduction

We can propose an open problem as follows: Whether and how can the generalized Tetranacci sequence W_n at negative indices be expressed by the sequence itself at positive indices?

We present our main result as follows which completely solves the above problem for the generalized Tetranacci sequence W_n.

THEOREM 1. *For $n \in \mathbb{Z}$, we have*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Note that H_n can be written in terms of W_n using Remark 3 below.

The generalized (r, s, t, u) sequence (or generalized Tetranacci sequence or generalized 4-step Fibonacci sequence) $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \qquad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1.1}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1,2,3,5,11,12,10] and references therein. The sequence $\{W_n\}_{n\geq 0}$

can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers n.

In the following Table 1 we present some special cases of generalized Tetranacci sequence.

Table 1 A few special case of generalized Tetranacci sequences.

No	Sequences (Numbers)	Notation
1	Generalized Tetranacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3; 1,1,1,1)\}$
2	Generalized Fourth Order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3; 2,1,1,1)\}$
3	Generalized Fourth Order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3; 1,1,1,2)\}$
4	Generalized 4-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3; 2,3,5,7)\}$

In literature, for example, the following names and notations (see Table 2) are used for the special case of □□□□□□□□□□□and initial values.

Table 2 A few special case of generalized (r,s,t,u) (generalized Tetranacci) sequence

No	Sequences (Numbers)	Notation	OEIS [6]	Ref.
1	Tetranacci	$\{M_n\} = \{W_n(0,1,1,2; 1,1,1,1)\}$	A000078	[7]
2	Tetranacci-Lucas	$\{R_n\} = \{W_n(4,1,3,7; 1,1,1,1)\}$	A073817	[7]
3	fourth order Pell	$\left\{P_n^{(4)}\right\} = \{W_n(0,1,2,5; 2,1,1,1)\}$	A103142	[8]
4	fourth order Pell-Lucas	$\left\{Q_n^{(4)}\right\} = \{W_n(4,2,6,17; 2,1,1,1)\}$	A331413	[8]
5	modified fourth order Pell	$\left\{E_n^{(4)}\right\} = \{W_n(0,1,1,3; 2,1,1,1)\}$	A190139	[8]
6	fourth order Jacobsthal	$\left\{J_n^{(4)}\right\} = \{W_n(0,1,1,1; 1,1,1,2)\}$	A007909	[4]
7	fourth order Jacobsthal-Lucas	$\left\{j_n^{(4)}\right\} = \{W_n(2,1,5,10; 1,1,1,2)\}$	A226309	[4]
8	modified fourth order Jacobsthal	$\left\{K_n^{(4)}\right\} = \{W_n(3,1,3,10; 1,1,1,2)\}$		[4]
9	fourth-order Jacobsthal Perrin	$\left\{q_n^{(4)}\right\} = \{W_n(3,0,2,8; 1,1,1,2)\}$		[4]
10	adjusted fourth-order Jacobsthal	$\left\{S_n^{(4)}\right\} = \{W_n(0,1,1,2; 1,1,1,2)\}$		[4]
11	modified fourth-order Jacobsthal-Lucas	$\left\{R_n^{(4)}\right\} = \{W_n(4,1,3,7; 1,1,1,2)\}$		[4]
12	4-primes	$\{G_n\} = \{W_n(0,0,1,2; 2,3,5,7)\}$		[9]
13	Lucas 4-primes	$\{H_n\} = \{W_n(4,2,10,41; 2,3,5,7)\}$		[9]
14	modified 4-primes	$\{E_n\} = \{W_n(0,0,1,1; 2,3,5,7)\}$		[9]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences. For easy writing, from now on, we drop the superscripts from the sequences, for example we write J_n for $J_n^{(4)}$.

It is well known that the generalized (r, s, t, u) numbers (the generalized Tetranacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n$$

where

$$\begin{aligned} A_1 &= \frac{W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}, \\ A_2 &= \frac{W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

and $\alpha, \beta, \gamma, \delta$ are the roots of characteristic equation of W_n which is given by

$$x^4 - rx^3 - sx^2 - tx - u = 0 \tag{1.2}$$

Note that we have the following identities

$$\begin{cases} \alpha+\beta+\gamma+\delta = r, \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = -s, \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = t, \\ \alpha\beta\gamma\delta = -u. \end{cases} \tag{1.3}$$

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers n. Now we define two special cases of the generalized (r, s, t, u) sequence $\{W_n\}$. (r, s, t, u) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r, s, t, u) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, \\ G_0 &= 0, G_1 = 1, G_2 = r, G_3 = r^2 + s, \\ H_{n+4} &= rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, \\ H_0 &= 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t, \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, \ldots$ respectively.

Some special cases of (r, s, t, u) sequence $\{G_n(0, 1, r, r^2 + s; r, s, t, u)\}$ and Lucas (r, s, t, u) sequence $\{H_n(4, r, 2s + r^2, r^3 + 3sr + 3t; r, s, t, u)\}$ are as follows:

(1) $G_n(0, 1, 1, 2; 1, 1, 1, 1) = M_n$, Tetranacci sequence,

(2) $H_n(4, 1, 3, 7; 1, 1, 1, 1) = R_n$, Tetranacci-Lucas sequence,

(3) $G_n(0, 1, 2, 5; 2, 1, 1, 1) = P_n$, fourth-order Pell sequence,

(4) $H_n(4, 2, 6, 17; 2, 1, 1, 1) = Q_n$, fourth-order Pell-Lucas sequence,

(5) $G_n(0,1,1,2;1,1,1,2) = S_n$, adjusted fourth-order Jacobsthal sequence,

(6) $H_n(4,1,3,7;1,1,1,2) = R_n$, modified fourth-order Jacobsthal-Lucas sequence.

For all integers n, (r,s,t,u) and Lucas (r,s,t,u) numbers can be expressed using Binet's formulas as

$$
\begin{aligned}
G_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\
H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n,
\end{aligned}
$$

respectively.

2. The Proof of Theorem 1

To prove Theorem 1, we need following lemma.

LEMMA 2. *For $n \in \mathbb{Z}$, denote*

$$S_n = \alpha^n\beta^n\gamma^n + \alpha^n\beta^n\delta^n + \alpha^n\gamma^n\delta^n + \beta^n\gamma^n\delta^n$$

where α, β, γ and δ are as in defined in Formula (1.3). Then the followings hold:

(a): *For $n \in \mathbb{Z}$, we have $S_n = (-u)^n H_{-n}$ and $S_{-n} = (-u)^{-n} H_n$.*

(b): *S_n has the recurrence relation so that*

$$S_n = tS_{n-1} - suS_{n-2} + ru^2 S_{n-3} + u^3 S_{n-4}$$

with the initial conditions $S_0 = 4$, $S_1 = t$, $S_2 = t^2 - 2su$, $S_3 = t^3 - 3stu + 3ru^2$. The sequence at negative indices is given by

$$S_{-n} = -\frac{ru^2}{u^3} S_{-(n-1)} - \frac{-su}{u^3} S_{-(n-2)} - \frac{t}{u^3} S_{-(n-3)} + \frac{1}{u^3} S_{-(n-4)}, \text{ for } n = 1,2,3,....$$

(c): *S_n has the identity so that*

$$S_n = \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)$$

Proof.

(a): From the definition of S_n and H_n, we obtain

$$H_{-n} = \alpha^{-n} + \beta^{-n} + \gamma^{-n} + \delta^{-n} = \frac{\alpha^n\beta^n\gamma^n + \alpha^n\beta^n\delta^n + \alpha^n\gamma^n\delta^n + \beta^n\gamma^n\delta^n}{(-u)^n} = \frac{S_n}{(-u)^n}.$$

i.e., $S_n = (-u)^n H_{-n}$ and so

$$S_{-n} = (-u)^{-n} H_n$$

(b): With Formula (1.3) or using the formula $S_n=(-u)^nH_{-n}$, we obtain initial values of S_n as

$$\begin{aligned}
S_0 &= (-u)^0H_0=4,\\
S_1 &= (-u)^1H_{-1}=-u\times(-\frac{t}{u})=t,\\
S_2 &= (-u)^2H_{-2}=u^2\times\frac{1}{u^2}(t^2-2su)=t^2-2su,\\
S_3 &= (-u)^3H_{-3}=-u^3\times(-\frac{1}{u^3})(t^3-3stu+3ru^2)=t^3-3stu+3ru^2,
\end{aligned}$$

For $n\geq 4$, we have

$$\begin{aligned}
tS_{n-1} &= S_1S_{n-1}\\
&= (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)(\alpha^{n-1}\beta^{n-1}\gamma^{n-1}+\alpha^{n-1}\beta^{n-1}\delta^{n-1}+\alpha^{n-1}\gamma^{n-1}\delta^{n-1}+\beta^{n-1}\gamma^{n-1}\delta^{n-1})\\
&= -S_{n-4}u^3-rS_{n-3}u^2+sS_{n-2}u+S_n\\
&\Rightarrow\\
tS_{n-1} &= -S_{n-4}u^3-rS_{n-3}u^2+sS_{n-2}u+S_n\\
&\Rightarrow\\
S_n &= tS_{n-1}-suS_{n-2}+ru^2S_{n-3}+u^3S_{n-4}
\end{aligned}$$

(c): From the definition of S_n we get

$$\begin{aligned}
H_n^3 &= (\alpha^n+\beta^n+\gamma^n+\delta^n)(\alpha^n+\beta^n+\gamma^n+\delta^n)(\alpha^n+\beta^n+\gamma^n+\delta^n)\\
&= 6S_n-2H_{3n}+3H_{2n}H_n\\
&\Rightarrow\ S_n=\frac{1}{6}(H_n^3+2H_{3n}-3H_{2n}H_n).\ \square
\end{aligned}$$

Now, we shall complete the proof of Theorem 1.

The Proof of Theorem 1:

Note that for $n\in\mathbb{Z}$, we have the following:

$$\begin{aligned}
\alpha^n\beta^n+\alpha^n\gamma^n+\alpha^n\delta^n+\beta^n\gamma^n+\beta^n\delta^n+\gamma^n\delta^n &= \frac{1}{2}(H_n^2-H_{2n})\\
A_1+A_2+A_3+A_4 &= W_0\\
A_4\alpha^n\beta^n\gamma^n+A_3\alpha^n\beta^n\delta^n+A_2\alpha^n\gamma^n\delta^n+A_1\beta^n\gamma^n\delta^n &= (-u)^nW_{-n}
\end{aligned}$$

Now, for $n\in\mathbb{Z}$, we obtain

$$\begin{aligned}
W_n\times\frac{1}{2}(H_n^2-H_{2n}) &= (A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n)(\alpha^n\beta^n+\alpha^n\gamma^n+\alpha^n\delta^n+\beta^n\gamma^n+\beta^n\delta^n+\gamma^n\delta^n)\\
&= (W_{2n}H_n-W_{3n})+(S_nW_0-(-u)^nW_{-n})
\end{aligned}$$

By Lemma 2 (c) (using $S_n = \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)$), it follows that

$$
\begin{aligned}
W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\
&= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \square
\end{aligned}
$$

Next, we present a remark which presents how H_n can be written in terms of W_n.

REMARK 3. *To express W_{-n} by the sequence itself at positive indices we need that H_n can be written in terms of W_n. For this, writing*

$$H_n = a \times W_{n+3} + b \times W_{n+2} + c \times W_{n+1} + d \times W_n$$

and solving the system of equations

$$
\begin{aligned}
H_0 &= a \times W_3 + b \times W_2 + c \times W_1 + d \times W_0 \\
H_1 &= a \times W_4 + b \times W_3 + c \times W_2 + d \times W_1 \\
H_2 &= a \times W_5 + b \times W_4 + c \times W_3 + d \times W_2 \\
H_3 &= a \times W_6 + b \times W_5 + c \times W_4 + d \times W_3
\end{aligned}
$$

or

$$
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} W_3 & W_2 & W_1 & W_0 \\ W_4 & W_3 & W_2 & W_1 \\ W_5 & W_4 & W_3 & W_2 \\ W_6 & W_5 & W_4 & W_3 \end{pmatrix}^{-1} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \\ H_3 \end{pmatrix}
$$

we find a, b, c, d so that H_n can be written in terms of W_n and we can replace this H_n in Theorem 1.

Using Theorem 1 and Remark 3, we have the following corollary.

COROLLARY 4. *For $n \in \mathbb{Z}$, we have*

(a): $G_{-n} = -(3ru^2 + t^3 - 3stu)^2G_n^3 - (2su - t^2)^2G_{n+3}^2G_n - (-rt^2 - tu + 2rsu)^2G_{n+2}^2G_n - (-st^2 + 2s^2u + 4u^2 + rtu)^2G_{n+1}^2G_n + 2(3ru^2 + t^3 - 3stu)((-2su + t^2)G_{n+3} + (-rt^2 - tu + 2rsu)G_{n+2} + (-st^2 + 2s^2u + 4u^2 + rtu)G_{n+1})G_n^2 + 2(2su - t^2)(-rt^2 - tu + 2rsu)G_{n+3}G_{n+2}G_n + 2(2su - t^2)(-st^2 + 2s^2u + 4u^2 + rtu)G_{n+3}G_{n+1}G_n - 2(-st^2 + 2s^2u + 4u^2 + rtu)(-rt^2 - tu + 2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su + t^2)G_{2n+3}G_n + u^2(-rt^2 - tu + 2rsu)G_{2n+2}G_n + u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n+1}G_n - 2u^2(2su - t^2)G_{2n}G_{n+3} + 2u^2(-rt^2 - tu + 2rsu)G_{2n}G_{n+2} + 2u^2(-st^2 + 2s^2u + 4u^2 + rtu)G_{2n}G_{n+1} - 3u^2(3ru^2 + t^3 - 3stu)G_{2n}G_n.$

(b): $H_{-n} = \frac{1}{6}(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

Using Theorem 1 and Remark 3 (or using the last corollary for (a) and (b)'s in the next corollaries), we can give some formulas for the special cases of generalized Tetranaci sequence (generalized (r,s,t,u)-sequence) as follows.

We have the following corollary which gives the connection between the special cases of generalized Tetranacci sequence at the positive index and the negative index.

COROLLARY 5. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *Tetranacci sequence:*

$$M_{-n} = \frac{1}{2(-1)^n}(-M_n^3 - M_{n+3}^2 M_n - 36M_{n+1}^2 M_n - 2M_n^2 M_{n+3} + 12M_n^2 M_{n+1} - 2M_{3n} - M_{2n+3}M_n + 6M_{2n+1}M_n - 2M_{2n}M_{n+3} + 12M_{2n}M_{n+1} - 3M_{2n}M_n + 12M_{n+3}M_{n+1}M_n).$$

(b): *Tetranacci-Lucas sequence:*

$$R_{-n} = \frac{1}{6(-1)^n}(R_n^3 + 2R_{3n} - 3R_{2n}R_n).$$

The following corollary illustrates the connection between the special cases of generalized fouth-order Pell sequence at the positive index and the negative index.

COROLLARY 6. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *fourth order Pell sequence:*

$$P_{-n} = \frac{1}{2(-1)^n}(-16P_n^3 - P_{n+3}^2 P_n - P_{n+2}^2 P_n - 49P_{n+1}^2 P_n - 8P_n^2 P_{n+3} + 8P_n^2 P_{n+2} + 56P_n^2 P_{n+1} - 2P_{3n} - P_{2n+3}P_n + P_{2n+2}P_n + 7P_{2n+1}P_n - 2P_{2n}P_{n+3} + 2P_{2n}P_{n+2} + 14P_{2n}P_{n+1} - 12P_{2n}P_n + 2P_nP_{n+2}P_{n+3} + 14P_nP_{n+1}P_{n+3} - 14P_nP_{n+1}P_{n+2}).$$

(b): *fourth order Pell-Lucas sequence:*

$$Q_{-n} = \frac{1}{6(-1)^n}(Q_n^3 + 2Q_{3n} - 3Q_{2n}Q_n).$$

(c): *modified fourth order Pell sequence:*

$$E_{-n} = \frac{1}{32(-1)^n}(-9E_n^3 - E_{n+3}^2 E_n - 9E_{n+2}^2 E_n - 484E_{n+1}^2 E_n + 6E_n^2 E_{n+3} + 18E_n^2 E_{n+2} - 132 E_n^2 E_{n+1} - 32E_{3n} - 4E_{2n+3}E_n - 12E_{2n+2}E_n + 88E_{2n+1}E_n - 8E_{2n}E_{n+3} - 24E_{2n}E_{n+2} + 176E_{2n}E_{n+1} + 36E_{2n}E_n - 6E_nE_{n+2}E_{n+3} + 44E_nE_{n+1}E_{n+3} + 132E_nE_{n+1}E_{n+2}).$$

The following corollary presents the connection between the special cases of generalized fouth-order Jacobsthal sequence at the positive index and the negative index.

COROLLARY 7. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *fourth-order Jacobsthal Perrin sequence:*

$$Q_{-n} = \frac{1}{35152(-2)^n}(-125Q_{n+3}^3 + 5832Q_{n+2}^3 - 6859Q_{n+1}^3 + 12\,960Q_n^3 + 1350Q_{n+3}^2 Q_{n+2} - 1425Q_{n+3}^2 Q_{n+1} + 2050Q_{n+3}^2 Q_n - 4860Q_{n+2}^2 Q_{n+3} - 18\,468Q_{n+2}^2 Q_{n+1} + 26568Q_{n+2}^2 Q_n - 5415Q_{n+1}^2 Q_{n+3} + 19494Q_{n+1}^2 Q_{n+2} - 10080Q_n^2 Q_{n+3} + 36288Q_n^2 Q_{n+2} - 38304Q_n^2 Q_{n+1} + 29602Q_{n+1}^2 Q_n - 6760Q_{3n+3} + 24336Q_{3n+2} - 25688Q_{3n+1} + 13520Q_{3n} - 1950Q_{2n+3}Q_{n+3} + 7020Q_{2n+3}Q_{n+2} - 7410Q_{2n+3}Q_{n+1} + 10660Q_{2n+3}Q_n + 7020Q_{2n+2}Q_{n+3} - 25272Q_{2n+2}Q_{n+2} + 26676Q_{2n+2}Q_{n+1} - 38376Q_{2n+2}Q_n - 7410Q_{2n+1}Q_{n+3} + 26676Q_{2n+1}Q_{n+2} - 28158Q_{2n+1}Q_{n+1} + 40508Q_{2n+1}Q_n + 7280Q_{2n}Q_{n+3} - 26208Q_{2n}Q_{n+2} + 27664Q_{2n}Q_{n+1} - 28080Q_{2n}Q_n + 10260Q_{n+1}Q_{n+2}Q_{n+3} - 14760Q_nQ_{n+2}Q_{n+3} + 15580Q_nQ_{n+1}Q_{n+3} - 56088Q_nQ_{n+1}Q_{n+2}).$$

(b): *adjusted fourth-order Jacobsthal sequence:*

$S_{-n} = \frac{1}{2^{n+8}(-1)^n}(-49S_n^3 - 9S_{n+3}^2S_n - S_{n+2}^2S_n - 441S_{n+1}^2S_n - 42S_n^2S_{n+3} + 14S_n^2S_{n+2} + 294$
$S_n^2S_{n+1} - 32S_{3n} - 12S_{2n+3}S_n + 4S_{2n+2}S_n - 24S_{2n}S_{n+3} + 8S_{2n}S_{n+2} + 84S_{2n+1}S_n + 168S_{2n}S_{n+1} -$
$84S_{2n}S_n + 6S_nS_{n+2}S_{n+3} + 126S_nS_{n+1}S_{n+3} - 42S_nS_{n+1}S_{n+2})$.

(c): *modified fourth-order Jacobsthal-Lucas sequence:*

$$R_{-n} = \frac{1}{6(-2)^n}(R_n^3 + 2R_{3n} - 3R_{2n}R_n).$$

The following corollary illustrates the connection between the special cases of generalized generalized 4-primes sequence at the positive index and the negative index.

COROLLARY 8. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *4-primes sequence:*

$G_{-n} = \frac{1}{98(-7)^n}(-25G_{n+3}^2G_n - 1444G_{n+2}^2G_n - 729G_{n+1}^2G_n - 170G_n^2G_{n+3} + 1292G_n^2G_{n+2} - 918$
$G_n^2G_{n+1} - 289G_n^3 - 98G_{3n} - 35G_{2n+3}G_n + 266G_{2n+2}G_n - 189G_{2n+1}G_n - 70G_{2n}G_{n+3} + 532G_{2n}G_{n+2} -$
$378G_{2n}G_{n+1} - 357G_{2n}G_n + 380G_nG_{n+2}G_{n+3} - 270G_nG_{n+1}G_{n+3} + 2052G_nG_{n+1}G_{n+2})$.

(b): *Lucas 4-primes sequence:*

$$H_{-n} = \frac{1}{6(-7)^n}(H_n^3 + 2H_{3n} - 3H_{2n}H_n).$$

(c): *modified 4-primes sequence:*

$E_{-n} = \frac{1}{512(-7)^n}(-169E_{n+3}^2E_n - 5929E_{n+2}^2E_n - 400E_{n+1}^2E_n - 286E_n^2E_{n+3} + 1694E_n^2E_{n+2} +$
$440E_n^2E_{n+1} - 121E_n^3 - 512E_{3n} - 208E_{2n+3}E_n + 1232E_{2n+2}E_n + 320E_{2n+1}E_n - 416E_{2n}E_{n+3} +$
$2464E_{2n}E_{n+2} + 640E_{2n}E_{n+1} - 528E_{2n}E_n + 2002E_nE_{n+2}E_{n+3} + 520E_nE_{n+1}E_{n+3} - 3080E_nE_{n+1}$
$E_{n+2})$.

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