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On the Problem of Tangency of Hyperbolic Curve

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Abstract – Associated to a given hyperbolic curve and a given point in the exterior region, there are always two tangents from the given point to the hyperbolic curve by yielded Critical Equation and discussing discriminant.

Keywords: Hyperbola, Tangents, Exterior region.

I. INTRODUCTION

In 2002, David R. Duncan and Bonnie H. Litwiller have studied concerning the numbers of tangent to parabola $y = kx^2$ wth $k > 0$ throught the given point in the exterior region by discussing discriminant and Critical Equation see more detail in [1].

In 2017, Apisit Pakapongpun has generalized analogous problem for the tangent of the given parabola $Ax^2 + Bx + D$ to the given point in the exterior region, where A, B and D are constants and $A \neq 0$ see more detail in [2].

In 2018, Apisit Pakapongpun and Saowaros Srisuk have studied analogous problem for ellipse curve. Given an ellipse curve and given a point in the exterior region, there are always two tangents from the given point to the given curve see more detail in [3].

In this article we continue our study of the tangents which through a given curve and a given point. Thus we consider a given hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$ through the given point, where a and b are constants such that $ab \neq 0$.

The set of exterior points is represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} < 1,$ the set of interior points is represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} > 1,$ and the set of points on the hyperbola is represented by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

II. MAIN RESULTS

Let $P(c,d)$ be a point not on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $Q(e, f)$ be a point on the hyperbola, the line PQ is the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at Q see Fig.1.

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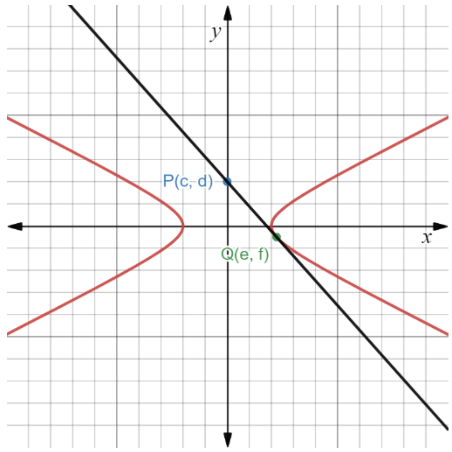

Fig 1: The tangent line PQ

The slope of the line PQ is $\dfrac{d-f}{c-e}$ and by calculus, the slope of the line through Q and tangent to the hyperbola is the derivative of $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$. We will evaluate at the point $Q(e,f)$.

Since $y'=\dfrac{b^2x}{a^2y}$, the slope is $\dfrac{b^2e}{a^2f}$. This gives us

$$\frac{d-f}{c-e}=\frac{b^2e}{a^2f}. \tag{1}$$

Thus

$$b^2e^2-a^2f^2=b^2ce-a^2df. \tag{2}$$

From (1) and (2), we obtain

$$a^2b^2-b^2ce=-da^2f.$$

Square both sides, we have

$$a^4b^4-2a^2b^4ce+b^4c^2e^2=a^4d^2f^2. \tag{3}$$

Since $Q(e,f)$ is lied on the hyperbola, the equation $\dfrac{e^2}{a^2}-\dfrac{f^2}{b^2}=1$ is held. Hence,

$$b^2e^2-a^2f^2=a^2b^2$$

rewriting

$$f^2=\frac{b^2}{a^2}(e^2-a^2) \tag{4}$$

and substituting f^2 into (3), we obtain the Critical Equation

$$(b^4c^2-a^2b^2d^2)e^2-2a^2b^4ce+(a^4b^4+a^4b^2d^2)=0.$$

Thus, we get the solutions

$$e=\frac{-B\pm\sqrt{B^2-4AC}}{2A} \tag{5}$$

where

$$A = b^4c^2 - a^2b^2d^2$$
$$B = -2a^2b^4c$$
$$C = a^4b^4 + a^4b^2d^2 \text{ and } c \neq \pm\frac{ad}{b}.$$

To find this quadratic equation has solutions for e, regard its discriminant as follows

$$\begin{aligned} D &= B^2 - 4AC \\ &= (-2a^2b^4c)^2 - 4(b^4c^2 - a^2b^2d^2)(a^4b^4 + a^4b^2d^2) \\ &= 4a^4b^4d^2(a^2b^2 + a^2d^2 - b^2c^2). \end{aligned}$$

Hence, two cases arise:

Case I: If $P(c,d)$ is in the interior region of hyperbola , then $\frac{c^2}{a^2} - \frac{d^2}{b^2} > 1$ thus, $a^2b^2 + a^2d^2 - b^2c^2 < 0$ implies that $D \leq 0$. Then, there are no real solutions to the critical equation. Therefore, no tangents to the hyperbola pass through interior point $P(c,d)$.

Case II: If $P(c,d)$ is in the exterior region of hyperbola , then $\frac{c^2}{a^2} - \frac{d^2}{b^2} < 1$ thus, $a^2b^2 + a^2d^2 - b^2c^2 > 0$ implies that $D \geq 0$. Therefore, there are two tangents to the hyperbola pass through the point $P(c,d)$.

This conclusion follows from the fact that a given point $P(c,d)$ which is not on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If $c \neq \pm\frac{ad}{b}$ then there are two tangents to the hyperbola pass through the point $P(c,d)$. Furthermore, on the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we can obtain a similar result as the following.

Remark 2.1: Let $P(c,d)$ be a point not on the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, if $c \neq \pm\frac{ad}{b}$ then there are two tangents to the hyperbola pass through the point $P(c,d)$ established by

$$e = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

and

$$f^2 = \frac{a^2}{b^2}(b^2 + e^2) \ ,$$

where

$$A = a^4c^2 - a^2b^2d^2$$
$$B = -2a^4b^2c$$
$$C = a^4b^4 + a^2b^4d^2 \ .$$

III. Numerical Example

Example 3.1 Find a points of the tangents from the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ through the point $P(1,0)$.

Soltion: the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ we know that $a=2$, $b=1$, $c=1$ and $d=0$, we obtain

$$A=b^4c^2-a^2b^2d^2=1$$
$$B=-2a^2b^4c=-8$$
$$C=a^4b^4+a^4b^2d^2=16.$$

Evaluating the solutions,

$$e=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$
$$=\frac{8\pm\sqrt{64-4(1)(16)}}{2(1)}=4.$$

So, we find f from equation (4)

$$f^2=\frac{b^2}{a^2}(e^2-a^2)$$
$$f=\pm\sqrt{3}.$$

Therefore, $Q_1(4,\sqrt{3})$ and $Q_2(4,-\sqrt{3})$ are two points of the tangents to the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ through the point $P(1,0)$ see Fig.2.

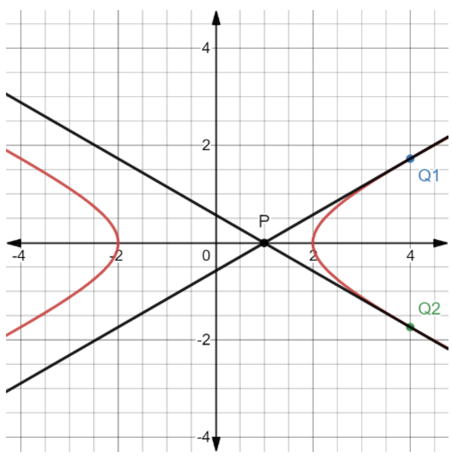

Fig. 2: Picture of example 3.1

Example 3.2 Find a points of the tangents from the parabola $\frac{x^2}{4}-\frac{y^2}{1}=1$ through the point $P(0,2)$.

Soltion: the hyperbola $\frac{x^2}{4}-\frac{y^2}{1}=1$ we know that $a=2$, $b=1$, $c=0$ and $d=2$, we obtain

$$A=b^4c^2-a^2b^2d^2=-16$$
$$B=-2a^2b^4c=0$$
$$C=a^4b^4+a^4b^2d^2=80.$$

Evaluating the solutions,

$$e=\frac{-B\pm\sqrt{B^2-4AC}}{2A}=\pm\sqrt{5}.$$

So, we find f from equation (4)

$$f^2 = \frac{b^2}{a^2}(e^2 - a^2)$$
$$f = \pm\frac{1}{2}.$$

Thus, we obtain four points $Q_1(\sqrt{5}, -\frac{1}{2})$, $Q_2(-\sqrt{5}, -\frac{1}{2})$, $Q_3(\sqrt{5}, \frac{1}{2})$ and $Q_4(-\sqrt{5}, \frac{1}{2})$. But if we check from the equation (1), then $Q_3(\sqrt{5}, \frac{1}{2})$ and $Q_4(-\sqrt{5}, \frac{1}{2})$ are not true. Therefore, only $Q_1(\sqrt{5}, -\frac{1}{2})$ and $Q_2(-\sqrt{5}, -\frac{1}{2})$ are points for the tangents to the hyperbola $\frac{x^2}{4} - \frac{y^2}{1} = 1$ through the point $P(0,2)$ see Fig.3.

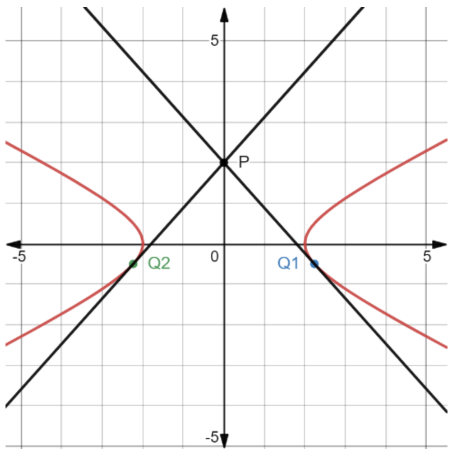

Fig. 3: Picture of example 3.2

Example 3.3 Find a points of the tangents from the parabola $\frac{y^2}{4} - \frac{x^2}{1} = 1$ through the point $P(0,1)$.

Soltion: the hyperbola $\frac{y^2}{4} - \frac{x^2}{1} = 1$ we know that $a = 2$, $b = 1$, $c = 0$ and $d = 1$, we obtain

$$A = a^4c^2 - a^2b^2d^2 = -4,$$
$$B = -2a^4b^2c = 0,$$
$$C = a^4b^4 + a^2b^4d^2 = 12 .$$

Evaluating the solutions,

$$e = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \pm\sqrt{5}.$$

So, we find f from equation (4)

$$f^2 = \frac{a^2}{b^2}(b^2 + e^2)$$
$$f = \pm 4.$$

Thus, we obtain four points $Q_1(\sqrt{3},4)$, $Q_2(-\sqrt{3},4)$, $Q_3(\sqrt{3},-4)$ and $Q_4(-\sqrt{3},-4)$. But if we check from the equation (1), then $Q_3(\sqrt{3},-4)$ and $Q_4(-\sqrt{3},-4)$ are not true. Therefore, only $Q_1(\sqrt{3},4)$ and $Q_2(-\sqrt{3},4)$ are points for the tangents to the hyperbola $\frac{y^2}{4}-\frac{x^2}{1}=1$ through the point $P(0,1)$ see Fig.4.

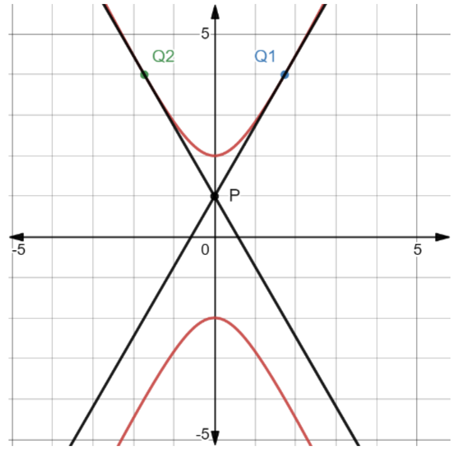

Fig. 4: Picture of example 3.3

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