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GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF THE ISOMETRIC ADDITIVE MAPPING IN QUASI-BANACH SPACES

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Abstract. In this paper, we study to solve the Hyers-Ulam-Rassias stability of the isometric additive mappings in quasi-Banach spaces, associated to additive functional equation with 2k-variables. First are investigated results the Hyers-Ulam-Rassias stability of of the isometric in quasi-Banach spaces, and last are investigated isometric in p-Banach spaces. Then I will show that the solutions of equation are additive mapping. These are the main results of this paper.

Keywords: Cauchy type additive, functional equation, Jensen functional equation isometric in quasi-Banach spaces, Hyers-Ulam-Rassias, stability; p-Banach spaces
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1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f:\mathbf{X}\to\mathbf{Y}$ be a mapping. We use the notation$\|\cdot\|_{\mathbf{X}}$ $\Big(\|\cdot\|_{\mathbf{Y}}\Big)$ for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate the stability of isometric when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a quasi-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$
or when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$.
In fact, when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a quasi-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$
when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$.

we solve and prove the Hyers-Ulam-Rassias type stability of the isomoetric in quasi-Banach spaces, associated to the Cauchy type additive functional equation and Jensen type additive functional equation

$$f\left(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\right)=\sum_{j=1}^{k}f\Big(x_j\Big)+\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) \tag{1.1}$$

$$2kf\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k^2}\sum_{j=1}^{k}x_{k+j}\right)=\sum_{j=1}^{k}f\Big(x_j\Big)+\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) \tag{1.2}$$

The study of the functional equation stability originated from a question of S.M. Ulam [22], concerning the stability of group homomorphisms. Let $\Big(\mathbb{G},*\Big)$ be a group and let

$\left(\mathbb{G}', \circ, d\right)$ be a metric group with metric $d\left(\cdot, \cdot\right)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbb{G} \to \mathbb{G}'$ satisfies

$$d\left(f\left(x * y\right), f\left(x\right) \circ f\left(y\right)\right) < \delta$$

for all $x, y \in \mathbb{G}$ then there is a homomorphism $h : \mathbb{G} \to \mathbb{G}'$ with

$$d\left(f\left(x\right), h\left(x\right)\right) < \epsilon$$

for all $x \in \mathbb{G}$?, if the answer, is affimartive, we would say that equation of homomophism $h\left(x * y\right) = h\left(y\right) \circ h\left(y\right)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers[11] gave a first affirmative answes the questionor of Ulam as follows:
(D. H. Hyers) Let $\mathbf{X}$,and $\mathbf{Y}$ be Banach space. Assume that $f : \mathbf{X} \to \mathbf{Y}$ satisfies

$$\left\|f\left(x + y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \epsilon,$$

for all $x, y \in \mathbf{X}$ and some $\epsilon \geq 0$. Then there exists a unique additive mapping $T : \mathbf{X} \to \mathbf{Y}$, such that

$$\left\|f\left(x\right) - T\left(x\right)\right\| \leq \epsilon, \forall x \in \mathbf{X}.$$

Next Th. M. Rassias [18] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:
(Th. M. Rassias.) Consider $\mathbf{X}$ and $\mathbf{Y}$ to be two Banach spaces, and let $f : \mathbf{X} \to \mathbf{Y}$ be a mapping such that $f\left(tx\right)$ is continous in t for each fixed x. Assume that there exist $\theta > 0$ and $p \in [0, 1]$ such that

$$\left\|f\left(x + y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \epsilon\left(\left\|x\right\|^p + \left\|y\right\|^p\right), \forall x, y \in \mathbf{X}.$$

then there exists a unique linear $L : \mathbf{X} \to \mathbf{Y}$ satifies

$$\left\|f\left(x\right) - L\left(x\right)\right\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p, x \in \mathbf{X}.$$

Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta following Th.M. Rassias approach for the stability of the linear mapping between Banach spaces obtained a generalization of Th.M. Rassias Theorem. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2,...,11]). More special in 2008 Chun-Gil $Park^{1*}$ and Themistocles M.Rassia [10] have established the and investigated the Hyers-Ulam-Rassias stability of the isomoetric in quasi-Banach spaces concerning to the following Cauchy functional equation and Jensen functional equation

$$f\left(x + y\right) = f\left(x\right) + f\left(y\right)$$

$$2f\left(\frac{x+y}{2}\right) = f\left(x\right) + f\left(y\right)$$

. Recently, in [2-11] the authors studied the on Hyers-Ulam-Rassias type stability the isometric in quasi-Banach spaces, associated to the Cauchy type following additive functional equation and Jensen type additive functional equation.

$$f\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f\left(x_j\right) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)$$

and

$$2kf\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) = \sum_{j=1}^{k} f\left(x_j\right) + \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right)$$

,

ie the functional equation with $2k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional (**??**) and (**??**). Thus, the results in this paper are generalization of those in [2-11] for functional equation with $2k$-variables.
The paper is organized as followns:

In section preliminarie we remind some basic notations in [12-17] such as Banach space, quasi-Banach space, p-Banach spaces, generalized quasi-normed space, generalized quasi-Banach space, normed linear space, isometry, preserves distance for the mapping $f$and solutions of the Cauchy function equation.

Section 3 is devoted to prove the Hyers-Ulam-Rassias type stability of the isometric in quasi-Banach space of the additive functional equations when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a quasi-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$

Section 4 is devoted to prove the Hyers-Ulam-Rassias type stability of the isometri in quasi-Banach spacess of the additive functional equations when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p-Banach space with quasi-norm with quasi-norm $\|\cdot\|_{\mathbf{Y}}$.

2. PRELIMINARIES

2.1. Banach spaces.

Definition 2.1. Let $\left\{x_n\right\}$ be a sequence in a normed space $\mathbf{X}$.

(1) A sequence $\left\{x_n\right\}_{n=1}^{\infty}$ in a space $\mathbf{X}$ is a Cauchy sequence iff the sequence $\left\{x_{n+1} - x_n\right\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\left\{x_n\right\}_{n=1}^{\infty}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in \mathbf{X}$ such that

$$\left\|x_n - x\right\| \leq \epsilon. \forall n \geq N,$$

for all n,m $\geq N$. Then the point x$\in \mathbf{X}$ is called the limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in $\mathbb{X}$ converger, then the normed space$\mathbf{X}$ is called a Bnanch space.

Definition 2.2. Let $\mathbf{X}$ be a real linear space. A quasi-norm is a real-valued function on $\mathbf{X}$ satisfying the following :

(1) $\Big\|x\Big\| \geq 0$ for all $x \in \mathbf{X}$ and $\Big\|x\Big\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = \Big|\lambda\Big|\Big\|x\Big\|$ for all $\lambda \in \mathbf{R}$ and all $x \in \mathbf{X}$.
(3) There is a constant $K \geq 1$ such that

$$\Big\|x + y\Big\| \leq K\Big(\Big\|x\Big\| + \Big\|y\Big\|\Big), \forall x, y \in \mathbf{X}.$$

The pair $\Big(\mathbf{X}, \Big\|\cdot\Big\|\Big)$ is called a quasi-normed space if $\Big\|\cdot\Big\|$ is a quasi-norm on $\mathbf{X}$.
The smallest possible K is called the modulus of concavity of $\Big\|\cdot\Big\|$.
A quasi-Banach space is a complete quasi-normed space.
A quasi-norm $\Big\|\cdot\Big\|$ is called a p-norm $\Big(0 < p \leq 1\Big)$ if

$$\Big\|x + y\Big\|^p \leq \Big\|x\Big\|^p + \Big\|y\Big\|^p \forall x, y \in \mathbf{X}.$$

In this case, a quasi-Banach space is called is called a p-Banach space

Definition 2.3. Let $\mathbf{X}$ be a real linear space. A generalized quasi-normed space is a real-valued function on $\mathbf{X}$ satisfying the following :

(1) $\Big\|x\Big\| \geq 0$ for all $x \in \mathbf{X}$ and $\Big\|x\Big\| = 0$ if and only if $x = 0$.
(2) $\|\lambda x\| = \Big|\lambda\Big|\Big\|x\Big\|$ for all $\lambda \in \mathbf{R}$ and all $x \in \mathbf{X}$.
(3) There is a constant $K \geq 1$ such that

$$\Big\|\sum_{j=1}^{\infty} x_j\Big\| \leq K\sum_{j=1}^{\infty}\Big\|x_j\Big\|, \forall x_1, x_2, ... \in \mathbf{X}.$$

The pair $\Big(\mathbf{X}, \Big\|\cdot\Big\|\Big)$ is called a generalized quasi-normed space if $\Big\|\cdot\Big\|$ is a generalized quasi-norm on $\mathbf{X}$. The smallest possible K is called the modulus of concavity of $\Big\|\cdot\Big\|$.
A generalized quasi-Banach space is a complete generalized quasi-normed space.
A generalized quasi-norm $\Big\|\cdot\Big\|$ is called a p-norm $\Big(0 < p \leq 1\Big)$ if

$$\Big\|x + y\Big\|^p \leq \Big\|x\Big\|^p + \Big\|y\Big\|^p \forall x, y \in \mathbf{X}.$$

In this case, a generalized quasi-Banach space is called is called a generalized p-Banach space

Definition 2.4. Let $\mathbf{X}, \mathbf{Y}$ be metric space. A mapping $f : \mathbf{X} \to \mathbf{Y}$ is called an isometry if f satisfies

$$d_{\mathbf{Y}}\Big(f(x), f(y)\Big) = d_{\mathbf{X}}\Big(x, y\Big), \forall x, y \in \mathbf{X}.$$

Where $d_{\mathbf{X}}\Big(\cdot, \cdot\Big), d_{\mathbf{Y}}\Big(\cdot, \cdot\Big)$, denote the metrics in the space $\mathbf{X}, \mathbf{Y}$, respecttively.

Definition 2.5. For r be a fixed positive number, suppose that f preserves distance r, ie, for all $x, y \in \mathbf{X}$ with $d_{\mathbf{X}}\Big(x, y\Big) = r$, we have $\Big(f(x), f(y)\Big) = r$. Then r is called a preserves distance for the mapping f.

Definition 2.6. Let $\Big(\mathbf{X}, \Big\|\cdot\Big\|\Big)$ and $\Big(\mathbf{Y}, \Big\|\cdot\Big\|\Big)$be normed space. A mapping $H : \mathbf{X} \to \mathbf{Y}$ is called an isometry if

$$\|H(x) - H(y)\| = \|x - y\|, \forall x, y \in \mathbf{X}.$$

2.2. **Solutions of the inequalities.** The functional equation

$$f\Big(x + y\Big) = f\Big(x\Big) + f\Big(y\Big)$$

is called the Cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

3. Stability of equation

Now, we first study the solutions of (1.1) and (1.2). Note that for this equations when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a quasi-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mapping satisfying (**??**) and (**??**) is additive. These results are give in the following.

Theorem 3.1. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping such that*

$$\left\|f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f\Big(x_j\Big) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \le \theta\Big(\sum_{j=1}^{n}\Big\|x_j\Big\|_{\mathbb{X}}^{r} + \sum_{j=1}^{n}\Big\|x_{k+j}\Big\|_{\mathbb{X}}^{r}\Big) \tag{3.1}$$

$$\left|\Big\|f\Big(x\Big)\Big\|_{\mathbb{Y}} - \Big\|x\Big\|_{\mathbb{X}}\right| \le \Big(n + n^{r+1}\Big)\theta\Big\|x\Big\|_{\mathbb{X}}^{r} \tag{3.2}$$

for all $x, x_j,\ x_{j+n} \in X$ for all $j = 1 \to n$. Then there exists a unique isometric Cauchy type additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\Big\|f(x) - H(x)\Big\|_{\mathbb{Y}} \le \frac{\Big(n + n^{r+1}\Big)\cdot \mathbf{K}\theta}{\Big(2n\Big)^{r} - 2n}\Big\|x\Big\|_{\mathbb{X}}^{r}, \forall x \in \mathbb{X}. \tag{3.3}$$

Proof. Letting $x_j = x, x_{n+j} = nx$ for all $j = 1 \to n$ by the hypothesis (**??**), we have

$$\left\|f\Big(2nx\Big) - 2nf\Big(x\Big)\right\|_{\mathbb{Y}} \le \Big(n + n^{r+1}\Big)\theta\Big\|x\Big\|_{\mathbb{X}}^{r}. \tag{3.4}$$

for all $x \in \mathbb{X}$. So

$$\left\| f\left(x\right) - 2nf\left(\frac{x}{2n}\right)\right\|_{Y} \leq \frac{\left(n+n^{r+1}\right)\theta}{\left(2n\right)^{r}}\left\|x\right\|_{\mathbb{X}}^{r}.$$

for all $x \in \mathbb{X}$. So

$$\left\|\left(2n\right)^{l} f\left(\frac{x}{\left(2n\right)^{l}}\right) - \left(2n\right)^{m} f\left(\frac{x}{\left(2n\right)^{m}}\right)\right\|_{\mathbb{Y}} \leq \mathbf{K}\sum_{j=l+1}^{m}\frac{(2n)^{j}\theta}{\left(2n\right)^{jr}}\left\|x\right\|_{\mathbb{X}}^{r}. \tag{3.5}$$

for all nonnegative integers m and l with $m > l$ and $\forall x \in X$. It follows from (**??**) that the sequence $\left\{\left(2n\right)^{h} f\left(\frac{x}{\left(2n\right)^{h}}\right)\right\}$ is a cauchy sequence for all $x \in X$. Since Y is complete space, the sequence $\left\{\left(2n\right)^{h} f\left(\frac{x}{\left(2n\right)^{h}}\right)\right\}$ coverges.

So one can define the mapping $H : \mathbb{X} \to \mathbb{Y}$ by

$$H\left(x\right) := \lim_{h\to\infty}\left(2n\right)^{h} f\left(\frac{x}{\left(2n\right)^{h}}\right)$$

for all $x \in X$. By (**??**)

$$\begin{aligned}
&\left\|H\left(\sum_{j=1}^{n}x_j + \frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right) - \sum_{j=1}^{n}H\left(x_j\right) - \sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&= \lim_{h\to\infty}\left(2n\right)^{h}\left\| f\left(\frac{1}{\left(2n\right)^{h}}\left(\sum_{j=1}^{n}x_j + \frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right) - \sum_{j=1}^{n} f\left(\frac{1}{\left(2n\right)^{h}}x_j\right)\right.\right.\\
&\left.\left. - \sum_{j=1}^{n} f\left(\frac{1}{\left(2n\right)^{h}}\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&\leq \lim_{h\to\infty}\frac{\theta\left(2n\right)^{h}}{\left(2n\right)^{hr}}\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbb{X}}^{r} + \sum_{j=1}^{n}\left\|x_{k+j}\right\|_{\mathbb{X}}^{r}\right)\\
&= 0,
\end{aligned}$$

for all x_j, $x_{j+n} \in \mathbf{X}$ for all $j = 1 \to n$. So

$$H\left(\sum_{j=1}^{n}x_j + \frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right) = \sum_{j=1}^{n}H\left(x_j\right) + \sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (**??**), we get (**??**). Now we prove the uniqueness of H. Assume that $H_1 : \mathbb{X} \to \mathbb{Y}$ is an additive mapping satisfying (**??**). Then we have

$$\begin{aligned}
&\left\|H(x) - H_1\left(x\right)\right\|_{\mathbb{Y}}\\
&= \left(2n\right)^h \left\|H\left(\frac{1}{\left(2n\right)^h}x\right) + H_1\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_{\mathbb{Y}}\\
&\leq \left(2n\right)^h \mathbf{K}\left(\left\|H\left(\frac{1}{\left(2n\right)^h}x\right) - f\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_{\mathbb{Y}} + \left\|f\left(\frac{1}{\left(2n\right)^h}x\right) + H_1\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_{\mathbb{Y}}\right)\\
&\leq \frac{2\left(2n\right)^h \cdot \mathbf{K^2}\theta}{\left(\left(2n\right)^r - 2n\right)\left(2n\right)^{hr}}\left\|x\right\|_{\mathbb{X}}^r
\end{aligned}$$

as which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that

$$H\left(x\right) = H_1\left(x\right)$$

This proves the uniqueness of H. It follows from (**??**) that

$$\begin{aligned}
\left|\left\|\left(2n\right)^h f\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_{\mathbb{Y}} - \left\|x\right\|_{\mathbb{X}}\right| &= \left(2n\right)^h \left|\left\|f\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_{\mathbb{Y}} - \left\|\frac{1}{\left(2n\right)^h}x\right\|_{\mathbb{X}}\right|\\
&\leq (n + n^{r+1})\theta\frac{\left(2n\right)^h}{\left(2n\right)^{hr}}\left\|x\right\|_{\mathbb{X}}^r \qquad (3.6)
\end{aligned}$$

which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So

$$\left\|H\left(x\right)\right\|_{\mathbb{Y}} := \lim_{h\to\infty}\left\|\left(2n\right)^h f\left(\frac{x}{\left(2n\right)^h}\right)\right\|_{\mathbb{Y}} = \left\|x\right\|_{\mathbb{X}}$$

for all $x \in \mathbb{X}$. Since H is additive,

$$\left\|H\left(x\right) - H\left(y\right)\right\|_{\mathbb{Y}} = \left\|H\left(x - y\right)\right\|_{\mathbb{Y}} = \left\|x - y\right\|_{\mathbb{X}}$$

For all $x, y \in \mathbf{X}$, as desired. □

Theorem 3.2. *Let $r < 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping such that*

$$\begin{aligned}
&\left\|f\left(\sum_{j=1}^n x_j + \frac{1}{n}\sum_{j=1}^n x_{n+j}\right) - \sum_{j=1}^n f\left(x_j\right) - \sum_{j=1}^n f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&\leq \theta\left(\sum_{j=1}^n \left\|x_j\right\|_{\mathbb{X}}^r + \sum_{j=1}^n \left\|x_{k+j}\right\|_{\mathbb{X}}^r\right) \qquad (3.7)
\end{aligned}$$

$$\left|\left\|f\left(x\right)\right\|_{\mathbb{Y}} - \left\|x\right\|_{\mathbb{X}}\right| \leq \left(n + n^{r+1}\right)\theta\left\|x\right\|_{\mathbb{X}}^r \qquad (3.8)$$

for all x_j, $x_{j+n} \in X$ for all $j = 1 \to n$. Then there exists a unique isometric additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\| f(x) - H(x) \right\|_{\mathbb{Y}} \leq \frac{\left(n + n^{r+1}\right) \cdot \mathbf{K}\theta}{2n - \left(2n\right)^{r}} \left\| x \right\|_{\mathbb{X}}^{r}, \forall x \in \mathbb{X}. \tag{3.9}$$

Proof. Letting $x_j = x, x_{n+j} = nx$ for all $j = 1 \to n$ by the hypothesis (**??**), we have

$$\left\| f\left(2nx\right) - 2nf\left(x\right) \right\|_{\mathbb{Y}} \leq \left(n + n^{r+1}\right)\theta \left\| x \right\|_{\mathbb{X}}^{r}. \tag{3.10}$$

for all $x \in \mathbb{X}$. So

$$\left\| f\left(x\right) - \frac{1}{2n} f\left(2nx\right) \right\|_{\mathbb{Y}} \leq \frac{\left(n + n^{r+1}\right)\theta}{2n} \left\| x \right\|_{\mathbb{X}}^{r}.$$

for all $x \in \mathbb{X}$. So

$$\left\| \frac{1}{\left(2n\right)^{l}} f\left(\left(2n\right)^{l} x\right) - \frac{1}{\left(2n\right)^{m}} f\left(\left(2n\right)^{m} x\right) \right\|_{\mathbb{Y}} \leq \frac{\left(n + n^{r+1}\right)\mathbf{K}}{2n} \sum_{j=l}^{m-1} \frac{2n\theta}{\left(2n\right)^{r}} \left\| x \right\|_{\mathbb{X}}^{r}. \tag{3.11}$$

for all nonnegative integers m and l with $m > l$ and $\forall x \in \mathbb{X}$. It follows from (**??**) that the sequence $\left\{ \frac{1}{\left(2n\right)^{h}} f\left(\left(2n\right)^{h} x\right) \right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{X}$ is complete space, the sequence $\left\{ \frac{1}{\left(2n\right)^{h}} f\left(\left(2n\right)^{h} x\right) \right\}$ coverges.

So one can define the mapping $H : \mathbb{X} \to \mathbb{Y}$ by

$$H\left(x\right) := \lim_{h \to \infty} \frac{1}{\left(2n\right)^{h}} f\left(\left(2n\right)^{h} x\right)$$

for all $x \in \mathbb{X}$. The restof the proof is similar to the proof of theorem 3.1.

□

Theorem 3.3. *Let $r < 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping with $f\left(0\right) = 0$ satisfying*

$$\left\| 2nf\left(\frac{1}{2n}\sum_{j=1}^{n} x_j + \frac{1}{2n^2}\sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f\left(x_j\right) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right) \right\|_{\mathbb{Y}} \leq \theta\left(\sum_{j=1}^{n} \left\| x_j \right\|_{\mathbb{X}}^{r} + \sum_{j=1}^{n} \left\| x_{k+j} \right\|_{\mathbb{X}}^{r}\right) \tag{3.12}$$

and

$$\left| \left\| f\left(x\right) \right\|_{\mathbb{Y}} - \left\| x \right\|_{\mathbb{X}} \right| \leq \left((3^r + 1)n^{r+1} + 2n \right)\theta \left\| x \right\|_{\mathbb{X}}^{r} \tag{3.13}$$

for all x_j, $x_{j+n} \in X$ for all $j = 1 \to n$. Then there exists a unique isometric Jensen additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{\left((3^r+1)n^{r+1} + 2n\right) \cdot \mathbf{K^2}\theta}{3-3^r}\left\|x\right\|_{\mathbb{X}}^r, \forall x \in \mathbb{X}. \tag{3.14}$$

Proof. Letting $x_j = -x, x_{n+j} = nx$ for all $j = 1 \to n$ by the hypothesis (**??**), we have

$$\left\|-nf\left(-x\right) - nf\left(x\right)\right\|_{\mathbb{Y}} \leq \left(n + n^{r+1}\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{3.15}$$

for all $x \in \mathbb{X}$. So Letting $x_{n+j} = 3nx$ and replacing x_j by $-x$ for all $j = 1 \to n$ in the hypothesis (**??**), we have

$$\left\|2nf\left(x\right) - nf\left(-x\right) - nf\left(3x\right)\right\|_{\mathbb{Y}} \leq \left(n + n\left(3n\right)^r\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{3.16}$$

for all $x \in \mathbb{X}$. So

$$\left\|3nf\left(x\right) - nf\left(3x\right)\right\|_{\mathbb{Y}} \leq \mathbf{K}\left((3^r+1)n^{r+1} + 2n\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{3.17}$$

for all $x \in \mathbb{X}$. So

$$\left\|f\left(x\right) - \frac{1}{3}f\left(3x\right)\right\|_{\mathbb{Y}} \leq \frac{\mathbf{K^2}}{3n}\left((3^r+1)n^{r+1} + 2n\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{3.18}$$

for all $x \in \mathbb{X}$. So

So

$$\left\|\frac{1}{3^l}f\left(3^l x\right) - \frac{1}{3^m}f\left(3^m x\right)\right\|_{\mathbb{Y}} \leq \frac{\mathbf{K^2}}{3n}\left((3^r+1)n^{r+1} + 2n\right)\sum_{j=l}^{m-1}\frac{3^{jr}\theta}{3^j}\left\|x\right\|_{\mathbb{X}}^r. \tag{3.19}$$

for all nonnegative integers m and l with $m > l$ and $\forall x \in_{\mathbb{X}}$. It follows from (**??**) that the sequence $\left\{\frac{1}{3^h}f\left(3^h x\right)\right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since Y is complete space, the sequence $\left\{\frac{1}{3^h}f\left(3^h x\right)\right\}$ coverges.

So one can define the mapping $H : \mathbb{X} \to \mathbb{Y}$ by

$$H\left(x\right) := \lim_{h\to\infty}\frac{1}{3^h}f\left(3^h x\right)$$

for all $x \in X$. By (??)

$$\begin{aligned}
&\left\|2nH\left(\frac{1}{2n}\sum_{j=1}^{n}x_j+\frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}H\left(x_j\right)-\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&=\lim_{h\to\infty}\frac{1}{3^h}\left\|2nf\left(3^h\left(\frac{1}{2n}\sum_{j=1}^{n}x_j+\frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}f\left(3^hx_j\right)\right.\right.\\
&\left.-\sum_{j=1}^{n}f\left(3^h\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&\le\lim_{h\to\infty}\theta\frac{3^{hr}}{3^h}\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbb{X}}^r+\sum_{j=1}^{n}\left\|x_{k+j}\right\|_{\mathbb{X}}^r\right)\\
&=0,
\end{aligned}$$

for all x_j, $x_{j+n} \in \mathbb{X}$ for all $j = 1 \to n$. So

$$2nH\left(\frac{1}{2n}\sum_{j=1}^{n}x_j+\frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right)=\sum_{j=1}^{n}H\left(x_j\right)+\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (??), we get (??). Now we prove the uniqueness of H. Assume that $H_1 : \mathbb{X} \to \mathbb{Y}$ is an additive mapping satisfying (??). Then we have

$$\begin{aligned}
&\left\|H(x)-H_1\left(x\right)\right\|_{\mathbb{Y}}\\
&=\frac{1}{3^h}\left\|H\left(3^hx\right)+H_1\left(3^hx\right)\right\|_{\mathbb{Y}}\\
&\le\frac{1}{3^h}\mathbf{K}\left(\left\|H\left(3^hx\right)-f\left(3^hx\right)\right\|_{\mathbb{Y}}+\left\|f\left(3^hx\right)+H_1\left(3^hx\right)\right\|_{\mathbb{Y}}\right)\\
&\le 2\frac{\left((3^r+1)n^{r+1}+2n\right)\cdot\mathbf{K^2}\theta}{\left(3-3^r\right)3^h}\left\|x\right\|_{\mathbb{X}}^r
\end{aligned}$$

.as which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that

$$H\left(x\right)=H_1\left(x\right)$$

This proves the uniqueness of H. It follows from (??) that

$$\begin{aligned}
\left|\left\|\frac{1}{3^h}f\left(3^hx\right)\right\|_{\mathbb{Y}}-\left\|x\right\|_{\mathbb{X}}\right|&=\frac{1}{3^h}\left|\left\|f\left(3^hx\right)\right\|_{\mathbb{Y}}-\left\|\frac{x}{3^h}\right\|_{\mathbb{X}}\right|\\
&\le\left((3^r+1)n^{r+1}+2n\right)\theta\frac{3^{hr}}{3^h}\left\|x\right\|_{\mathbb{X}}^r
\end{aligned}\tag{3.20}$$

which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So

$$\left\|H\left(x\right)\right\|_{\mathbb{Y}} := \lim_{h\to\infty}\left\|\frac{1}{3^h}f\left(3^h\right)\right\|_{\mathbb{Y}} = \left\|x\right\|_{\mathbb{X}}$$

for all $x \in \mathbb{X}$. Since H is additive ,

$$\left\|H\left(x\right) - H\left(y\right)\right\|_{\mathbb{Y}} = \left\|H\left(x-y\right)\right\|_{\mathbb{Y}} = \left\|x-y\right\|_{\mathbb{X}}$$

For all $x, y \in \mathbf{X}$, as desired.

□

Theorem 3.4. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping with $f\left(0\right) = 0$ satifsfying*

$$\left\|2nf\left(\frac{1}{2n}\sum_{j=1}^{n}x_j + \frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right) - \sum_{j=1}^{n}f\left(x_j\right) - \sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \theta\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbb{X}}^{r} + \sum_{j=1}^{n}\left\|x_{k+j}\right\|_{\mathbb{X}}^{r}\right) \tag{3.21}$$

and

$$\left|\left\|f\left(x\right)\right\|_{\mathbb{Y}} - \left\|x\right\|_{\mathbb{X}}\right| \leq \left((3^r+1)n^{r+1} + 2n\right)\theta\left\|x\right\|_{\mathbb{X}}^{r} \tag{3.22}$$

for all x_j, $x_{j+n} \in X$ for all $j = 1 \to n$. Then there exists a unique isometric Jensen additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{\left((3^r+1)n^{r+1} + 2n\right)\cdot \mathbf{K^2}\theta}{3^r - 3}\left\|x\right\|^{r}, \forall x \in \mathbb{X}. \tag{3.23}$$

Proof. Letting $x_j = -x, x_{n+j} = nx$ for all $j = 1 \to n$ by the hypothesis (3.13), we have

$$\left\|-nf\left(-x\right) - nf\left(x\right)\right\|_{Y} \leq \left(n + n^{r+1}\right)\theta\left\|x\right\|^{r}. \tag{3.24}$$

for all $x \in \mathbb{X}$. So Letting $x_{n+j} = 3nx$ and replacing x_j by $-x$ for all $j = 1 \to n$ in the hypothesis (3.13), we have

$$\left\|2nf\left(x\right) - nf\left(-x\right) - nf\left(3x\right)\right\|_{Y} \leq \left(n + n\left(3n\right)^{r}\right)\theta\left\|x\right\|_{\mathbb{X}}^{r}. \tag{3.25}$$

for all $x \in \mathbb{X}$. So

$$\left\|3nf\left(x\right) - nf\left(3x\right)\right\|_{Y} \leq \mathbf{K}\left((3^r+1)n^{r+1} + 2n\right)\theta\left\|x\right\|^{r}. \tag{3.26}$$

for all $x \in \mathbb{X}$. So

$$\left\|f\left(x\right) - 3f\left(\frac{x}{3}\right)\right\|_{Y} \leq \frac{\mathbf{K}}{3^r n}\left((3^r+1)n^{r+1} + 2n\right)\theta\left\|x\right\|^{r}. \tag{3.27}$$

for all $x \in \mathbb{X}$. So

So

$$\left\|3^l f\left(\frac{1}{3^l}x\right) - 3^m f\left(\frac{1}{3^m}x\right)\right\|_Y \leq \frac{\mathbf{K^2}}{3^r n}\Big((3^r+1)n^{r+1} + 2n\Big)\sum_{j=l}^{m-1}\frac{3^j\theta}{3^{rj}}\Big\|x\Big\|^r. \tag{3.28}$$

for all nonnegative integers m and l with $m > l$ and $\forall x \in X$. It follows from (3.20) that the sequence $\left\{3^h f\left(\frac{1}{3^h}x\right)\right\}$ is a cauchy sequence for all $x \in X$. Since Y is complete space, the sequence $\left\{3^h f\left(\frac{1}{3^h}x\right)\right\}$ coverges.
So one can define the mapping $H : \mathbb{X} \to \mathbb{Y}$ by

$$H(x) := \lim_{h\to\infty} 3^h\Big(\frac{1}{3^h}x\Big)$$

for all $x \in X$. The rest of the proof is similar to the proof of theorem 3.3 □

4. Stability of the isometric additive mapping in generalized p-Banach space

Now, we first study the solutions of (1.1) and (1.2). Note that for this equations when $\mathbf{X}$ is a quasi-normed vector space with quasi-norm $\|\cdot\|_{\mathbf{X}}$ and that $\mathbf{Y}$ is a p-Banach space with quasi-norm $\|\cdot\|_{\mathbf{Y}}$. Under this setting, we can show that the mapping satisfying (**??**) and (**??**) is additive. These results are give in the following.

Theorem 4.1. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping such that*

$$\left\|f\left(\sum_{j=1}^{n}x_j + \frac{1}{n}\sum_{j=1}^{k}x_{n+j}\right) - \sum_{j=1}^{k}f\Big(x_j\Big) - \sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}$$
$$\leq \theta\left(\sum_{j=1}^{n}\Big\|x_j\Big\|_{\mathbb{X}}^r + \sum_{j=1}^{n}\Big\|x_{n+j}\Big\|_{\mathbb{X}}^r\right) \tag{4.1}$$

$$\left|\Big\|f\Big(x\Big)\Big\|_{\mathbb{Y}} - \Big\|x\Big\|_{\mathbb{X}}\right| \leq \Big(n+n^{r+1}\Big)\theta\Big\|x\Big\|_{\mathbb{X}}^r \tag{4.2}$$

for all $x, x_j,\ x_{n+j} \in \mathbb{X}$ for all $j = 1 \to n$. then there exists a unique isometric Cauchy type additive $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\Big\|f(x) - H(x)\Big\|_{\mathbb{Y}} \leq \frac{\Big(n+n^{r+1}\Big)\theta}{\left(\Big(2n\Big)^{pr} - \Big(2n\Big)^{p}\right)^{\frac{1}{p}}}\Big\|x\Big\|_{\mathbb{X}}^r, \forall x \in \mathbb{X}. \tag{4.3}$$

Proof. Letting $x_j = x, x_{n+j} = nx$ for all $j = 1 \to n$ by the hypothesis (**??**), we have

$$\Big\|f\Big(2nx\Big) - 2nf\Big(x\Big)\Big\|_{\mathbb{Y}} \leq \Big(n+n^{r+1}\Big)\theta\Big\|x\Big\|_{\mathbb{X}}^r. \tag{4.4}$$

for all $x \in \mathbb{X}$. So

$$\left\|f\left(x\right)-2nf\left(\frac{x}{2n}\right)\right\|_{\mathbb{Y}}\leq\left(n+n^{r+1}\right)\frac{\theta}{\left(2n\right)^{r}}\left\|x\right\|_{\mathbb{X}}^{r}.$$

for all $x\in\mathbb{X}$. Sence $\mathbb{Y}$ is a p-Banach space,

$$\begin{aligned}&\left\|\left(2n\right)^{l}f\left(\frac{x}{\left(2n\right)^{l}}\right)-\left(2n\right)^{m}f\left(\frac{x}{\left(2n\right)^{m}}\right)\right\|_{\mathbb{Y}}^{p}\\&\leq\sum_{j=l}^{m-1}\left\|\left(2n\right)^{j}f\left(\frac{x}{\left(2n\right)^{j}}\right)-\left(2n\right)^{j+1}f\left(\frac{x}{\left(2n\right)^{j+1}}\right)\right\|_{\mathbb{Y}}^{p}\\&\leq\left(n+n^{r+1}\right)^{p}\frac{\theta^{p}}{\left(2n\right)^{pr}}\sum_{j=l}^{m-1}\frac{\left(2n\right)^{pj}}{\left(2n\right)^{prj}}\left\|x\right\|^{pr}.\end{aligned}\tag{4.5}$$

for all $x\in\mathbf{X}$. Sence $\mathbf{Y}$ is a p-Banach spaces

for all nonnegative integers m and l with $m>l$ and $\forall x\in\mathbb{X}$. It follows from (**??**) that the sequence $\left\{\left(2n\right)^{h}f\left(\frac{x}{\left(2n\right)^{h}}\right)\right\}$ is a cauchy sequence for all $x\in\mathbb{X}$. Since $\mathbb{Y}$ is complete, the sequence $\left\{\left(2n\right)^{h}f\left(\frac{x}{\left(2n\right)^{h}}\right)\right\}$ coverges.
So one can define the mapping $H:\mathbb{X}\to\mathbb{Y}$ by

$$H\left(x\right):=\lim_{h\to\infty}\left(2n\right)^{h}f\left(\frac{x}{\left(2n\right)^{h}}\right)$$

for all $x\in X$. It follows from (**??**) that

$$\begin{aligned}&\left\|H\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}H\left(x_j\right)-\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\&=\lim_{h\to\infty}\left(2n\right)^{h}n\left\|f\left[\frac{1}{\left(2n\right)^{h}}\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)\right]-\sum_{j=1}^{k}f\left(\frac{1}{\left(2n\right)^{h}}x_j\right)\right.\\&\left.-\sum_{j=1}^{n}f\left(\frac{1}{\left(2n\right)^{h}}\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\&\leq\lim_{n\to\infty}\frac{\theta\left(2n\right)^{h}}{\left(2n\right)^{hr}}\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbf{X}}^{r}+\sum_{j=1}^{n}\left\|x_{n+j}\right\|_{\mathbf{X}}^{r}\right)\\&=0,\end{aligned}$$

for all x_j, $x_{n+j}\in X$ for all $j=1\to n$. So

$$H\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)=\sum_{j=1}^{n}H\left(x_j\right)+\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (**??**), we get (**??**).

Now we prove the uniqueness of H. Assume that $H_1 : X \to Y$ is an additive mapping satisfing (**??**). Then we have

$$\begin{aligned}
&\left\|H\left(x\right) - H_1\left(x\right)\right\|_Y^p \\
&= \left(2n\right)^{ph}\left\|H\left(\frac{1}{\left(2k\right)^n}x\right) - H_1\left(\frac{1}{\left(2n\right)^{ph}}x\right)\right\|_Y^p \\
&\leq \left(2n\right)^h\mathbf{K}^p\left(\left\|H\left(\frac{1}{\left(2n\right)^h}x\right) - f\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_Y^p + \left\|f\left(\frac{1}{\left(2n\right)^h}x\right) - H_1\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_Y^p\right) \\
&\leq 2\frac{\mathbf{K}^p\left(n + n^{r+1}\right)^p\theta^p}{\left(\left(2n\right)^{pr} - \left(2n\right)^p\right)^{\frac{1}{p}}}\frac{\left(2n\right)^{ph}}{\left(2n\right)^{phr}}\left\|x\right\|_{\mathbf{X}}^{pr}
\end{aligned}$$

. which tends to zero as $n \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that $H(x) = H_1(x)$ for all $x \in \mathbf{X}$. This proves the uniqueness of H.
It follows from (**??**) that

$$\begin{aligned}
\left|\left\|\left(2n\right)^h f\left(\frac{1}{\left(2n\right)^p}x\right)\right\|_{\mathbb{Y}} - \left\|x\right\|_{\mathbb{X}}\right| &= \left(2n\right)^h\left|\left\|f\left(\frac{1}{\left(2n\right)^h}x\right)\right\|_{\mathbb{Y}} - \left\|\frac{1}{\left(2n\right)^h}x\right\|_{\mathbb{X}}\right| \\
&\leq \left(n + n^{r+1}\right)\theta\frac{\left(2n\right)^h}{\left(2n\right)^{hr}}\left\|x\right\|_{\mathbb{X}}^r \qquad (4.6)
\end{aligned}$$

which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So

$$\left\|H\left(x\right)\right\|_{\mathbb{Y}} := \lim_{h\to\infty}\left\|\left(2n\right)^h f\left(\frac{x}{\left(2n\right)^h}\right)\right\|_{\mathbb{Y}} = \left\|x\right\|_{\mathbb{X}}$$

for all $x \in \mathbb{X}$. Since H is additive,

$$\left\|H\left(x\right) - H\left(y\right)\right\|_{\mathbb{Y}} = \left\|H\left(x - y\right)\right\|_{\mathbb{Y}} = \left\|x - y\right\|_{\mathbb{X}}$$

For all $x, y \in \mathbf{X}$, as desired. □

Theorem 4.2. *Let $r < 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping such that*

$$\left\| f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{k} x_{n+j}\right) - \sum_{j=1}^{k} f\left(x_j\right) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \theta\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbb{X}}^{r} + \sum_{j=1}^{n}\left\|x_{n+j}\right\|_{\mathbb{X}}^{r}\right) \quad (4.7)$$

$$\left|\left\|f\left(x\right)\right\|_{\mathbb{Y}} - \left\|x\right\|_{\mathbb{X}}\right| \leq \left(n + n^{r+1}\right)\theta\left\|x\right\|_{\mathbb{X}}^{r} \quad (4.8)$$

for all $x, x_j,\ x_{n+j} \in \mathbb{X}$ for all $j = 1 \to n$. then there exists a unique isometric Cauchy type additive $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{\left(n + n^{r+1}\right)\theta}{\left(\left(2n\right)^{p} - \left(2n\right)^{pr}\right)^{\frac{1}{p}}}\left\|x\right\|_{\mathbb{X}}^{r}, \forall x \in \mathbb{X}. \quad (4.9)$$

The rest of the proof is similar to the proof of theorem 3.2.

Theorem 4.3. *Let $r < 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping with $f\left(0\right) = 0$ satisfying*

$$\left\|2nf\left(\frac{1}{2n}\sum_{j=1}^{n} x_j + \frac{1}{2n^2}\sum_{j=1}^{n} x_{n+j}\right) - \sum_{j=1}^{n} f\left(x_j\right) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \theta\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbb{X}}^{r} + \sum_{j=1}^{n}\left\|x_{k+j}\right\|_{\mathbb{X}}^{r}\right) \quad (4.10)$$

and

$$\left|\left\|f\left(x\right)\right\|_{\mathbb{Y}} - \left\|x\right\|_{\mathbb{X}}\right| \leq \left((3^r + 1)n^{r+1} + 2n\right)\theta\left\|x\right\|_{\mathbb{X}}^{r} \quad (4.11)$$

for all $x_j,\ x_{j+n} \in X$ for all $j = 1 \to n$. Then there exists a unique isometric Jensen additive mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{\left((3^r + 1)n^{r+1} + 2n\right) \cdot \mathbf{K^2}\theta}{\left(3^p - 3^{pr}\right)^{\frac{1}{p}}}\left\|x\right\|_{\mathbb{X}}^{r}, \forall x \in \mathbb{X}. \quad (4.12)$$

Proof. Letting $x_j = -x, x_{n+j} = nx$ for all $j = 1 \to n$ by the hypothesis (**??**), we have

$$\left\|-nf\left(-x\right) - nf\left(x\right)\right\|_{\mathbb{Y}} \leq \left(n + n^{r+1}\right)\theta\left\|x\right\|_{\mathbb{X}}^{r}. \quad (4.13)$$

for all $x \in \mathbb{X}$. So Letting $x_{n+j} = 3nx$ and replacing x_j by $-x$ for all $j = 1 \to n$ in the hypothesis (**??**), we have

$$\left\|2nf\left(x\right)-nf\left(-x\right)-nf\left(3x\right)\right\|_{\mathbb{Y}} \leq \left(n+n\left(3n\right)^r\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{4.14}$$

for all $x \in \mathbb{X}$. So

$$\left\|3nf\left(x\right)-nf\left(3x\right)\right\|_{\mathbb{Y}} \leq \mathbf{K}\left(\left(3^r+1\right)n^{r+1}+2n\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{4.15}$$

for all $x \in \mathbb{X}$. So

$$\left\|f\left(x\right)-\frac{1}{3}f\left(3x\right)\right\|_{\mathbb{Y}} \leq \frac{\mathbf{K}}{3n}\left(\left(3^r+1\right)n^{r+1}+2n\right)\theta\left\|x\right\|_{\mathbb{X}}^r. \tag{4.16}$$

for all $x \in \mathbb{X}$. So

So

$$\left\|\frac{1}{3^l}f\left(3^lx\right)-\frac{1}{3^m}f\left(3^mx\right)\right\|_{\mathbb{Y}}^p \leq \frac{\mathbf{K}^p}{3n}\left(\left(3^r+1\right)n^{r+1}+2n\right)\sum_{j=l}^{m-1}\frac{3^{jr}\theta}{3^j}\left\|x\right\|_{\mathbb{X}}^r. \tag{4.17}$$

for all nonnegative integers m and l with $m > l$ and $\forall x \in_{\mathbb{X}}$. It follows from (**??**) that the sequence $\left\{\frac{1}{3^h}f\left(3^hx\right)\right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since Y is complete space, the sequence $\left\{\frac{1}{3^h}f\left(3^hx\right)\right\}$ coverges.
So one can define the mapping $H : \mathbb{X} \to \mathbb{Y}$ by

$$H\left(x\right) := \lim_{h\to\infty}\frac{1}{3^h}f\left(3^hx\right)$$

for all $x \in X$. By (**??**)

$$\begin{aligned}
&\left\|2nH\left(\frac{1}{2n}\sum_{j=1}^{n}x_j+\frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}H\left(x_j\right)-\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&=\lim_{h\to\infty}\frac{1}{3^h}\left\|2nf\left(3^h\left(\frac{1}{2n}\sum_{j=1}^{n}x_j+\frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right)-\sum_{j=1}^{n}f\left(3^hx_j\right)\right.\right.\\
&\left.-\sum_{j=1}^{n}f\left(3^h\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\\
&\leq\lim_{h\to\infty}\theta\frac{3^{hr}}{3^h}\left(\sum_{j=1}^{n}\left\|x_j\right\|_{\mathbb{X}}^r+\sum_{j=1}^{n}\left\|x_{k+j}\right\|_{\mathbb{X}}^r\right)\\
&=0,
\end{aligned}$$

and so for all x_j, $x_{j+n} \in \mathbb{X}$ for all $j = 1 \to n$.

$$2nH\left(\frac{1}{2n}\sum_{j=1}^{n}x_j+\frac{1}{2n^2}\sum_{j=1}^{n}x_{n+j}\right)=\sum_{j=1}^{n}H\left(x_j\right)+\sum_{j=1}^{n}H\left(\frac{x_{n+j}}{n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (??), we get (??). Now we prove the uniqueness of H. Assume that $H_1 : \mathbb{X} \to \mathbb{Y}$ is an additive mapping satisfying (??). Then we have

$$\begin{aligned}
&\left\| H(x) - H_1\Big(x\Big) \right\|_{\mathbb{Y}} \\
&= \frac{1}{3^h} \left\| H\Big(3^h x\Big) + H_1\Big(3^h x\Big) \right\|_{\mathbb{Y}} \\
&\le \frac{1}{3^h} \mathbf{K} \left(\left\| H\Big(3^h x\Big) - f\Big(3^h x\Big) \right\|_{\mathbb{Y}} + \left\| f\Big(3^h x\Big) + H_1\Big(3^h x\Big) \right\|_{\mathbb{Y}} \right) \\
&\le 2 \frac{\left(\Big(3^r + 1\Big) n^{r+1} + 2n \right) \cdot \mathbf{K^2} \theta}{\Big(3 - 3^r\Big) 3^h} \left\| x \right\|_{\mathbb{X}}^r
\end{aligned}$$

.as which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So we can conclude that

$$H\Big(x\Big) = H_1\Big(x\Big)$$

This proves the uniqueness of H. It follows from (??) that

$$\begin{aligned}
\left| \left\| \frac{1}{3^h} f\Big(3^h x\Big) \right\|_{\mathbb{Y}} - \left\| x \right\|_{\mathbb{X}} \right| &= \frac{1}{3^h} \left| \left\| f\Big(3^h x\Big) \right\|_{\mathbb{Y}} - \left\| \frac{x}{3^h} \right\|_{\mathbb{X}} \right| \\
&\le \left(\Big(3^r + 1\Big) n^{r+1} + 2n \right) \theta \frac{3^{hr}}{3^h} \left\| x \right\|_{\mathbb{X}}^r
\end{aligned} \tag{4.18}$$

which tends to zero as $h \to \infty$ for all $x \in \mathbf{X}$. So

$$\left\| H\Big(x\Big) \right\|_{\mathbb{Y}} := \lim_{h \to \infty} \left\| \frac{1}{3^h} f\Big(3^h\Big) \right\|_{\mathbb{Y}} = \left\| x \right\|_{\mathbb{X}}$$

for all $x \in \mathbb{X}$. Since H is additive ,

$$\left\| H\Big(x\Big) - H\Big(y\Big) \right\|_{\mathbb{Y}} = \left\| H\Big(x - y\Big) \right\|_{\mathbb{Y}} = \left\| x - y \right\|_{\mathbb{X}}$$

For all $x, y \in \mathbf{X}$, as desired.

□

Theorem 4.4. *Let $r > 1$ and θ be positive real numbers,and $f : \mathbb{X} \to \mathbb{Y}$ be a mapping with $f\Big(0\Big) = 0$ satisfying*

$$\begin{aligned}
\left\| 2nf\left(\frac{1}{2n} \sum_{j=1}^{n} x_j + \frac{1}{2n^2} \sum_{j=1}^{n} x_{n+j} \right) - \sum_{j=1}^{n} f\Big(x_j\Big) - \sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n} \right) \right\|_{\mathbb{Y}} \\
\le \theta \left(\sum_{j=1}^{n} \left\| x_j \right\|_{\mathbb{X}}^r + \sum_{j=1}^{n} \left\| x_{k+j} \right\|_{\mathbb{X}}^r \right)
\end{aligned} \tag{4.19}$$

and

$$\left|\left\|f\left(x\right)\right\|_{\mathbb{Y}}-\left\|x\right\|_{\mathbb{X}}\right|\leq\left(\left(3^{r}+1\right)n^{r+1}+2n\right)\theta\left\|x\right\|_{\mathbb{X}}^{r}\tag{4.20}$$

for all $x_j,\ x_{j+n}\in X$ for all $j=1\rightarrow n$. Then there exists a unique isometric Jensen additive mapping $H:\mathbb{X}\rightarrow\mathbb{Y}$ such that

$$\left\|f\left(x\right)-H\left(x\right)\right\|_{\mathbb{Y}}\leq\frac{\left(\left(3^{r}+1\right)n^{r+1}+2n\right)\cdot\mathbf{K^2}\theta}{\left(3^{pr}-3^{p}\right)^{\frac{1}{p}}}\left\|x\right\|_{\mathbb{X}}^{r},\forall x\in\mathbb{X}.\tag{4.21}$$

The rest of the proof is similar to the proofs of theorems 3.1 and 4.5.

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