International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373/doi:10.14445/22315373/IJMTT-V67I9P507
Volume 67 Issue 9, 65-71, September, 2021

Solution of Nonlinear System of Fractional Differential Equations

E. A. A. Ziada

Nile Higher Institute for Engineering & Technology, Mansoura, Egypt.

Abstract **-** *In this paper, we apply the Adomian decomposition method (ADM) for solving nonlinear system of fractional differential equations (FDEs) of sequential Riemann-Liouville sense. The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are discussed.*

Keywords - *Fractional differential equations; Adomian decomposition Method; Existence; Uniqueness; Error analysis.*

I. Introduction

Fractional Differential equations have many applications in engineering and science, including electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems ([1]-[11]). In this paper, Adomian decomposition method (ADM) ([12]-[19]) is used to solve these type of equations. This method has many advantages; it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization.

The paper is organized as follows: In section two ADM is applied to the problem under consideration. In section three uniqueness, convergence; error analysis and stability are discussed. Finally, five numerical examples are presented by using MATHEMATICA package.

II. Formulation of the problem

Consider a system of nonlinear FDEs of the form,

$$ {}_0\mathcal{D}_t^{\sigma_n} y_i(t) + \beta_i(t) f_i(\overline{y}) = x_i(t), \tag{1} $$

subject to the initial conditions,

$$ [{}_0\mathcal{D}_t^{\sigma_n} y_i(t)]_{t=0} = c_{ij}, \quad i,j = 1,2,\dots,n. \tag{2} $$

where,

$$ \overline{y} = \{y_1(t), y_2(t), \dots, y_n(t)\}, $$
$$ {}_0\mathcal{D}_t^{\sigma_n} \equiv \quad {}_0D_t^{\alpha_n} \quad {}_0D_t^{\alpha_{n-1}} \quad {}_0D_t^{\alpha_{n-2}} \cdots \quad {}_0D_t^{\alpha_1}, $$
$$ {}_0\mathcal{D}_t^{\sigma_{n-1}} \equiv \quad {}_0D_t^{\alpha_{n-1}} \quad {}_0D_t^{\alpha_{n-2}} \quad {}_0D_t^{\alpha_{n-3}} \cdots \quad {}_0D_t^{\alpha_1}, $$
$$ \sigma_n = \sum_{k=1}^{n} \alpha_k, \quad 0 \le \alpha_k \le 1, $$

The fractional derivative in this system can be of the sequential (Riemann-Liouville or Grünwald-Letnikov) sense. Now performing subsequently the fractional integration of orders $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$, this reduces the system (1)-(2) to its equivalent system of FIEs,

$$ y_i(t) = \sum_{j=1}^{n} \frac{c_{ij}}{\Gamma(\sigma_j)} t^{\sigma_j - 1} + \frac{1}{\Gamma(\sigma_n)} \int_0^t (t-\tau)^{\sigma_n - 1} x_i(\tau) d\tau $$
$$ - \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau) (t-\tau)^{\sigma_n - 1} f_i(\overline{y}) d\tau \tag{3} $$

Assume that $x_i(t)$ is bounded $\forall\ t \in J = [0,T], T \in R^+$, $|\beta_i(\tau)| \le M_i \ \ \forall\ \ 0 \le \tau \le t \le T$, M_i are finite constants and $f_i(\overline{y})$ satisfy Lipschitz condition with Lipschitz constants L_i such as,

$$ |f_i(\overline{y}) - f_i(\overline{z})| \le L_i |\overline{y} - \overline{z}| \tag{4} $$

and has Adomian polynomials representation,

$$ f_i(\overline{y}) = \sum_{k=0}^{\infty} A_{ik}(y_{i0}, y_{i1}, \dots, y_{in}) \tag{5} $$

where,

$$A_{ik} = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[f_i\left(\sum_{j=0}^{\infty}\lambda^j y_j\right)\right]_{\lambda=0} \qquad (6)$$

Substitute from equation (5) into equation (3), we get

$$y_i(t) = \sum_{j=1}^{n}\frac{c_{ij}}{\Gamma(\sigma_j)}t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)}\int_0^t (t-\tau)^{\sigma_n-1}x_i(\tau)\,d\tau$$

$$-\frac{1}{\Gamma(\sigma_n)}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}\sum_{k=0}^{\infty}A_{ik}\,d\tau. \qquad (7)$$

Let $y_i(t) = \sum_{k=0}^{\infty} y_{ik}(t)$ in (7) we get,

$$y_{i0}(t) = \sum_{j=1}^{n}\frac{c_{ij}}{\Gamma(\sigma_j)}t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)}\int_0^t (t-\tau)^{\sigma_n-1}x_i(\tau)\,d\tau, \qquad (8)$$

$$y_{ik}(t) = -\frac{1}{\Gamma(\sigma_n)}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}A_{i(k-1)}\,d\tau, \quad k \geq 1. \qquad (9)$$

Finally, the solution is,

$$y_i(t) = \sum_{k=0}^{\infty} y_{ik}(t). \qquad (10)$$

III. Analysis of Convergence

A. The Uniqueness of Solution

In the previous section, we find the series solution (10) of the system (1)-(2) and here we want to prove the existence and uniqueness of this series solution.

Theorem 1 *If* $0 < \alpha < 1$ *where* $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}$, *then the series (10) is the solution of the system (1)-(2) and this solution is unique, where* $L = \max\ \{L_1, L_2, \dots, L_n\}$, $M = \max\ \{M_1, M_2, \dots, M_n\}$.

Proof *For existence,*

$$\begin{aligned}
y_i(t) &= \sum_{k=0}^{\infty} y_{ik}(t) \\
&= y_{i0}(t) + \sum_{k=1}^{\infty} y_{ik}(t) \\
&= y_{i0}(t) - \frac{1}{\Gamma(\sigma_n)}\sum_{k=1}^{\infty}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}A_{i(k-1)}d\tau \\
&= y_{i0}(t) - \frac{1}{\Gamma(\sigma_n)}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}\sum_{k=1}^{\infty}A_{i(k-1)}\,d\tau \\
&= y_{i0}(t) - \frac{1}{\Gamma(\sigma_n)}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}\sum_{k=0}^{\infty}A_{ik}\,d\tau \\
&= \sum_{j=1}^{n}\frac{c_{ij}}{\Gamma(\sigma_j)}t^{\sigma_j-1} + \frac{1}{\Gamma(\sigma_n)}\int_0^t (t-\tau)^{\sigma_n-1}x_i(\tau)d\tau \\
&\quad -\frac{1}{\Gamma(\sigma_n)}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}f_i(\overline{y})d\tau
\end{aligned}$$

then the Adomian's series solution satisfy equation (3) which is the reduced system of FIEs to the system (1)-(2).

For uniqueness of the solution: Assume that $\overline{y}$ and $\overline{z}$ are two different solutions to the system (1)-(2) and hence,

$$\begin{aligned}
|\overline{y}-\overline{z}| &= \left|\frac{1}{\Gamma(\sigma_n)}\int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1}[f_i(\overline{y}) - f_i(\overline{z})]d\tau\right| \\
&\leq \frac{1}{\Gamma(\sigma_n)}\int_0^t (t-\tau)^{\sigma_n-1}|\beta_i(\tau)||f_i(\overline{y}) - f_i(\overline{z})|d\tau \\
&\leq \frac{L_i M_i}{\Gamma(\sigma_n)}|\overline{y}-\overline{z}|\int_0^t (t-\tau)^{\sigma_n-1}d\tau
\end{aligned}$$

$$\leq \frac{L_i M_i T^{\sigma_n}}{\Gamma(\sigma_n+1)} |\overline{y}-\overline{z}|$$
$$\leq \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)} |\overline{y}-\overline{z}|$$

Let $\frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)} = \alpha$ where, $0 < \alpha < 1$ then,

$$|\overline{y}-\overline{z}| \leq \alpha \ |\overline{y}-\overline{z}|$$
$$(1-\alpha) \ |\overline{y}-\overline{z}| \leq 0$$

but, $(1-\alpha) \ |\overline{y}-\overline{z}| \geq 0$ and since, $(1-\alpha) \neq 0$ then, $|\overline{y}-\overline{z}| = 0$ this imply that, $\overline{y} = \overline{z}$ and this completes the proof. ■

B. Proof of Convergence

Theorem 2 *The series solution (10) of the system (1)-(4) using ADM converges if* $|y_{i1}| < \infty$ *and* $0 < \alpha < 1$, $\alpha = \frac{LMT^{\sigma_n}}{\Gamma(\sigma_n+1)}$, *where* $L = \max \ \{L_1, L_2, \dots, L_n\}$, $M = \max \ \{M_1, M_2, \dots, M_n\}$.

Proof Define the Banach space $(C[J], \|\cdot\|)$, the space of all continuous functions on J with the norm $\|f_1(t) - f_2(t)\| = \max_{t\in J} |f_1(t) - f_2(t)|$ and a sequence $\{S_{in}\}$ such that, $S_{in} = \sum_{k=0}^{n} y_{ik}(t)$. We have,

$$f(S_{in}) = \sum_{k=0}^{n} A_{ik}\,(y_{i0}, y_{i1}, \dots, y_{in})$$

Let, S_{in} and S_{im} be two arbitrary partial sums with, $n \geq m$. Now, we are going to prove that $\{S_{in}\}$ is a Cauchy sequence in this Banach space.

$$\|S_{in} - S_{im}\| = \max_{t\in J} |S_{in} - S_{im}|$$
$$= \max_{t\in J} \left| \sum_{k=m+1}^{n} y_{ik}(t) \right|$$
$$= \max_{t\in J} \left| \sum_{k=m+1}^{n} -\frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n - 1} A_{i(k-1)} d\tau \right|$$

$$\|S_{in} - S_{im}\| = \max_{t\in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1} \sum_{k=m+1}^{n} A_{i(k-1)}\, d\tau \right|$$
$$= \max_{t\in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1} \sum_{k=m}^{n-1} A_{ik}\, d\tau \right|$$
$$= \max_{t\in J} \left| \frac{1}{\Gamma(\sigma_n)} \int_0^t \beta_i(\tau)(t-\tau)^{\sigma_n-1} [f(S_{i(n-1)}) - f(S_{i(m-1)})] d\tau \right|$$
$$\leq \frac{1}{\Gamma(\sigma_n)} \max_{t\in J} \int_0^t (t-\tau)^{\sigma_n-1} |\beta_i(\tau)| \left| f(S_{i(n-1)}) - f(S_{i(m-1)}) \right| d\tau$$
$$\leq \frac{L_i M_i}{\Gamma(\sigma_n)} \max_{t\in J} \left| S_{i(n-1)} - S_{i(m-1)} \right| \int_0^t (t-\tau)^{\sigma_n-1} d\tau$$
$$\leq \frac{LMT^{\sigma_n}}{\sigma_n \Gamma(\sigma_n)} \left\| S_{i(n-1)} - S_{i(m-1)} \right\|$$
$$\leq \alpha \left\| S_{i(n-1)} - S_{i(m-1)} \right\|$$

Let $n = m+1$ then,

$$\left\| S_{i(m+1)} - S_{im} \right\| \leq \alpha \left\| S_{im} - S_{i(m-1)} \right\| \leq \alpha^2 \left\| S_{i(m-1)} - S_{i(m-2)} \right\| \leq \cdots \leq \alpha^m \|S_{i1} - S_{i0}\|$$

Using the triangle inequality,

$$\|S_{in} - S_{im}\| \leq \left\| S_{i(m+1)} - S_{im} \right\| + \left\| S_{i(m+2)} - S_{i(m+1)} \right\| + \cdots + \left\| S_{in} - S_{i(n-1)} \right\|$$
$$\leq [\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}] \|S_{i1} - S_{i0}\|$$
$$\leq \alpha^m [1 + \alpha + \cdots + \alpha^{n-m-1}] \|S_{i1} - S_{i0}\|$$
$$\leq \alpha^m \left[\frac{1-\alpha^{n-m}}{1-\alpha} \right] \|y_{i1}(t)\|$$

Since, $0 < \alpha < 1$, and $n \geq m$ then, $(1 - \alpha^{n-m}) \leq 1$. Consequently,

$$\begin{aligned} \|S_{in} - S_{im}\| &\leq \frac{\alpha^m}{1-\alpha} \|y_{i1}(t)\| \\ &\leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_{i1}(t)| \end{aligned}$$

but, $|y_{i1}(t)| \leq \infty$ and as $m \to \infty$ then, $\|S_{in} - S_{im}\| \to 0$ and hence, $\{S_{in}\}$ is a Cauchy sequence in this Banach space so, the series $\sum_{k=0}^{\infty} y_{ik}(t)$ converges and the proof is complete. ■

C. Error Analysis

For ADM, we can estimate the maximum absolute truncated error of the Adomian's series solution in the following theorem.
Theorem 3 *The maximum absolute truncation error of the series solution (10) to the system (1)-(2) is estimated to be,*

$$\max_{t\in J} \left| y_i(t) - \sum_{k=0}^{m} y_{ik}(t) \right| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_{i1}(t)|.$$

Proof From Theorem 2 we have,

$$\|S_{in} - S_{im}\| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_{i1}(t)|.$$

But, $S_{in} = \sum_{k=0}^{n} y_{ik}(t)$ as $n \to \infty$ then, $S_{in} \to y_i(t)$ so,

$$\|y_i(t) - S_{im}\| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_{i1}(t)|.$$

So, the maximum absolute truncation error in the interval J is,

$$\max_{t\in J} \left| y_i(t) - \sum_{k=0}^{m} y_{ik}(t) \right| \leq \frac{\alpha^m}{1-\alpha} \max_{t\in J} |y_{i1}(t)|$$

and this completes the proof. ■

4. Numerical Examples

Example 1 Consider the following nonlinear system of FDEs,

$$\begin{aligned} \boldsymbol{D}^{0.5} y_1 &= \Gamma(1.5) + y_2^2 - t^4, \\ \boldsymbol{D}^{1.5} y_2 &= \frac{2}{\Gamma(1.5)} y_1 + y_1^4 - t^2, \\ \boldsymbol{D}^{2.5} y_3 &= \frac{24}{\Gamma(2.5)} y_1^3, \end{aligned} \tag{11}$$

subject to the initial conditions,

$$\boldsymbol{D}^{-0.5} y_1|_{t=0} = 0, \boldsymbol{D}^{1.5-k} y_2|_{t=0} = 0, \quad k = 1,2.$$
$$\boldsymbol{D}^{2.5-m} y_3|_{t=0} = 0, \quad m = 1,2,3.$$

which has the exact solution $y_1(t) = \sqrt{t}$, $y_2(t) = t^2$ and $y_3(t) = t^4$.
Using ADM to system (11) leads to the following scheme,

$$y_{1,0} = t^{1/2} - \frac{24}{\Gamma(5.5)} t^{4.5}, \quad y_{1,j+1} = J^{1/2}(A_{1,j}), \tag{12}$$

$$y_{2,0} = -\frac{2}{\Gamma(4.5)} t^{3.5}, \quad y_{2,j+1} = \frac{2}{\Gamma(1.5)} J^{1.5}(y_{1,j}) + J^{1.5}(A_{2,j}), \tag{13}$$

$$y_{3,0} = 0, \quad y_{3,j+1} = \frac{24}{\Gamma(2.5)} J^{2.5}(A_{3,j}), \tag{14}$$

where $A_{1,j}$, $A_{2,j}$ and $A_{3,j}$ represent the Adomian polynomials of the nonlinear terms y_2^2, y_1^4 and y_1^3 respectively.
Using relations (12)-(14), the first two-terms of the series solution are,

$$\begin{aligned} y_1 &= t^{0.5} - 0.458516 t^{4.5} + 0.0106171 t^{7.5} + \cdots, \\ y_2 &= -0.171943 t^{3.5} + t^2 + 0.171943 t^{3.5} - 0.0752253 t^6 - 0.094092 t^{7.5} \\ &\quad + 0.0334503 t^{11.5} - 0.00647686 t^{15.5} + 0.000523436 t^{19.5} + \cdots, \\ y_3 &= t^4 - 0.177317 t^8 + 0.0269405 t^{12} - 0.00192082 t^{16} + \cdots. \end{aligned} \tag{15}$$

A comparison between ADM and exact solutions of y_1, y_2 and y_3 is given in figures 1.a-1.c ($n = 10$).

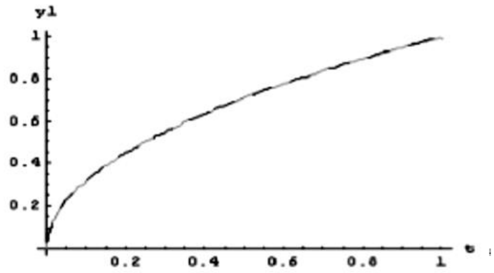

Fig. 1.a: ADM and Exact Sol.

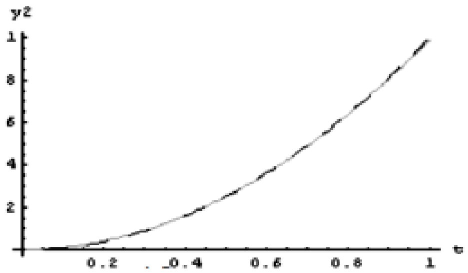

Fig. 1.b: ADM and Exact Sol.

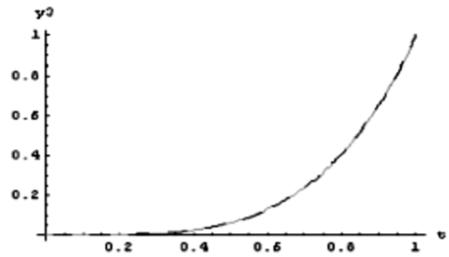

Fig. 1.c: ADM and Exact Sol.

Example 2 Consider the following nonlinear system of FDEs,

$$\begin{aligned} \boldsymbol{D}^{0.5}(\boldsymbol{D}^{0.5}y_1) &= 1 + y_2^3 - t^6, \\ \boldsymbol{D}^{0.5}(\boldsymbol{D}^{0.5}y_2) &= y_1 + t, \\ \boldsymbol{D}^{0.5}(\boldsymbol{D}^{0.5}y_3) &= 3y_1^2, \end{aligned} \tag{16}$$

subject to the initial conditions,

$$\boldsymbol{D}^{-0.5}y_1|_{t=0} = \boldsymbol{D}^{-0.5}y_2|_{t=0} = \boldsymbol{D}^{-0.5}y_3|_{t=0} = 0$$

which has the exact solution $y_1(t) = t,\ y_2(t) = t^2$ and $y_3(t) = t^3$.
Using ADM to system (16) leads to the following scheme,

$$y_{1,0} = t - \frac{t^7}{7}, \quad y_{1,j+1} = J^1(A_{1,j}), \tag{17}$$

$$y_{2,0} = \frac{t^2}{2}, \quad y_{2,j+1} = J^1(y_{1,j}), \tag{18}$$

$$y_{3,0} = 0, \quad y_{3,j+1} = 3J^1(A_{2,j}), \tag{19}$$

where $A_{1,j}$ and $A_{2,j}$ represent the Adomian polynomials of the nonlinear terms y_2^3 and y_1^2 respectively.
Using the relations (17)-(19), the first four-terms of the series solution are,

$$\begin{aligned} y_1 &= t - \frac{t^7}{7} + \frac{t^7}{56} - \frac{3t^7(-52 + t^6)}{2912} + \frac{3t^7(55328 - 1995t^6 + 26t^{12})}{3098368} + \cdots, \\ y_2 &= \frac{t^2}{2} - \frac{1}{56}t^2(-28 + t^6) + \frac{t^8}{448} - \frac{3t^8(-91 + t^6)}{40768} + \cdots, \\ y_3 &= t^3 - \frac{2t^9}{21} + \frac{t^{12}}{245} + \frac{t^9(35 - 3t^6)}{2940} + \frac{t^9(10192 - 973t^6 + 12t^{12})}{285376} \cdots. \end{aligned} \tag{20}$$

A comparison between ADM and exact solutions of y_1, y_2 and y_3 is given in figures 2.a-2.c ($n = 5$).

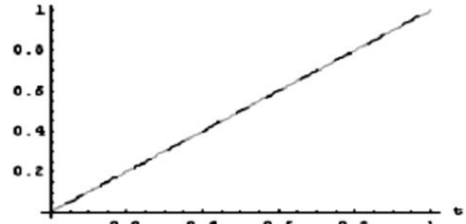

Fig. 2.a:ADM and Exact Sol.

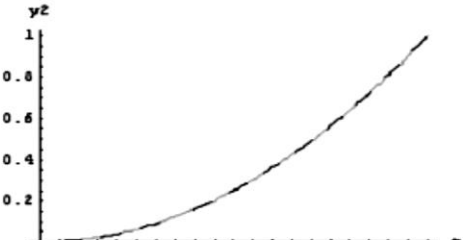

Fig. 2.b:ADM and Exact Sol.

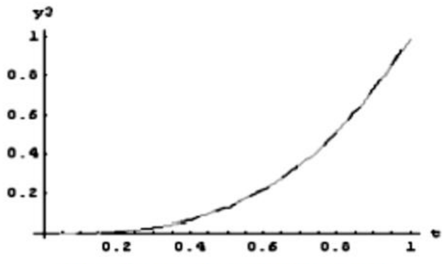

Fig. 2.c:ADM and Exact Sol.

Example 3 Consider the nonlinear system of FDEs,

$$\boldsymbol{D}^{3/2}y_1 = \frac{1}{8}y_2^2 + t,$$
$$\boldsymbol{D}^{3/2}y_2 = \frac{1}{6}y_1^4 + t^2, \qquad 0 < t \le 1, \tag{21}$$

subject to the initial conditions,

$$\boldsymbol{D}^{1.5-k}y_1|_{t=0} = 0, \boldsymbol{D}^{1.5-k}y_2|_{t=0} = 0, \quad k = 1,2.$$

Using ADM to the system (21), we get

$$y_{1,0} = J^{3/2}(t), \quad y_{1,j+1} = \frac{1}{8}J^{3/2}(A_{1,j}),$$
$$y_{2,0} = J^{3/2}(t^2), \quad y_{2,j+1} = \frac{1}{6}J^{3/2}(A_{2,j}), \tag{22}$$

From the relations (22), the first two-terms of the series solution are,

$$y_1 = \left(\frac{8t^{5/2}}{15\sqrt{\pi}}\right) + \left(\frac{1048576t^{17/2}}{1206079875\pi^{3/2}}\right) + \cdots,$$
$$y_2 = \left(\frac{32t^{7/2}}{105\sqrt{\pi}}\right) + \left(\frac{2147483648t^{23/2}}{3388222963125\pi^{5/2}}\right) + \cdots. \tag{23}$$

Figures 3.a and 3.b show ADM solution of y_1 and y_2 $(m = 5)$.

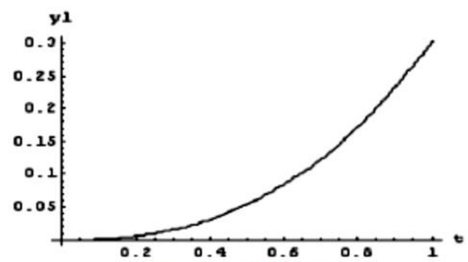

Fig. 3.a: ADM Sol.

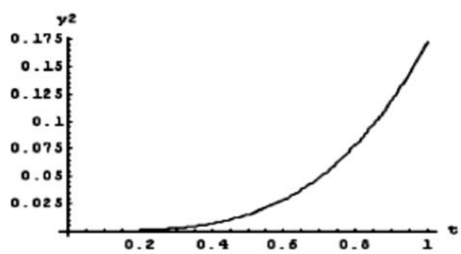

Fig. 3.b: ADM Sol.

Now, we will use Theorem 3 to evaluate the maximum absolute truncated error of the series solution (23).

- L_1: $|f_1(y) - f_1(z)| = |y^2 - z^2| = |y + z||y - z|$
$\le 2|y - z| \quad \Rightarrow \quad L_1 = 2.$
- M_1: $|g_1(\tau)| \le \frac{1}{8} \;\Rightarrow\; M_1 = \frac{1}{8}.$
- L_2: $|f_2(y) - f_2(z)| = |y^4 - z^4| = |y^2 + z^2||y + z||y - z|$
$\le 4|y - z| \quad \Rightarrow \quad L_2 = 4.$
- M_2: $|g_2(\tau)| \le \frac{1}{6} \;\Rightarrow\; M_2 = \frac{1}{6}.$
- β: $\beta = \frac{LMT^{\alpha}}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(5/2)}.$
- $\max_{t\in J} |y_{11}(t)| = \frac{1048576}{1206079875\pi^{3/2}}, \quad \max_{t\in J}|y_{21}(t)| = \frac{2147483648}{3388222963125\pi^{5/2}}.$

The maximum error of y_1: $\max_{t\in J} |y_1(t) - \sum_{k=0}^{m} y_{1k}(t)| \le \frac{\beta^m}{1-\beta} \max_{t\in J} |y_{11}(t)|$,

1. For $m = 5$: $\max_{t\in J} |y_1(t) - \sum_{k=0}^{5} y_{1k}(t)| \le 0.000151813,$
2. For $m = 10$: $\max_{t\in J} |y_1(t) - \sum_{k=0}^{10} y_{1k}(t)| \le 0.0000365703,$

3. For $m = 15$: $\max_{t\in J} \left|y_1(t) - \sum_{k=0}^{15} y_{1k}(t)\right| \le 8.80942 \times 10^{-6}$,
4. For $m = 20$: $\max_{t\in J} \left|y_1(t) - \sum_{k=0}^{20} y_{1k}(t)\right| \le 2.1221 \times 10^{-6}$.

The maximum error of y_2: $t \in J \max \left|y_2(t) - \sum_{k=0}^{m} y_{2k}(t)\right| \le \frac{\beta^m}{1-\beta} \max_{t\in J} \left|y_{21}(t)\right|$,

- For $m = 5$: $\max_{t\in J} \left|y_2(t) - \sum_{k=0}^{5} y_{2k}(t)\right| \le 0.0000352284$,
- For $m = 10$: $\max_{t\in J} \left|y_2(t) - \sum_{k=0}^{10} y_{2k}(t)\right| \le 8.48618 \times 10^{-6}$,
- For $m = 15$: $\max_{t\in J} \left|y_2(t) - \sum_{k=0}^{15} y_{2k}(t)\right| \le 2.04424 \times 10^{-6}$,
- For $m = 20$: $\max_{t\in J} \left|y_2(t) - \sum_{k=0}^{20} y_{2k}(t)\right| \le 4.92437 \times 10^{-7}$.

REFERENCES

[1] K. S. Miller, and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley-Interscience, New York (1993).
[2] I. Podlubny, Fractional Differential Equations, Academic Press, New York (1999).
[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional differential equations, Elsevier, New York (2006).
[4] Sh. A. Abd El-Salam, and A. M. A. El-Sayed, On the stability of some fractional-order non-autonomous systems, Electronic Journal of Qualitative Theory of Differential Equations, 6 (2007) 1-14.
[5] A. M. A. El-Sayed, and Sh.A. Abd El-Salam, On the stability of a fractional-order differential equation with nonlocal initial condition, Electronic Journal of Qualitative Theory of Differential Equations, 29 (2008) 1-8.
[6] D. J. Evans, and K. R. Raslan, The Adomian decomposition method for solving delay differential equation, International Journal of Computer Mathematics, (UK), 82 (2005) 49-54.
[7] D. Zwillinger, Handbook of Differential Equations, Academic Press (1997).
[8] B. Mensour, and A. Longtin, Chaos control in multistable delay-differential equations and their singular limit maps, Physical Review E, 58 (1998) 410-422.
[9] J. M. Hefferan, and R. M. Corless, Solving some delay differential equations with computer algebra, Applied Probability Trust, (2005) 1-22.
[10] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, Comput. and Appl. Math., 23(1) (2004) 33-54.
[11] E. M. El-Mesiry, A. M. A. El-Sayed and H. A. A. El-Saka, Numerical methods for multi-term fractional (arbitrary) orders differential equations, Appl. Math. and Comput., 160(3) (2005) 683-699.
[12] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer (1995).
[13] G. Adomian, Stochastic System, Academic press (1983).
[14] G. Adomian, Nonlinear Stochastic Operator Equations, Academic press, San Diego (1986).
[15] G. Adomian, Nonlinear Stochastic Systems: Theory and Applications to Physics, Kluwer (1989).
[16] K. Abbaoui, and Y. Cherruault, Convergence of Adomian's method applied to differential equations, Computers Math. Applic., 28 (1994) 103-109.
[17] Y. Cherruault, G. Adomian, K. Abbaoui, and R. Rach, Further remarks on convergence of decomposition method, International J. of Bio-Medical Computing., 38 (1995) 89-93.
[18] N. T. Shawaghfeh, Analytical approximate solution for nonlinear fractional differential equations, J. Appl. Math. Comput., 131 (2002) 517-529.
[19] I. L. El-kalla, Convergence of the Adomian method applied to a class of nonlinear integral equations, Applied Mathematics Letters, 21 (2008) 372-376.
[20] S. Momani, and K. Al-Khaled, Numerical solutions for systems of fractional differential equations by the decomposition method, J. Appl. Math. Comput. 162 (2005) 1351-1365.
[21] H. Jafari, and V. Daftardar-Gejji, Solving a system of nonlinear fractional differential equations using Adomian decomposition method, J. Appl. Math. Comput., 196 (2) (2006) 644–651.
[22] E. Ahmed, A. M. A. El-Sayed, and H. A. A. El-Saka Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, J. Math. Anal. Appl., (2007) 542–553.