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Review Article

Euler Line And The Nine-Point Circle

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Abstract *- We will show how the way of defining mathematical concepts significantly affects the possibility of generalising them. The method of a proof provides an opportunity for a better understanding of the nature of problems and their solutions. The paper will present a slightly different way of understanding the tetrahedron orthogonality concept. Various ways and possibilities of generalisation are also considered.*

Keywords - *Euler line, Feuerbach circle, Monge point, Nine-point circle.*

I. INTRODUCTION

In science, there is a constant aspiration for continuous research, not only for new cognition, but also the need to reconsider already existing claims, no matter how long ago they were discovered. The topic of this paper has long been known and loved by many mathematicians. There are thousands of scientific studies written on this topic [21]. In this paper, we will consider two very well-known claims. They are the following two theorems.

Theorem 1. (Euler line). In every triangle: the centre of the circumscribed circle, the centroid and the orthocenter are collinear.

Theorem 2. (Nine-point circle). In each triangle: the midpoints of the edges, the bases of the altitudes and the midpoints between the vertex to the orthocenter lie on the same circle. The centre of that circle is the midpoint between the orthocenter and the centre of the circumscribed circle. The radius of this circle is equal to half of the radius of the circumcircle of the given triangle.

Only a few essentially various proofs can be found in literature. Often a completely different approach can reveal a hidden property that is not visible in the evidence already derived. We will try to look at the problem from as many different angles as possible and provide a few various proofs for this claim, which can serve as an example for others to continue this series. Due to the limited space, we gave a different variant of a similar type of evidence only as a sketch or idea. We will then set up a possible generalization of these claims concerning dimensionality. In that case, we will replace the triangle with the term n-simplex, which for $n = 2$ represents a triangle; $n = 3$ a tetrahedron; and $n = 4$ a four-dimensional pyramid, etc. [1]. Observing the generalization in this way, we will try to generalize some properties spatially.

The main goal of this paper is to contribute to the philosophy of natural sciences and a model of how to interest the reader for further independent work.

II. VARIOUS PROOFS OF THE EULER THEOREM

A) The most substantial and uncomplicated proof of Euler line and the of the nine-point circle can be derived using homothety [13].

Proof of Theorem 1. We observe a homothety with the centre in T and coefficient -1/2, $\mathcal{H}\left(\mathrm{T},-\frac{1}{2}\right)$. With this homothety, the points: A, B and C are mapped to the points: A_1, B_1 and C_1 (Figure 1). The lines AP, BQ and CQ pass into the lines OA_1, OB_1 and OC_1. Therefore, the intersection of these lines, point H, is mapped to point O. And yet, the points H, T and O are collinear and HT = 2TO. □

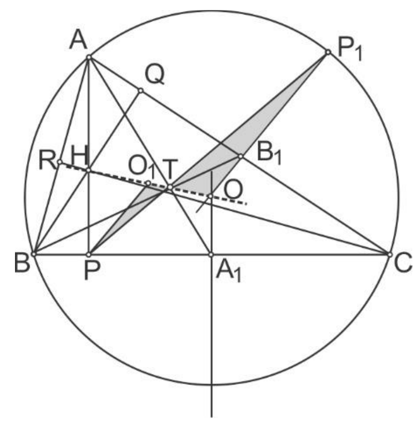

Figure 1.

Proof of Theorem 2. i) Let O_1 be the midpoint of OH. The same homothety maps the point O to O_1. The circle around $\Delta A_1B_1C_1$ is homothetic to the circle around ΔABC and has a radius equal to its half.

ii) Let PT intersect the circumscribed circle in P_1. With homothety $\mathcal{H}\left(\mathrm{T},-\frac{1}{2}\right)$. the point P_1 is mapped to P. Similar for Q and R, because $\Delta TO_1P \sim \Delta TOP_1$ (one angle is equal and the ratio of the overlapping sides is proportional).

iii) Similarly observe $MT \cap \mathcal{K}=N$ and $\Delta O_1MT \sim \Delta$ NOT because of $\mathcal{H}\left(\mathrm{T},-\frac{1}{2}\right)$ and point N passes into M. □

With this proof, Euler's line and the nine-point circle have a simple explanation. Moreover, with this kind of proof, these two claims get their interpretation. We can state that the described circle of a triangle and the circle of nine points are two homothetic circles. By applying homothety, we not only prove, but also discover one of their interpretations, which will not be the case with each of the following proofs.

Remark. We could also observe the homothety $\mathcal{H}(\mathrm{T},-2)$. The direct consequence is to prove that the altitudes of a triangle intersect at one point. There are several different ways to use this idea with homothety in proof. Also, it is possible to observe similar triangles ΔABC and ΔXUV and homothety $\mathcal{H}\left(\mathrm{T},\frac{1}{2}\right)$, where all points of the described circle ΔABC are mapped into points of nine-point circle described around ΔXUV with the centre of homothety in the orthocenter H. Homothety returns the seeming multitude of objects returns to unity, to the monolithic nature of all things, supporting the idea of the Big Bang - the image of the universe is a point. Everything arises from a point and can return to it isomorphically [22].

B) Vector proof of this statement requires knowledge of Hamilton's theorem.

Hamilton's theorem. Let the points A, B and C be the vertices of the triangle ΔABC. If O is the centre of the circumscribed circle, and H is the orthocenter of that triangle, then: $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$.

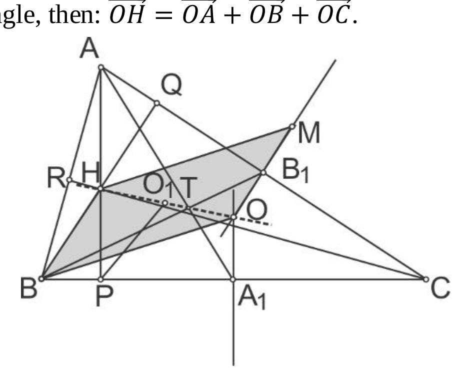

Figure 2.

Proof. Let M be an axisymmetric point of the point O with respect to AC. The quadrilateral □OCMA is a parallelogram. Therefore $\overrightarrow{OC}+\overrightarrow{OA}=\overrightarrow{OM}$.

On the other hand, the quadrilateral □BOMH (Figure 2) is also a parallelogram. To prove this, let U and V be the midpoint of AH and BH, respectively. Now $UV=A_1B_1$, because these are the midlines of the triangles ΔABC and ΔABH on the same base. Therefore, the triangles A_1B_1O and UVH are congruent ($UV = A_1B_1$, and the angles on that base are equal because $AP \parallel OA_1$ and $VH \parallel OB_1$). From this we conclude BH = OM and BH∥OM. So, a quadrilateral □BOMH is a parallelogram. Now $\overrightarrow{OM}+\overrightarrow{OB}=\overrightarrow{OH}$, and, $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OH}$. □

After proving Hamilton's theorem, it is possible to elegantly prove the existence of Euler's line. Let T be the centroid of the triangle ΔABC. Now

$$\left.\begin{array}{l}\overrightarrow{OT}=\overrightarrow{OA}+\overrightarrow{AT}\\ \overrightarrow{OT}=\overrightarrow{OB}+\overrightarrow{BT}\\ \overrightarrow{OT}=\overrightarrow{OC}+\overrightarrow{CT}\end{array}\right\}+$$

$$3\overrightarrow{OT}=\overbrace{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}^{\overrightarrow{OH}}+\overbrace{\overrightarrow{AT}+\overrightarrow{BT}+\overrightarrow{CT}}^{\vec{0}}$$

We conclude, $\overrightarrow{3OT}=\overrightarrow{OH}$. Thus, points O, T, and H are collinear, and point T divide segment OH in a 1: 3 ratio. □

Hamilton's theorem is a physical explanation of the behavior of a system with three material points (triangles) that move away from a common starting point (the center of the circumscribed circle). The result of that system is the vector $\overrightarrow{OH}$ (H orthocenter of the triangle). In the case of an equilateral triangle (which does not have Euler's line!), only the points move away from each other, and the system remains stationary. In the case of other triangles, the whole system moves in the direction of the vector $\overrightarrow{OH}$.

Such a proof has a completely different interpretation and meaning thanks to a different observation and method of proof. It allows for a completely different kind of generalization and understanding of the nature of these claims. So, if the three objects A, B and C that form the vertices of a triangle move in the directions $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, where O is the point from which the vertices are equidistant then the whole system moves along Euler's line make intensity $\overrightarrow{OH}$, where H is the orthocenter of the triangle.

From Figure 2 we find:

i) $\overrightarrow{O_1V}=\frac{1}{2}\overrightarrow{OB}, \overrightarrow{O_1U}=\frac{1}{2}\overrightarrow{OA}\Rightarrow|\overrightarrow{O_1V}|=\frac{1}{2}|\overrightarrow{OB}|=\frac{1}{2}R$

The centers of AH, BH and CH are on a circle whose radius is $\frac{1}{2}R$.

ii) $\overrightarrow{O_1B_1}=\frac{1}{2}\overrightarrow{HM}=\frac{1}{2}\overrightarrow{BO}$ and similarly the other centers of the pages AB, BC and CA are on the same circle.

iii) The altitudes of the triangle intersect the opposite edges at the points P, Q and R. Let us prove the following lemma.

Lemma. The points symmetric to the orthocenter with respect to the edges of the triangle lie on the circumscribed circle.

Proof. If H_1 is the intersection of the altitude AP with the described circle then it is easily proved that ΔBPH1 ≅ΔBPH, which is enough for a proof. Hence $\overrightarrow{OP}=\frac{1}{2}\overrightarrow{OH_1}$ and for other points the procedure is similar. So, all nine points are on one circle (Nine-point circle). The centre is the midpoint of OH, and the radius is half of R.

C) Proofs of theorem 1 using planimetry dominates in older books [17].

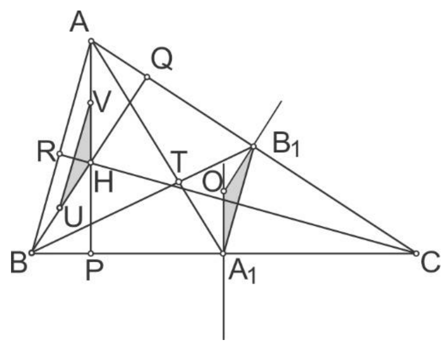

Figure 3.

Let U be the midpoint of BH and V the midpoint of AH, then ΔUHV ≅ΔA_1OB_1 ⇒ AH = $2OA_1$. It follows that ΔAHT ≅ΔA_1TO, (Figure 3) because they have the same angle ∢OA_1T = ∢HAT, and the affected sides are proportional. That is why the other angles are equal and therefore ∢ATH = ∢ A_1TO and that is why the points H, T and O are collinear, and because of the similarity we have HT: TO = 2: 1. □

Proof of the existence of a nine-point circle uses the property of chordal quadrilaterals. (Here, too, there are different variants of the same proof, say that a quadrilateral $\square C_1UXB_1$ is a rectangle, etc.) A quadrilateral is chordal if its opposite angles are supplemental. So, we have:

The quadrilateral □$PA_1B_1C_1$ is chordal. The angle ∢$C_1B_1A_1$ coincides with the angle ß. The triangle Δ BC_1P is isosceles, so

∢ BPC_1 = ß. Therefore, ∢P + ∢B = 180 °. In conclusion, all the bases of the heights and the middle of the pages lie on the same circle.

The quadrilateral □$UA_1B_1C_1$ is chordal. The angle □$UA_1B_1C_1$ is a right-angle, $C_1U \parallel AP$ and $C_1B_1 \parallel BC$. Also, ∢UA_1B_1 is a right-angle, $U_1A_1 \parallel CH$ and $A_1B_1 \parallel AB$. In conclusion, the midpoints of the vertex to the orthocenter lie on the same circle that contains the midpoints of the edges. Hence all nine points lie on the same circle. □

D) Something less common in literature can be found using analytical geometry with the help of affixes, i.e., coordinates expressed using complex numbers [13], [20].

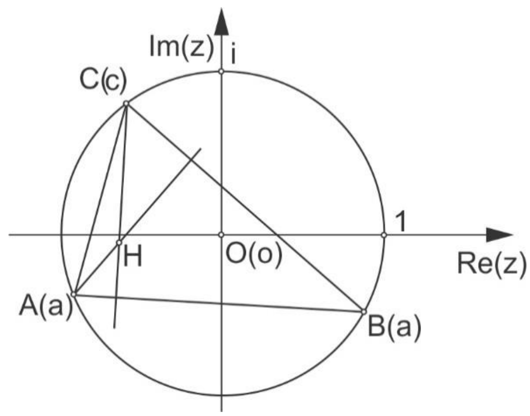

Figure 4.

Consider an arbitrary triangle whose vertices are on a unit circle with center at the coordinate origin (Figure 4). Let the affixes of the vertices of the triangle ΔABC be in the order: a, b and c. It is quite clear that the affix of the center of gravity is $\frac{a+b+c}{3}$. Let's find the orthocenter affix. The direction coefficient of the line BC is $\gamma = \frac{c-b}{\bar{c}-\bar{b}} = -bc$. The coefficient of the height direction from the vertex A is $\gamma_1 = -\gamma = bc$. Completely analogously, we find that the coefficient of the direction of the line AC is equal to -ac, and hence the coefficient of the direction of the height from the vertex B is equal to ac. Now the equations of the line containing the heights respectively from vertices A and B are given by the equations:

$$z - a = bc(\bar{z} - \bar{a}),$$
$$z - b = ac\left(\bar{z} - \bar{b}\right).$$

By eliminating $\bar{z}$ from the previous equations, we get the affix of their intersection, which is the orthocenter whose affix is $a + b + c$. Thus, we find that $\overrightarrow{OH} = 3\overrightarrow{OT}$, and therefore, the points O, T and H are collinear and the point T divides segment OH in the ratio 1: 2. □

In this case, the proof of the existence of the nine-point circle consists in finding the affix of all the necessary points and showing that they are all equally distant from the middle of the longer OH. The affix of the middle of face BC is $\frac{b+c}{2}$, and affixes of other means are easy to find. The affix of the base of the height from vertex A to BC is $\frac{bc}{2a}$, and other affixes would be found similarly. The affixes of the midpoint from the vertex to the orthocenter are: $\frac{2a+b+c}{2}$, $\frac{a+2b+c}{2}$ and $\frac{a+b+2c}{2}$. It is easy to check that all distances to the middle longer than OH are equal to 1/2, therefore, a circle of nine points is a circle whose radius is equal to half the radius of the described circle. □

Note. This approach imposes the following interpretation. An equilateral triangle has no Euler line while all other triangles have it. The vertices of a triangle are obtained from three points of a circle which in a certain position form an equilateral triangle, while in all others they form an irregular one. It is imposed as natural, for the measure of irregularity (deviation from an equilateral triangle) to take the length $d = |OH| = |a + b + c|$. So, the longer the length d, the more irregular the triangle is compared to the equilateral one. When d = 0 the triangle is equilateral (regular).

E) We could not find the proof using analytical geometry in the available literature. And if analytical geometry is often used in planimetric evidence, we are of the opinion that it is only in cases when the proof cannot be more easily implemented in another way. The main feature of analytical geometry is that in the absence of an elementary geometric approach, it offers very diverse algebraic methods for proving any theorem. It is completely justified to observe the vertices of an arbitrary triangle within the Cartesian right-angled system at the points: A (0,0), B (1,0) and C (a, 1). By translation, it is possible to bring each triangle to the position that its base A is in the coordinate origin. Then, by rotating the same triangle, it is always possible to achieve that the side AB coincides with the x axis. By homothety with the center in the coordinate origin and the coefficient $k = \frac{1}{AB}$, it is always possible to achieve the position from the previous position: A (0,0), B (1,0) and C (a, b). And finally, by introducing an additional homothety with the center at the coordinate origin and the coefficient $k_1 = \frac{1}{BC}$, we arrive at a model with vertices: A (0,0), B (1,0) and C (a, 1).

It is very important for analytical geometry to keep the number of parameters as small as possible, because its formulas very quickly form long and opaque expressions. Therefore, instead of the previously proposed model, the literature suggests that each triangle can be observed with vertices at points: A (0,0), B (1,0) and C (0,1) [13].

Determine the coordinates of the relevant points of the triangle ΔABC required for the proof. The coordinates of the center of gravity are determined most simply and are given with $T\left(\frac{1+a}{3}, \frac{1}{3}\right)$. The coordinates of the center of the circumscribed circle are obtained using the bisectors of the sides:

$$s_{AB}: x^2 + y^2 = (x-1)^2 + y^2;\ x = \frac{1}{2}$$

$$s_{AC}: (x-a)^2 + (y-1)^2 = x^2 + y^2;\ 2y + 2xa - a^2 - 1 = 0.$$

Thus, we find that the coordinates of the center of the circumscribed circle are $O\left(\frac{1}{2}, \frac{a^2+1-a}{2}\right)$. The coordinates of the orthocenter are found as the intersection of the heights

$$h_c: x = a$$

$$\text{BC}: y = \frac{1}{a-1}(x-1);\ h_a: y = (1-a)x$$

We find that $(a, a(1-a))$. Now the proof can be continued in very colourful ways. We will list a few possibilities.

E1) If the area of the triangle ΔOTH is equal to zero then the points are collinear. Indeed,

$$P(OTH) = \frac{1}{2}\begin{vmatrix} \frac{1+a}{3} & \frac{1}{3} & 1 \\ a & a(1-a) & 1 \\ \frac{1}{2} & \frac{a^2+1-a}{2} & 1 \end{vmatrix} = 0$$

E2) If the sum of the segments OT and TH is equal to OH then the points are collinear. It is easily checked that: $d(OT) + d(TH) = d(OH)$.

E3) Collinearity can also be determined using the surfaces of rectangular trapezoids, whose area is simply calculated, i.e.. determining that:

$$P(H_1T'TH) + P(T'A_1OT) = P(H_1A_1OH)$$

E4) The points O, T and H are collinear if the coefficients of the lines OT and OH are equal. This is easy to determine from the given coordinates, etc.

For the existence of a nine-point circle, the equation of a circle is used in a similar way to the previous approach with affixes.

Note. Today's mathematicians rarely use Descartes' analytical geometry. Instead of Cartesian coordinates, they use: trilinear and barycentric coordinates. Let the reference triangle ΔABC be given, with sides of lengths a, b, c. Each point of the plane of the triangle ΔABC has homogeneous trilinear coordinates, or simply trilinears of the form $x' : y' : z'$. For this it is necessary to determine the nonzero function $x\ (a, b, c)$ such that: $x = hx'$, $y = hy'$, $x = hz'$, where x', y', z' the distances of that point from the sides of the triangle. Similarly, barycentric coordinates use the surfaces of triangles that form a point with vertices as a measure. Each significant point of the triangle is given a unique notation $X(n)$. Thus, the center of gravity has the designation $X(2)$ and trilinear coordinates $\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$. $X(3)$ is the center of the circumscribed circle and has trilinear coordinates $cosA: cosB: cosC$. $X(4)$ is an orthocenter with coordinates $secA$: sec$B$: $secC$. $X(5)$ is the center of a nine-point circle and has the coordinates $cos(B-C)$: $cos(C-A)$: cos$(A-B)$, etc. In this way, more than 5,000 significant points of the triangle are described in great detail! Together with them, all their known features are systematized. Among other points are: collinear, harmonic, on the same circle, ... [23].

F) Proof of theorem 1 using trigonometry.

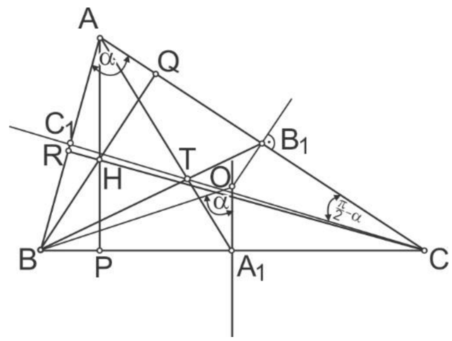

Figure 5.

∢ AHQ= γ, ∢ QHC= α, ∢ A1BO= α – (central angle for α).

From the triangle ΔAHC

$$\frac{AH}{\sin\left(\frac{\pi}{2}-\alpha\right)} = \frac{b}{\sin(\alpha+\gamma)} = \frac{b}{sin(\pi-\beta)} = \frac{b}{\sin\beta} = 2R.$$

We conclude that $AH = 2R\cos\alpha$. (Figure 5)

From the triangle ΔBA1O is: $0A_1 = R\cos\alpha$. It follows that AH = 2 · A_1 O.

According to the cosine theorem applied to the triangles ΔAHT and ΔA_1OT we conclude

$$HT^2 = AH^2 + \left(\frac{2}{3}t_a\right)^2 - 2\cdot AH\cdot\frac{2}{3}t_a\cdot\cos\varphi$$

$$OT^2 = {OA_1}^2 + \left(\frac{1}{3}t_a\right)^2 - 2\cdot OA_1\cdot\frac{1}{3}t_a\cdot\cos\varphi$$

and therefore HT = 2 · OT. By the same theorem, we conclude that ∢ ATH = ∢ A_1TO, so the points H, T and O are collinear.

Points A_1, B_1 and C_1 belong to one circle. The angle over the chord A_1B_1 of that circle is $\sin\gamma = \frac{h_a}{b}$. From the triangle Δ$PA_1B_1$ is with $∢ A_1PB_1 = \frac{\frac{h_a}{2}}{PB_1} = \frac{h_a}{b}$. Thus, the point P belongs to that circle, and similarly to Q and R.

If X is the midpoint of CH, then $\cos∢ A_1XB_1 = \cdots = \cos\gamma$ and the point X belongs to the circle, similarly to the other points.

Dimensional generalization of the previous theorem. The possible generalisation of these theorems can be observed as varied. For example, whether these statements are valid in some other geometry [5], [11] or whether there is another statement such that this is its special case [8], or in the direction to the attempt to generalise dimensionally [15], [16]. Even when we determine the direction of generalisation, there are always more possibilities. It is possible to generalise these statements when the vertices of a triangle are in a space whose dimension is arbitrary [12], but also when the term triangle is replaced by the term n-simplex [1], [2]. Here we will deal with generalisation when a triangle is replaced by an n-simplex. The spatial counterpart of a triangle is a tetrahedron (3-simplex). A tetrahedron does not have all the properties of a triangle, while it has many properties that a triangle does not have. For example, not every tetrahedron has an orthocenter. Tetrahedrons that contain an orthocenter are called orthogonal. There are a number of definitions corresponding to orthogonal tetrahedron [3]. It seems to us the most natural one is to say that a tetrahedron is orthogonal if at least one of its themes is projected into the orthocenter of the opposite side (Figure 6). Then this is the case with all other vertices and the orthocenter exists in the tetrahedron as a single point. Such a definition cannot be found in literature. We consider it the most natural. It shows how the notion of a triangle is dimensionally generalised. Imagine a plane containing the vertices of a triangle in three-dimensional space and a straight line outside it perpendicular to that plane passing through the orthocenter of the triangle. If we take the next point required for a tetrahedron on that normal, we get an orthogonal tetrahedron, and if it is outside that normal, then we get a tetrahedron that is not orthogonal [15].

What about the Euler line in the tetrahedron? It is determined with two points, because the centroid and the center of the described sphere exist in each tetrahedron. It is in this case that we can see how important the way an object is defined is. We have seen that in a triangle the orthocenter is a point homothetic to the center of the circumscribed circle with the centroid and the coefficient $-\frac{1}{2}$. If we similarly define a point homothetic to the center of the described sphere in relation to the centroid with coefficient -1, we get a point known as the Monge point. This point coincides with the orthocenter of the tetrahedron when it is orthogonal, but the Monge point also exists in every other tetrahedron and in that case could be called a pseudo-orthocenter [2]. The Monge point is usually defined as the intersection of six planes, each of which contains the midpoint of an edge and is perpendicular to the opposite edge. With this definition, at least six pages of text are needed to prove that they all have one point in common [6].

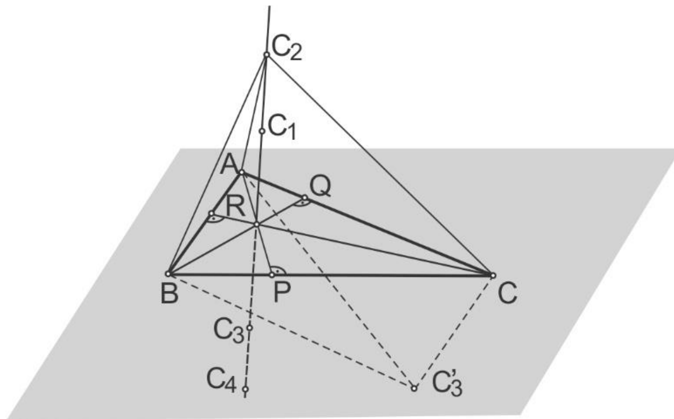

Figure 6.

The nine-point circle of the triangle touches the inscribed and all externally inscribed circles of the triangle. When stating this property, this circle is more often called Feuerbach's. The nine-point circle is called both Euler's and Feuerbach's circle. Different terms of the same object derive from various definitions. The Feuerbach circle is defined as a circle that touches the inscribed circle and all three externally inscribed circles of the triangle. In a plane they are: a nine-point circle and a Feuerbach circle are the same circles. The justification of different names can be most easily understood as an attempt to generalise the same term, given its definitions, it is not possible to reach similar further objects. The Feuerbach sphere exists in every tetrahedron [16]. The Euler sphere exists in every orthogonal tetrahedron. Then the Euler sphere contains all Euler circles in the triangular faces of the tetrahedron. The radius of this sphere is equal to half the radius of the described tetrahedron sphere. In orthogonal tetrahedra, the Euler sphere contains 27 known points plus 5 Feuerbach points in which that sphere touches: the inscribed and all four externally inscribed spheres. And in all other tetrahedrons, there is a sphere that contains pre-known points but not the same points as in the orthogonal ones. This sphere contains the centroids of the triangular faces and the points on the lines that connect the Monge point with the vertices and divide respective segments in the ratio 1: 2, and its radius is one-third of the radius of the described sphere. Then that sphere is homothetic with described tetrahedron sphere with coefficient $-\frac{1}{3}$ with the centre at the centroid of the tetrahedron. Then the number of points that we know in advance that they contain is only 8 points [15]. These spheres are not equal. It is now justified to call them differently.

Tetrahedron as a 3-D triangle (3-simplex). In three dimensions, one can still perceive spatial relations quite well and derive proofs in an elementary stereometric way. However, more complex claims are often much more convenient to prove using the model of analytical geometry. In this case, it is not justified to take an orthogonal tetrahedron for the initial model: A(0,0,0), B(1,0,0), C(0,1,0) and D(0,0,1). The reason is obvious there is an orthocenter in such a tetrahedron. A model like this does not include non-orthogonal tetrahedrons in which the orthocenter can take the Monge point. (Which represents the pseudo-orthocenter in tetrahedrons that do not have the orthocenter).

The center of the described sphere is in the intersection of six planes, each of which is a symmetrical plane of some of the edges of the tetrahedron. Let T be the centroid of the tetrahedron, then the homothetic image of all six symmetrical planes of the edges $\mathcal{H}(\mathrm{T},-1)$ represent six planes that have a Monge point in cross section.

For a non-orthogonal tetrahedron, the following starting tetrahedron in 3-D should be taken: A (0,0,0), B (1,0,0), C (0,1,0) and D (p, q, 1), (Figure 7) with a similar explanation as in the two-dimensional case. Here, even if the base is a right triangle, the case of non-orthogonal tetrahedron is not eliminated.

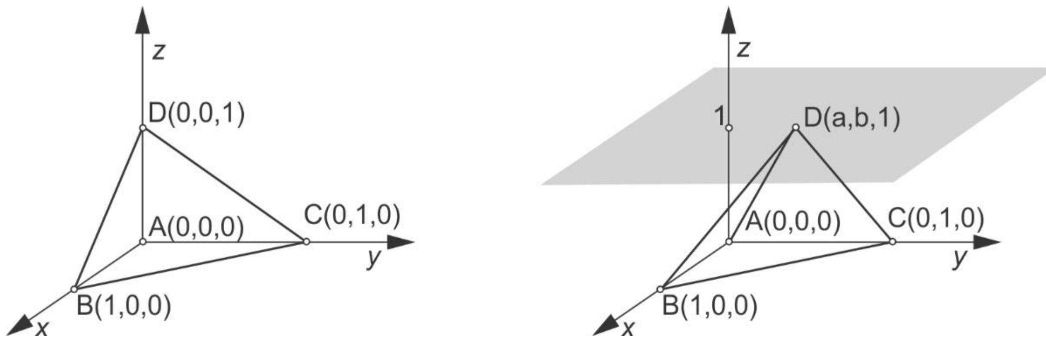

Figure 7.

For orthogonal tetrahedron, vertices can be given at points: A (0,0,0), B (1,0,0), C (0,1,0) and D (0,0,1). The coordinates of the center of the described sphere are: $\mathrm{O}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, centroid $\mathrm{T}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ and orthocenter H (0,0,0) = M. Then, Euler's = Feuerbach's sphere has the equation: $\left(x-\frac{1}{4}\right)^2+\left(y-\frac{1}{4}\right)^2+\left(z-\frac{1}{4}\right)^2=\frac{3}{16}$.

In the non-orthogonal tetrahedron is: $\mathrm{O}\left(\frac{1}{2},\frac{1}{2},\frac{\left(p-\frac{1}{2}\right)^2+\left(q-\frac{1}{2}\right)^2+\frac{1}{2}}{2}\right)$, $\mathrm{T}\left(\frac{1+p}{4},\frac{1+q}{4},\frac{1}{4}\right)$ and Monge's point is $\mathrm{M}\left(\frac{p}{2},\frac{q}{2},\frac{\frac{1}{2}-\left(p-\frac{1}{2}\right)^2-\left(q-\frac{1}{2}\right)^2}{2}\right)$. Points O, T and M define Euler's line. Then Feuerbach's sphere has the equation: $\left(x-\frac{1+p}{4}\right)^2+\left(y-\frac{1+q}{4}\right)^2+\left(z-\frac{1}{4}\right)^2=\frac{1}{2}+\left(\frac{\left(p-\frac{1}{2}\right)^2+\left(q-\frac{1}{2}\right)^2+\frac{1}{2}}{2}\right)^2$, in this case there is no Euler sphere.

4-D triangle, 4-simplex. In this case, too, it is clear that a distinction should still be made between orthogonal and non-orthogonal 4-simplexes. For orthogonal models one should take: A (0,0,0,0) B (1,0,0,0), C (0,1,0,0), D (0,0,1,0) and E (0,0,0,1). In the second case, the base may be an orthogonal tetrahedron, but the fifth theme should spoil that property. In this case 4-simplex is: A (0,0,0,0), B (1,0,0,0), C (0,1,0,0), D (0,0,1,0) and E (a, b, c, 1).

In the first case, the coordinates of significant points are given by:

$$\mathrm{O}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\ \mathrm{T}\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right),\ \mathrm{H}(0,0,0,0)=\mathrm{M}.$$

in the second case, the coordinates of the center of gravity are: $\mathrm{T}\left(\frac{1+a}{5},\frac{1+b}{5},\frac{1+c}{5},\frac{1}{5}\right)$ and the Monge point is obtained with $\mathcal{H}\left(\mathrm{T},-\frac{2}{3}\right)$ in relation to point O.

The Monge point in the n-simplex is obtained by the homothety $\mathcal{H}\left(\mathrm{T},-\frac{2}{n-1}\right)$ from the center of the described hypersphere [13].

In the general case, we have two types of hyperspheres for orthogonal and non-orthogonal simplexes. The Feuerbach hypersphere coincides with Euler's only in the case of orthogonal tetrahedron. In the second case, it is only the Feuerbach $2(n+1)$ hypersphere, which contains all the centroids of the faces of the simplex by a dimension smaller than it and all the points that connect the Monge point with the vertices and divide them in the ratio 1: n [11]. When the points of projection on the opposite (n-1) sides are added, that number becomes $3(n+1)$ [2]. Then this hypersphere is homothetic to that described with respect to the centroid of the simplex with coefficient $-\frac{1}{n}$.

III. CONCLUSION

We see that the possibility of generalising some concepts very much depends on the way we define them. In addition, the method of proving sometimes better reveals the true nature of the claim. In the case of these claims, the homothety had a somewhat better connection with the essence of the claim and very clearly indicated the way of further generalisation. The proper definition is crucial for the introduction of new concepts into science. This paper shows why it is not possible to discuss anything if the topic of discussion at the beginning is not precisely defined.

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