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Original Article

Solution of One – Dimensional Ground Water Recharge Through Porous Media Via Reduced Differential Transform Method

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Abstract *- This paper contains a problem of one-dimensional ground water recharge by spreading through porous media. The mathematical formulation is obtained with variable permeability and constant average coefficient of diffusivity over the entire range of moisture content. Reduced differential transform method (RDTM) is applied to obtain solutions of governing equation. Graphical and numerical representation of the variations in moisture content of soil with the depth and time have been discussed. For validation of the method, results obtained by RDTM are compared with numerical solutions. This research also emphasizes the applicability of RDTM for solving non-linear partial differential equation.*

Keywords — *Ground water,Moisture content, Porous media, Reduced Differential Transform Method*

MSC 2010: 76S05, 35A22

I. INTRODUCTION

The unstable and unsaturated flow of water through the soil occurs due to changes in content over time, and the entire pore area is not respectively filled with flowing liquid. These flows encounter infiltration systems and the treatment of underground sewage and wastewater, which are expressed by nonlinear partial differential equations. In the dry soil, in the unsaturated porous medium, the value of moisture content is considered as 0 as there is no moisture while it is considered as 1 for fully saturated porous medium by water. Therefore, here range of the moisture content is considered as [0, 1]. Many researchers analyzed this problem using different techniques like Finite element method, Adomian decomposition method**,** Optimal homotopy analysis method [8-16]. Verma obtained similarity solution using Laplace transformation [3,4]. Patel [19] used perturbation technique. Groundwater recharge occurs in such a geologically large basin, whose sides are restricted by rigid boundaries, so thick groundwater levels restrict the bottom. In this case, it is assumed that the flow is vertically downwards through unsaturated porous media [5]. Coefficient of diffusivity is assumed to be equivalent to its average value over the whole range of moisture content (small enough constant), and the permeability of the media is assumed to be a variable form of moisture content. Medium is considered with only continuous flowing water and empty voids with no air resistance to the flow. The soil properties and the moisture content at the soil surface is considered to be constant. To handle with non-linear partial differential equations is very difficult. Recently, many researchers have applied various techniques like Differential transform method [20], Reduced Differential transform method [11,17,18], Homotopy perturabation method [2] to solve non-linear partial differential equations. The aim the proposed work is to apply RDTM to solve a governing equation which is a nonlinear partial differential equation. To check the effectiveness of the proposed method, obtained values of moisture contents by RDTM are compared with the numerical results.

II. MATHEMATICAL FORMULATION

For an unsaturated medium, the continuity equation is given by [1]

$$\frac{\partial}{\partial t}(\rho_s \theta) = -\nabla \cdot M \tag{1}$$

where $M = \rho \vec{V}$

$\vec{V}$, the motion of water in a porous medium is defined using Darcy's law as,

$$\vec{V} = -k\nabla\varphi \tag{2}$$

From equations (1) and (2) we obtain,

$$\frac{\partial}{\partial t}(\rho_s \theta) = \nabla\cdot(\rho k \nabla \phi) \tag{3}$$

For the vertical downwards flow, the equation (3) reduces to,

$$\rho_s \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(\rho k \frac{\partial\psi}{\partial z}\right) - \frac{\partial}{\partial z}(\rho k g) \tag{4}$$

where $\phi = \psi - gz$ [3] and the upward direction of the Z-axis and gravity are equal.

Here θ and ψ are assumed to be connected by a one valued function, hence (4) reduces to,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right) - \frac{\rho g}{\rho_s}\left(\frac{\partial k}{\partial z}\right) \tag{5}$$

where $D = \frac{\rho}{\rho_s} k \frac{\partial\psi}{\partial\theta}$

Assume $k = (1-b\theta)^{-2}$, where $b > 0$ is any constant and replace D by average value D_a, from equation (5) we have,

$$\frac{\partial\theta}{\partial t} = D_a \frac{\partial^2\theta}{\partial z^2} - \frac{\rho g}{\rho_s}\frac{\partial}{\partial z}\left(1 + 2b\theta + 3b^2\theta^2\right) \tag{6}$$

(Neglecting third and higher order of θ)

Let $\alpha = \frac{2b\rho g}{\rho_s}$ and $\beta = \frac{3b^2\rho g}{\rho_s}$

$$\frac{\partial\theta}{\partial t} = D_a \frac{\partial^2\theta}{\partial z^2} - \alpha\frac{\partial\theta}{\partial z} - 2\beta\theta\frac{\partial\theta}{\partial z} \tag{7}$$

We choose dimensionless variables

$$\xi = \frac{z}{L},\ T = \frac{tD_a}{L^2},\ \alpha = \frac{D_a}{L},\ \beta = \frac{D_a}{2L} \tag{8}$$

Hence equation (7) becomes

$$\frac{\partial\theta}{\partial T} = \frac{\partial^2\theta}{\partial\xi^2} - \frac{\partial\theta}{\partial\xi} - \theta\frac{\partial\theta}{\partial\xi} \tag{9}$$

where ξ and T are dimensionless variables of penetration depth and time respectively.

At the top, (i.e. depth $z = 0$), in the unsaturated porous media, the moisture content, is chosen very small and at the bottom it is considered fully saturated. Also according to the physical behavior of the longitudinal dispersion phenomenon, the initial condition has been chosen as a negative exponential function of space variable-ξ.

Hence a set of appropriate initial and boundary conditions for equation (8) are considered as [6]

$$\begin{aligned}\theta(\xi,0) &= e^{-\xi}, 0 \le \xi \le 1\\ \theta(0,T) &= 1,\ 0.001 \le T \le 0.01\end{aligned} \tag{10}$$

III. REDUCED DIFFERENTIAL TRANSFORM METHOD

Definition : The function $f(\xi,T)$ which is analytical and differentiated continuously with respect to time T and space ξ in the domain of interest, is defined as

$$F_m(x) = \frac{1}{m!}\left[\frac{\partial^m}{\partial T^m} f(x,T)\right]_{T=0} \tag{11}$$

where T is the dimension spectrum function, $F_m(x)$ is the transformed function and $f(x,T)$ represents the original function.

The inverse transform of $F_m(x)$ as follows:

$$f(x,T) = \sum_{m=0}^{\infty} F_m(x) T^m \tag{12}$$

From equations (11) and (12), we write

$$f(x,T) = \sum_{m=0}^{\infty} \frac{1}{m!}\left[\frac{\partial^m}{\partial T^m} f(x,T)\right]_{t=0} T^m \tag{13}$$

Equation (13) represents the power series expansion.

TABLE-I: PROPERTIES OF RDTM

Functional form	Transform form
$\theta(\xi,T)$	$\Theta_m(\xi) = \frac{1}{m!}\left[\frac{\partial^m}{\partial T^m}\theta(\xi,T)\right]_{T=0}$
$\omega(\xi,T) = \theta(\xi,T) \pm \phi(\xi,T)$	$W_m(x) = \Theta_m(x) \pm \Phi_m(x)$
$\omega(\xi,T) = \alpha\theta(\xi,T)$	$W_m(x) = \alpha\Theta_m(x)$, α is a constant
$\omega(\xi,T) = \xi^r T^s$	$W_m(x) = x^r \delta(m-s)$, where $\delta(m) = \begin{cases} 1, m = 0 \\ 0, m \neq 0 \end{cases}$
$\omega(\xi,T) = \theta(\xi,T)\phi(\xi,T)$	$W_m(x) = \sum_{r=0}^{m} \Phi_r(x)\Theta_{m-r}(x)$
$\omega(\xi,T) = \frac{\partial^m}{\partial T^m}\theta(\xi,T)$	$W_m(x) = \frac{(m+r)!}{m!}\Theta_{m+r}(x)$
$\omega(\xi,T) = \frac{\partial}{\partial \xi}\theta(\xi,T)$	$W_m(x) = \frac{\partial}{\partial x}\Theta_m(x)$
$\omega(\xi,T) = \xi^r T^s \theta(\xi,T)$	$W_m(x) = x^r \Theta_{m-s}$

IV. SOLUTION PROCEDURE

Now applying the Reduced differential transform method [7,21-24] on equation (9), we get

$$\Theta_{m+1}(x) = \frac{1}{(m+1)}\left[\frac{\partial^2}{\partial x^2}\Theta_m(x) - \frac{\partial}{\partial x}\Theta_m(x) - \sum_{r=0}^{m}\Theta_r(x)\Theta_{m-r}(x)\right] \tag{14}$$

From, the initial condition equation (9), we write

$$\Theta_0(x) = e^{-x} \tag{15}$$

Substituting equation (15) into equation (14), we obtain the following $\Theta_m(x)$ values successively,

$$
\begin{aligned}
\Theta_1(\xi) &= 2e^{-\xi} + e^{-2\xi} \\
\Theta_2(\xi) &= 2e^{-\xi} + 5e^{-2\xi} + \frac{3}{2}e^{-3\xi} \\
\Theta_3(\xi) &= \frac{4}{3}e^{-\xi} + \frac{38}{3}e^{-2\xi} + 13e^{-3\xi} + \frac{8}{3}e^{-4\xi} \\
&\vdots
\end{aligned}
\tag{16}
$$

and so the approximate solution of equation (8) is

$$\theta(\xi,T) = \sum_{m=0}^{\infty} \Theta_m T^m$$

$$\theta(\xi,T) = e^{-\xi} + \left(2e^{-\xi} + e^{-2\xi}\right)T + \left(2e^{-\xi} + 5e^{-2\xi} + \frac{3}{2}e^{-3\xi}\right)T^2 + \left(\frac{4}{3}e^{-\xi} + \frac{38}{3}e^{-2\xi} + 13e^{-3\xi} + \frac{8}{3}e^{-4\xi}\right)T^3 + \cdots \tag{17}$$

Eq. (17) is the approximate analytical solution of equation (9).

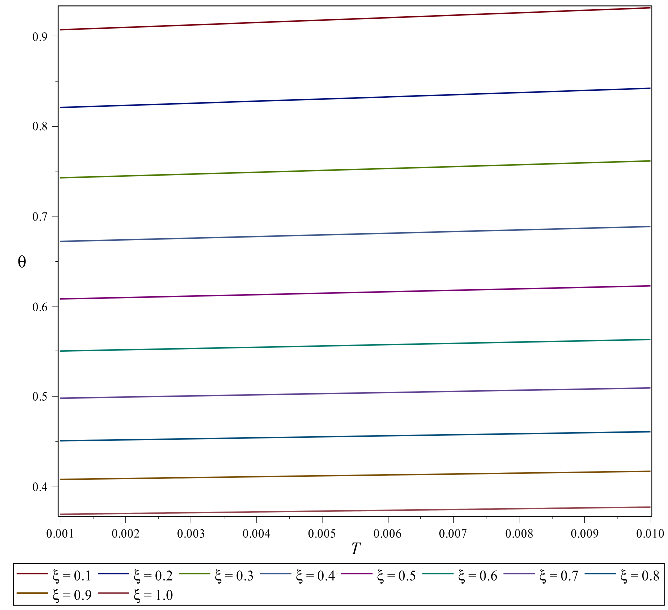

Figure- I: Penetration Moisture (θ) → Time (T)

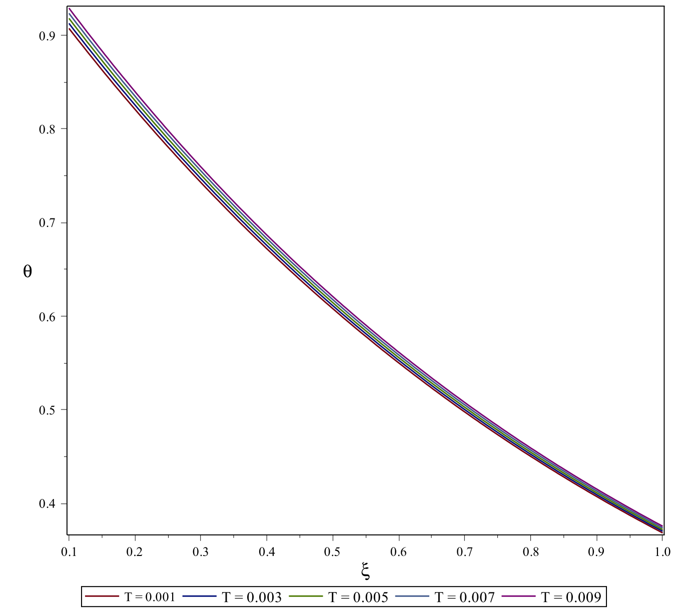

Figure- II: Penetration Moisture (θ) → depth (ξ)

Table II: Moisture content $\theta(\xi,T)$ for the depth ξ at different time levels T by RDTM

RDTM Solution										
T\\ξ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.001	0.90510	0.81896	0.74102	0.67050	0.60669	0.54895	0.49671	0.44944	0.40667	0.36797
0.002	0.90536	0.81919	0.74122	0.67068	0.60685	0.54909	0.49683	0.44955	0.40677	0.36805
0.003	0.90563	0.81942	0.74143	0.67086	0.60701	0.54923	0.49696	0.44966	0.40686	0.36814
0.004	0.90589	0.81965	0.74163	0.67104	0.60716	0.54937	0.49708	0.44977	0.40696	0.36823
0.005	0.90615	0.81989	0.74183	0.67122	0.60732	0.54951	0.49721	0.44988	0.40706	0.36832
0.006	0.90642	0.82012	0.74204	0.67140	0.60748	0.54965	0.49733	0.44999	0.40716	0.36840
0.007	0.90668	0.82035	0.74224	0.67158	0.60764	0.54979	0.49745	0.45010	0.40726	0.36849
0.008	0.90694	0.82058	0.74245	0.67175	0.60780	0.54993	0.49758	0.45021	0.40735	0.36858
0.009	0.90721	0.82081	0.74265	0.67193	0.60796	0.55007	0.49770	0.45032	0.40745	0.36866
0.010	0.90747	0.82104	0.74285	0.67211	0.60811	0.55021	0.49783	0.45043	0.40755	0.36875

Table III: Moisture content $\theta(\xi,T)$ for the depth ξ at different time levels T by Numerical solution

Numerical Solution										
T\ ξ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.001	0.90510	0.81896	0.74102	0.67050	0.60669	0.54895	0.49671	0.44944	0.40667	0.36788
0.002	0.90536	0.81919	0.74122	0.67068	0.60685	0.54909	0.49683	0.44955	0.40677	0.36788
0.003	0.90563	0.81942	0.74143	0.67086	0.60701	0.54923	0.49696	0.44966	0.40686	0.36788
0.004	0.90589	0.81965	0.74163	0.67104	0.60716	0.54937	0.49708	0.44977	0.40696	0.36788
0.005	0.90615	0.81989	0.74183	0.67122	0.60732	0.54951	0.49721	0.44988	0.40706	0.36788
0.006	0.90642	0.82012	0.74204	0.67140	0.60748	0.54965	0.49733	0.44999	0.40716	0.36788
0.007	0.90668	0.82035	0.74224	0.67158	0.60764	0.54979	0.49745	0.45010	0.40725	0.36788
0.008	0.90694	0.82058	0.74245	0.67175	0.60780	0.54993	0.49758	0.45021	0.40735	0.36788
0.009	0.90720	0.82081	0.74265	0.67193	0.60796	0.55007	0.49770	0.45032	0.40745	0.36788
0.010	0.90745	0.82104	0.74285	0.67211	0.60811	0.55021	0.49783	0.45043	0.40755	0.36788

V. RESULTS AND DISCUSSION

It is interpreted from the table-II and III that for a fixed time level, moisture content decreases as the depth increases, in the basin. Also for a fixed depth level, as time increases, moisture content increases. Figure I shows the changes in moisture content obtained for different time and figure II shows that changes in moisture content at different depth level. From both the graphs and tables, it can be easily seen that at specific time, optimum moisture content decreases with increase in depth, and for specific depth, it increases with increases in time, which is physically consistent.

The purpose of the paper is to apply Reduced differential transform method to obtain the solution of the problem of one-dimensional ground water recharge through porous media. For validation of the method, results obtained by RDTM are compared with numerical solutions. From table -II and III, We can observe that the errors are negligible. Therefore, we conclude that the proposed method is very effective.

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NOMENCLATURE

r_s - Bulk density of the medium

M - Mass flux of moisture

k - Co-efficient of aqueous conductivity

∇j - Moisture potential gradient

r - Flux density

$\vec{V}$ - Volume flux of moisture

y - Capillary potential or pressure potential

g - Gravitational constant

D - Diffusivity co-efficient

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