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Original Article

New Group Structure of Compatible Systems of First Order Partial Differential Equations

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Abstract - *Group theory plays a vital role in mathematics, physics, chemistry, and computer science. Group theory has applications in geometry, symmetry and transformation puzzles like Rubik's Cube. Partial differential equations are used in problems involving functions of several variables, such as heat or sound, elasticity, electrodynamics, fluid flow, etc. In this article we have established relation between first order partial differential equations and group theory. If* $g(x,y,z,p,q)$ *is the given first order partial differential equation, the set of all partial differential equations* $f(x,y,z,p,q)$ *which are compatible with* $g(x,y,z,p,q)$ *form group under usual addition of two functions. Furthermore this group forms an Abelian group.*

Keywords — *Group, Structure, Abelian, Compatible, Partial Differential*

Introduction

Group structure is used to creating algorithms for solving the problems of the group analysis of differential equations. Symmetry groups of linear partial differential equations and representation theory used the Laplace and Axially symmetric Wave Equations introduced by Craddock, Mark. (Craddock 2000). A new approach to resolving the problem to group classification of nonlinear partial differential equations given by LAHNO, P. BASARAB–HORWATH (LAHNO 2002).

In this article first we defined collection of all) partial differential equations $f(x,y,z,p,q)$ which are compatible with $g(x,y,z,p,q)$. In next part, by defining trivial operation of function we proved that it form Abelian Group Structure.

Basic Definitions

Group structure:

A non-empty set G with operation $*$ is said to be *group* if it satisfies following four conditions:

a) Closure property hold with respect to $*$ i.e. $x * y$ is in G ,for every $x , y \in$ G
b) Associativity property hold with respect to $*$ i.e.$(x * y) * z = x * (y * z)$ $for\ every\ x , y , z \in$ G
c) Identity element exits in G i.e. there is e in G such that $x * e = e * x = x$ for all x ∈ G
d) Inverse element exits in G i.e. there is x' in G such that $x * x' = x' * x = e$ for all x ∈ G.

Abelian Group: Group G is Abelian group if $x * y = y * x$, for every $x, y \in$ G.

Compatible:

Consider the partial differential equation $f(x, y, z, p, q) = 0$, where $z = z(x,y)$ and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$. The partial differential equations $f(x,y,z,p,q) = 0$ and $g(x,y,z,p,q) = 0$ are said to be *compatible* if they have a common solution.

New Group Structure of Compatible system:

The necessary and sufficient condition that the two partial differential equation $f(x, y, z, p, q) = 0$ and $g(x,y,z,p,q) = 0$ are compatible if $[f,g] = 0$.

Where, $[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$

$$= \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix} + p\begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} + q\begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= (g_p f_x - g_x f_p + \mathrm{p}\, g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q)$$

Result 1:

Consider the set $G = \{ f(x,y,z,p,q) = 0 : [f,g] = 0$, where g is $g(x,y,z,p,q) = 0 \}$

i.e. the set of all P.D.E.'s f which are compatible with g. Then the set G is a group with respect to trivial addition of functions.

Proof:

let $= f(x,y,z,p,q)$, $h = h(x,y,z,p,q) \in G$.

Therefore, $[f,g] = 0$ and $[h,g] = 0$

i.e. $g_p f_x - g_x f_p + \mathrm{p}\, g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q = 0$ and $g_p h_x - g_x h_p + \mathrm{p}\, g_p h_z - p g_z h_p + g_q h_y - g_y h_q + q g_q h_z - q g_z h_q = 0$ ------- (1)

Consider,

$$[f + h, g] = \frac{\partial(f+h,g)}{\partial(x,p)} + p\frac{\partial(f+h,g)}{\partial(z,p)} + \frac{\partial(f+h,g)}{\partial(y,q)} + q\frac{\partial(f+h,g)}{\partial(z,q)}$$

$$= \begin{vmatrix} f_x + h_x & f_p + h_p \\ g_x & g_p \end{vmatrix} + p\begin{vmatrix} f_z + h_z & f_p + h_p \\ g_z & g_p \end{vmatrix} +$$

$$\begin{vmatrix} f_y + h_y & f_q + h_q \\ g_y & g_q \end{vmatrix} + q\begin{vmatrix} f_z + h_z & f_q + h_q \\ g_z & g_q \end{vmatrix}$$

$$= g_p(f_x + h_x) - g_x(f_p + h_p) + \mathrm{p}\,[g_p(f_z + h_z) - g_z(f_p + h_p)] +$$

$g_q(f_y + h_y) - g_y(f_q + h_q) + q[g_q(f_z + h_z) + g_z(f_q + h_q)]$

$= g_p f_x + g_p h_x - g_x f_p - g_x h_p + \text{p } g_p f_z + \text{p } g_p h_z - p g_z f_p - p g_z h_p +$

$g_q f_y + g_q h_y - g_y f_q - g_y h_q + q g_q f_z + q g_q h_z - q g_z f_q - q g_z h_q$

$= (g_p f_x - g_x f_p + \text{p } g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q) +$

$(g_p h_x - g_x h_p + \text{p } g_p h_z - p g_z h_p + g_q h_y - g_y h_q + q g_q h_z - q g_z h_q)$

= 0 + 0 ---------- by (1)

= 0

Therefore, $f + h \in G$ ------- (a)

Now for any $f = f(x, y, z, p, q)$, $h = h(x, y, z, p, q)$ and $k = k(x, y, z, p, q) \in G$.

As, $f + (h + k) = (f + h) + k$ for any $, h, k$.

Therefore, associativity property holds in G. ------- (b)

Now consider, $0 = 0(x, y, z, p, q)$ and

$[0 , g] = \frac{\partial(0,\mathrm{g})}{\partial(\mathrm{x,p})} + p \frac{\partial(0,\mathrm{g})}{\partial(\mathrm{z,p})} + \frac{\partial(0,\mathrm{g})}{\partial(\mathrm{y,q})} + q \frac{\partial(0,\mathrm{g})}{\partial(\mathrm{z,q})}$

$= \begin{vmatrix} 0 & f_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} 0 & f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} 0 & f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} 0 & f_q \\ g_z & g_q \end{vmatrix}$

= 0 + 0 + 0 + 0

= 0

Therefore, $0 \in G$.

Also, $+ 0 = 0 + f = f$, for any $f \in G$.

Hence $e = 0$ is an identity element in G . ------- (c)

Now consider,

$[-f , g] = \frac{\partial(-\mathrm{f,g})}{\partial(\mathrm{x,p})} + p \frac{\partial(-\mathrm{f,g})}{\partial(\mathrm{z,p})} + \frac{\partial(-\mathrm{f,g})}{\partial(\mathrm{y,q})} + q \frac{\partial(-\mathrm{f,g})}{\partial(\mathrm{z,q})}$

$= \begin{vmatrix} -f_x & -f_p \\ g_x & g_p \end{vmatrix} + p \begin{vmatrix} -f_z & -f_p \\ g_z & g_p \end{vmatrix} + \begin{vmatrix} -f_y & -f_q \\ g_y & g_q \end{vmatrix} + q \begin{vmatrix} -f_z & -f_q \\ g_z & g_q \end{vmatrix}$

$= -g_p f_x + g_x f_p - \text{p } g_p f_z + p g_z f_p - g_q f_y + g_y f_q - q g_q f_z + q g_z f_q$

$= -\ (g_p f_x - g_x f_p + \mathrm{p}\, g_p f_z - p g_z f_p + g_q f_y - g_y f_q + q g_q f_z - q g_z f_q)$

$= 0$ --------by (1)

Therefore, $-f \in G.$

As , $f + (-f) = (-f) + f = 0 = e$, for any f

Hence inverse exists for every element in G . -------(d)

From (a), (b), (c) and (d) G is group w.r.t. usual addition of functions.

Result 2: The set G is an Abelian group w.r.t. usual addition of functions.

Proof: For any $f = f(x, y, z, p, q)\ and\ g = g(x, y, z, p, q)$

We have , $f(x, y, z, p, q) + g(x, y, z, p, q) = (f + g)(x, y, z, p, q)$

$= (g + f)(x, y, z, p, q)$

$= g(x, y, z, p, q) + f(x, y, z, p, q)$

Therefore , $f(x, y, z, p, q) + g(x, y, z, p, q) = g(x, y, z, p, q) + f(x, y, z, p, q)$

Hence G is Abelian group with respect to trivial addition of functions.

II. Conclusion and Future Work

In this article new set is defined which contain of all partial differential equations $f(x, y, z, p, q)$ which are compatible with fixed function $g(x, y, z, p, q)$. using trivial addition of functions the given set form a group. Furthermore, this new group structure forms an Abelian group. In Future we want to extend our work whether this set form Ring Structure, Vector Space, Field and Integral Domain etc.

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