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Stability and Hopf Bifurcation of PD-Controlled XCP Network Congestion Model

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Abstract ***—*** *Based on feedback control and bifurcation theory, a PD controller is proposed to solve the Hopf branch problem of XCP network congested power system. Firstly,* α *is selected as the branching parameter to obtain the critical value that keeps the original system and the controlled system stable. When the delay value passes the critical value, the system will lose stability at the equilibrium point and Hopf branch will be generated. Then, the addition of PD controller increases the critical value of system branch parameters, expands the stability region, and effectively alleviates the generation of Hopf branches. Finally, the feasibility of theoretical analysis is verified by numerical simulation with mathematical software.*

Keywords — *Hopf bifurcation, PD controller, Stability, network congestion*

I. INTRODUCTION

In recent years, with the increase of the number and types of wireless network technologies, when the total demand for a resource exceeds the available capacity of the resource, and the data received by routers in the network exceeds the data they can forward, Internet congestion will occur, and even the whole system may crash due to congestion collapse [1-2]. Network congestion control is a very important and challenging problem, which has always been the main subject of in-depth research. The nonlinear dynamic characteristics of congestion control systems urge researchers to use existing bifurcation control methods to improve the performance of related schemes.Subsequently, many papers have studied nonlinear behaviors such as bifurcation and chaos in the network system model [3-5]. Compared with the traditional TCP protocol, congestion control of XCP protocol is excellent in fairness, efficiency and flexibility, and can better adapt to the network environment. In this paper, Hopf bifurcation of XCP network congested power system is delayed by proportional differential controller.

II. MODEL BUILDING

According to literature [6], the congestion model of XCP network is presented:

$$\begin{cases} \dfrac{dy(t)}{dt} = -\dfrac{\alpha}{\tau}(y(t-\tau)-C) - \dfrac{\beta}{\tau^2}q(t-\tau) \\ \dfrac{dq(t)}{dt} = y(t) - C \end{cases} \tag{1}$$

where $y(t)$ represents total traffic, $q(t)$ represents average queue length (packets), C represents link capacity, and τ represents round-trip time (seconds).

First, let the equilibrium point of model (1) be (y_0, q_0), then it satisfies the following equation:

$$\begin{cases} -\dfrac{\alpha}{\tau}(y_0 - C) - \dfrac{\beta}{\tau^2}q_0 = 0 \\ y_0 - C = 0 \end{cases} \tag{2}$$

If we solve the above equation, we get:

$$y_0 = C \;,\;\; q_0 = 0 \;, \tag{3}$$

In recent years, many scholars have studied Hopf bifurcation in XCP network congestion model. In literature [7], the author studied the *Hopf* bifurcation problem after adding a hybrid controller to the wireless network congestion model. In literature [8], the author studied the XCP network congestion model by adding a state feedback controller.

Inspired by the above research, this paper adds proportional differential controller (PD) to the XCP network congestion model in order to delay the generation of *Hopf* branch. According to Equation (1), the controlled system added to PD controller is obtained as follows:

$$\begin{cases} \dfrac{dy(t)}{dt} = -\dfrac{\alpha}{\tau}(y(t-\tau)-C) - \dfrac{\beta}{\tau^2}q(t-\tau) + k_p(y(t)-y_0) + k_d\dfrac{d}{dt}(y(t)-y_0) \\ \dfrac{dq(t)}{dt} = y(t) - C \end{cases}$$

Then the above controlled system can be further rewritten as:

$$\begin{cases} \dfrac{dy(t)}{dt} = \dfrac{1}{1-k_d}[-\dfrac{\alpha}{\tau}(y(t-\tau)-C) - \dfrac{\beta}{\tau^2}q(t-\tau)] + \dfrac{k_p}{1-k_d}(y(t)-y_0) \\ \dfrac{dq(t)}{dt} = y(t) - C \end{cases} \tag{5}$$

III. STABPLPTY AND LOCAL HOPF BIFURCATION ANALYSIS

According to calculation, the equilibrium point of the controlled system (5) is the same as that of the original system (1), which means that the original system structure will not be changed after PD controller is added.

Let $y_1(t) = y(t) - y_0$, and $y_2(t) = q(t) - q_0$. After linearization of the controlled system (5) at the equilibrium point, the linearization equation is:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = \dfrac{k_p}{1-k_d}y_1(t) - \dfrac{1}{1-k_d}\dfrac{\alpha}{\tau}y_1(t-\tau) - \dfrac{1}{1-k_d}\dfrac{\beta}{\tau^2}y_1(t-\tau) \\ \dfrac{dy_2(t)}{dt} = y_1(t) \end{cases} \tag{6}$$

The characteristic equation of linearized equation (6) is:

$$\lambda^2 + a_1\lambda + a_2\lambda e^{-\lambda\tau} + a_3 e^{-\lambda\tau} = 0 \tag{7}$$

Among them:

$$a_1 = -\frac{k_p}{1-k_d} \;,\;\; a_2 = \frac{\alpha}{(1-k_d)\tau} \;,\;\; a_3 = \frac{\beta}{(1-k_d)\tau^2} \;.$$

***Lemma* 1**: When $\alpha > \alpha_c$ and $(H_1): a_3 b_2 < 0$, $a_2 < 0$, $a_2 a_3 b_2 > 0$, the controlled system (5) is locally asymptotically stable, otherwise the controlled system (5) is unstable.

Proof. Using quadratic approximation $e^{-\lambda\tau} = 1 - \lambda d + \dfrac{\lambda^2 d^2}{2}$, the above equation becomes:

$$(\frac{\alpha\tau}{2(1-k_d)})\lambda^3 + (1 - \frac{\alpha}{1-k_d} + \frac{\beta}{2(1-k_d)})\lambda^2 + (\frac{\alpha}{\tau(1-k_d)} - \frac{k_p}{1-k_d} - \frac{\beta}{\tau(1-k_d)}) + \frac{\beta}{\tau^2(1-k_d)} = 0 \;, \tag{8}$$

Routh-Hurwitz stability criterion shows that the closed-loop system is stable if and only if all values are greater than zero, that is, the following coefficient conditions are satisfied:

$$\frac{\alpha\tau}{2(1-k_d)} > 0\ ,\quad 1-\frac{\alpha}{1-k_d}+\frac{\beta}{2(1-k_d)} > 0\ ,\quad \frac{\alpha}{\tau(1-k_d)}-\frac{k_p}{1-k_d}-\frac{\beta}{\tau(1-k_d)} > 0\ ,\quad \frac{\beta}{\tau^2(1-k_d)} > 0\ ,$$

that is $2-2k_d-2\alpha+\beta>0$, $\alpha-k_p\tau-\beta>0$, At this point, the controlled system (5) is stable. The proof is done.

***Lemma* 2**: If $\alpha=\alpha_c$ is true, then the characteristic equation (7) has a pair of pure imaginary roots $\lambda=\pm i\omega_0(\omega_0>0)$, and we take α as the bifurcation parameter of the characteristic equation (7), where

$$\omega_0=\sqrt{\frac{(a_2{}^2-a_1{}^2)+\sqrt{(a_2{}^2-a_1{}^2)^2+4a_3{}^2}}{2}}\ ,$$

$$\alpha_c=\frac{\omega_0{}^2(1-k_d)\tau}{\omega_0\sin\ \omega_0\tau}-\frac{\beta\cos\ \omega_0\tau}{\tau\omega_0\sin\ \omega_0\tau}\ .$$

Proof. First we assume that $\lambda=i\omega_0(\omega>0)$ is a root of the characteristic equation (7), the following equation is satisfied after substituting it into the above characteristic equation:

$$-\omega_0{}^2+i\omega_0a_1+ia_2\omega_0(\cos\ \omega_0\tau-i\sin\ \omega_0\tau)+a_3(\cos\ \omega_0\tau-i\sin\ \omega_0\tau)=0\ , \tag{9}$$

the separation of the real and imaginary parts, it follows:

$$\begin{cases}-\omega_0{}^2+a_2\omega_0\sin\ \omega_0\tau+a_3\cos\ \omega_0\tau=0\\ a_1\omega_0+a_2\omega_0\cos\ \omega_0\tau-a_3\sin\ \omega_0\tau=0\end{cases} \tag{10}$$

add the left and right sides of equation (10) to get:

$$\omega_0{}^4+(a_1{}^2-a_2{}^2)\omega_0{}^2-a_3{}^2=0\ , \tag{11}$$

therefore, we can solve the above equation to get:

$$\omega_0{}^2=\frac{(a_2{}^2-a_1{}^2)\pm\sqrt{(a_2{}^2-a_1{}^2)^2+4a_3{}^2}}{2}\ ,$$

because of $\omega_0>0$, then

$$\omega_0=\sqrt{\frac{(a_2{}^2-a_1{}^2)+\sqrt{(a_2{}^2-a_1{}^2)^2+4a_3{}^2}}{2}}\ ,$$

so equation (10) is also obtained

$$\alpha_c=\frac{\omega_0{}^2(1-k_d)\tau}{\omega_0\sin\ \omega_0\tau}-\frac{\beta\cos\ \omega_0\tau}{\tau\omega_0\sin\ \omega_0\tau}\ .$$

Now we show that $\lambda=\pm i\omega_0$ is a simple root of the characteristic equation (7) if $\alpha=\alpha_c$ is true. First of all define:

$$Q(\lambda,\alpha)=\lambda^2+a_1\lambda+a_2\lambda e^{-\lambda\tau}+a_3e^{-\lambda\tau}\ ,$$

we can get:

$$\frac{dQ(\lambda,\alpha)}{d\lambda}=2\lambda+a_1+a_2e^{-\lambda\tau}-a_2\tau\lambda e^{-\lambda\tau}-a_3\tau e^{-\lambda\tau}\ ,$$

In this case, substitute $\lambda = \pm i\omega_0$ into the equation above to get $\frac{dQ(\lambda,\alpha)}{d\lambda}|_{\lambda=i\omega_0} \neq 0$, and $\frac{dQ(\lambda,\alpha)}{d\lambda}|_{\lambda=-i\omega_0} \neq 0$. So when $\alpha = \alpha_c, \lambda = \pm i\omega_0$ is a pair of pure imaginary roots of the characteristic equation (6). The proof is done.

***Lemma* 3**: If $\lambda(\alpha) = R(\alpha) + i\omega(\alpha)$ is the root of the characteristic equation (7) and the conditions $R(\alpha_c) = 0$ and $\omega(\alpha_c) = \omega_0$ are satisfied, then the transversely condition $\mathrm{Re}(\frac{d\lambda}{d\alpha_c})^{-1}|_{\alpha=\alpha_c} \neq 0$ is true.

Proof. By differentiating both sides of equation (7) with regard to α_c and applying the implicit function theorem, we have :

$$\frac{d\lambda}{d\alpha}|_{\alpha=\alpha_c} = \frac{\lambda}{-2(1-k_d)\tau\lambda e^{\lambda\tau} + k_p\tau e^{\lambda\tau} - \alpha + \alpha\lambda\tau + \beta}|_{\alpha=\alpha_c}$$

$$\mathrm{Re}(\frac{d\lambda}{d\alpha_c})^{-1} = -2(1-k_d)\tau\cos\omega\tau + \frac{k_p\tau\sin\omega\tau}{\omega} + \alpha_c$$

$$[\frac{d\lambda}{d\alpha}]^{-1}|_{\alpha=\alpha_c} = -2(1-k_d)\tau e^{\lambda\tau} + \frac{k_p\tau e^{\lambda\tau}}{\lambda} - \frac{\alpha}{\lambda} + \alpha\tau + \frac{\beta}{\lambda}|_{\alpha=\alpha_c}$$

$$= -2(1-k_d)\tau(\cos\omega\tau + i\sin\omega\tau) + \frac{-ik_p\tau\cos\omega\tau + k_p\tau\sin\omega\tau)}{\omega} - \frac{-i\beta + i\alpha_c}{\omega} + \alpha_c\tau$$

So

$$\mathrm{Re}(\frac{d\lambda}{d\alpha_c})^{-1} = -2(1-k_d)\tau\cos\omega\tau + \frac{k_p\tau\sin\omega\tau}{\omega} + \alpha_c ,$$

It can be known from Equation (10):

$$\mathrm{Re}(\frac{d\lambda}{d\alpha_c})^{-1} = \frac{\tau^4(1-k_d)^2[-2\beta\omega_0^2(1-k_d)^2 - k_p^2\beta - k_p\alpha_c\omega_0^2\tau(1-k_d)]}{\alpha_c^2\tau^2\omega_0^2 + \beta^2} \neq 0 .$$

Then the transversal condition holds. The proof is done.

***Lemma* 4**: When $\alpha > \alpha_c$, equation (7) has at least one root with a positive real part.

According to the above lemma and the *Hopf* branch theorem of delay differential equations in literature [9-10], we can get the following conclusions.

Theorem 1: For the controlled system (5), the following conclusion holds:

(1) When $\alpha < \alpha_c$, the controlled system is asymptotically and uniformly stable near the equilibrium point(y_0, q_0);

(2) When $\alpha = \alpha_c$, the controlled system generates Hopf branch at the equilibrium point (y_0, q_0);

(3) When $\alpha > \alpha_c$, the controlled system is unstable at the equilibrium point (y_0, q_0).

IV. NUMERICAL SIMULATION

In this section, we will verify the validity of the above theoretical analysis results through numerical simulation with mathematical software mathematica. In order to facilitate comparison, we selected the same parameters as those in literature [7]: $\tau = 0.2$, $C = 1000$, $\beta = 0.436$. When $k_d = k_p = 0$, system (5) is in the state of no control system, and $y_0 = 1000$, $p_0 = 0$, $\alpha_{c\min} = 0.555$, $\alpha_{c\max} = 1.274$ is obtained through calculation.

When $\alpha = 0.5 < \alpha_{c\,\min}$ is taken, the system (5) is asymptotically stable at the equilibrium point, as shown in FIG. 1. When $\alpha = 0.555 = \alpha_{c\,\min}$, The controlled system (5) generates a *Hopf* branch at the equilibrium point, as shown in FIG. 2. Next, the control effect is verified. The above parameters are still selected. By selecting an appropriate PD control coefficient of $k_d = -0.1$ and $k_p = -0.1$, when $\alpha = 0.555$, the system finally stabilizes at the equilibrium point, as shown in FIG. 3. However, as α continues to increase, as in $\alpha = 0.504$, the wireless network congestion model with PD controller added still generates *Hopf* branch, and the system loses stability, generating limit cycle, as shown in Figure 4. Therefore, *Hopf* bifurcation can be advanced by selecting an appropriate PD control coefficient.

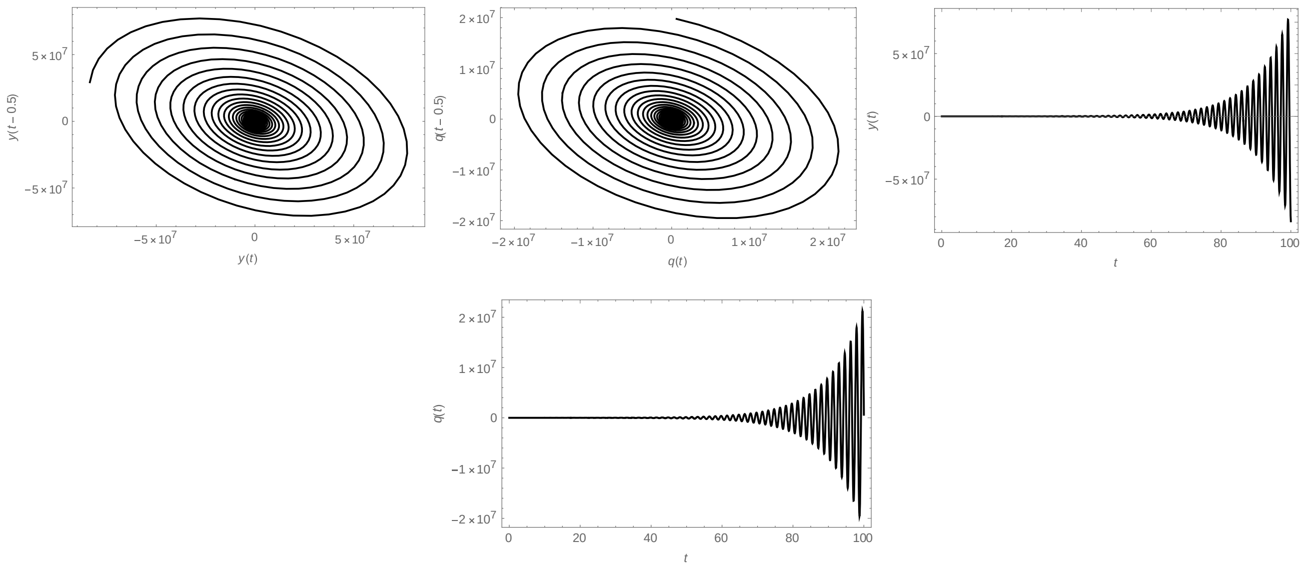


Fig. 1 State and Phase plot of $y(t)$ **and** $p(t)$ **with** $\alpha = 0.5 < \alpha_{c\,\min}$ **.**

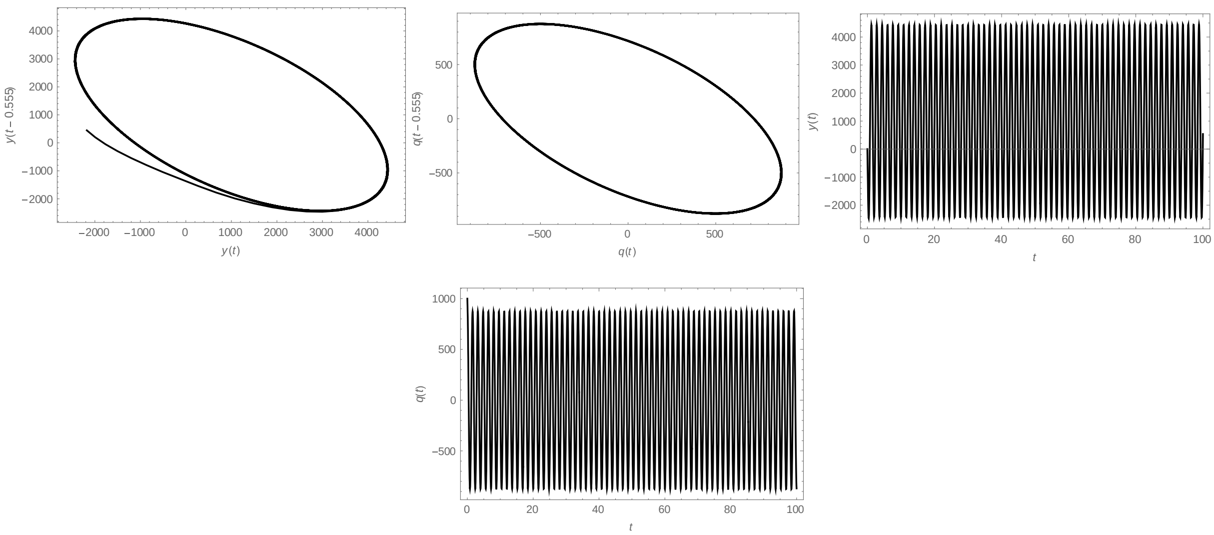


Fig. 2 State and Phase plot of $y(t)$ **and** $p(t)$ **with** $\alpha = 0.555 = \alpha_c$ **.**

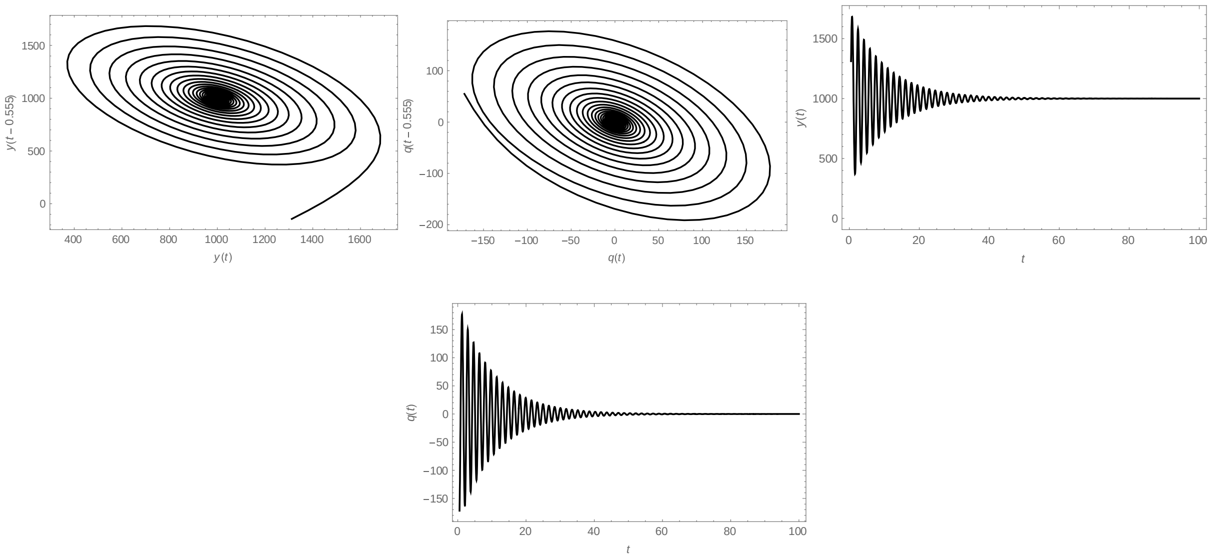


Fig. 3 State and Phase plot of $y(t)$ **and** $p(t)$ **with** $\alpha = 0.555$

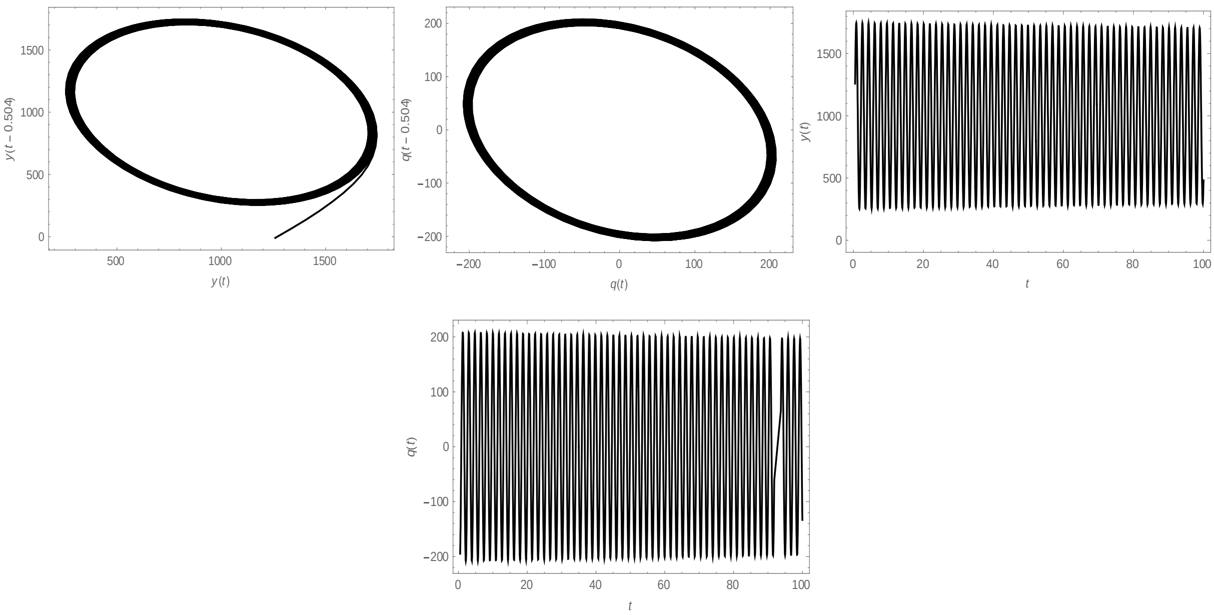
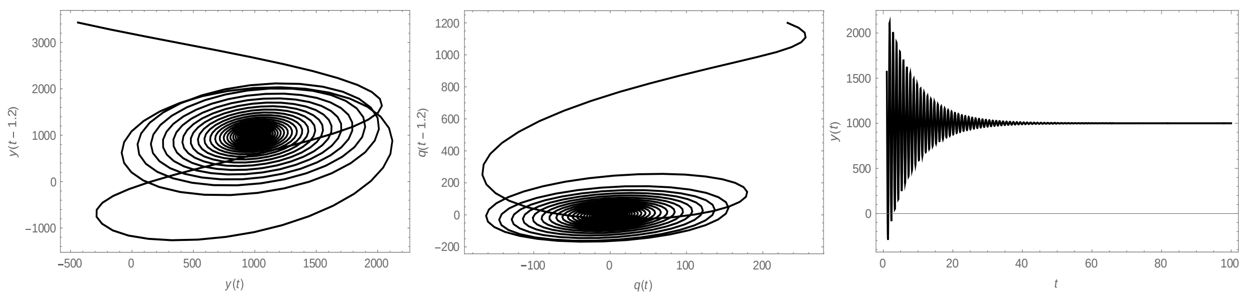


Fig. 4 State and Phase plot of $y(t)$ **and** $p(t)$ **with** $\alpha = 0.504$

Now take $\alpha = 1.2 < \alpha_c$, the controlled system (5) is asymptotically stable at the equilibrium point, as shown in FIG. 5. Now take $\alpha = 1.274 = \alpha_{c\max}$, the controlled system (5) generates *Hopf* branch at the equilibrium point, as shown in FIG. 6. Then verify the control effect. The above parameters are still selected. When $\alpha = 1.274$, the system finally stabilizes at the equilibrium point, as shown in FIG. 7. However, when α continues to increase, such as $\alpha = 1.4755$, the wireless network congestion model with PD controller added still generates *Hopf* branch, and the system loses stability, as shown in FIG. 8. Therefore, choosing an appropriate PD control coefficient can make *Hopf* branch delay.



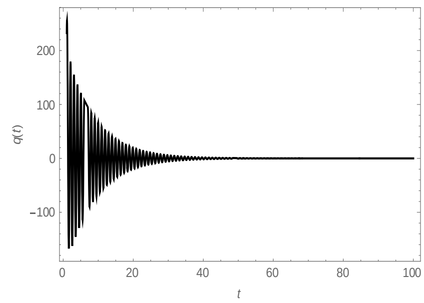


Fig. 5 State and Phase plot of $y(t)$ **and** $\mathrm{p}(t)$ **with** $\alpha = 1.2 < \alpha_{c\max}$

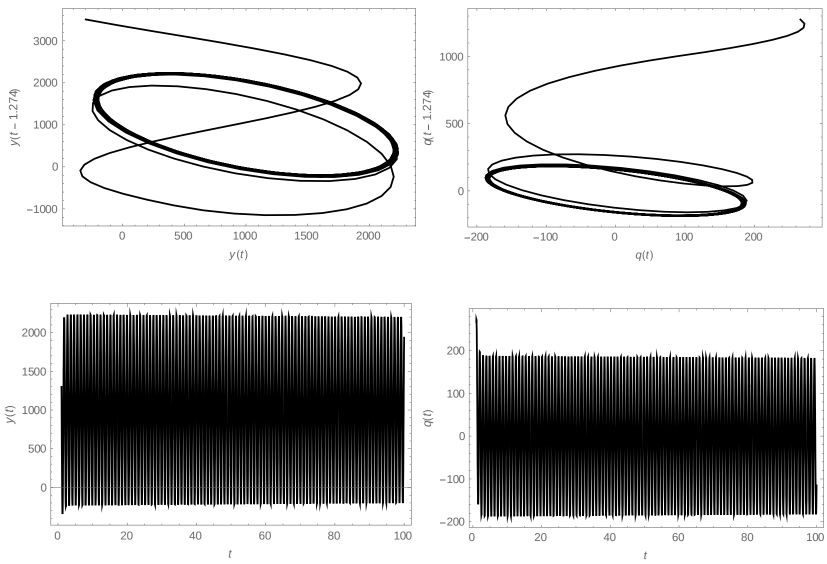


Fig. 6 State and Phase plot of $y(t)$ **and** $\mathrm{p}(t)$ **with** $\alpha = 1.274 = \alpha_{c\max}$

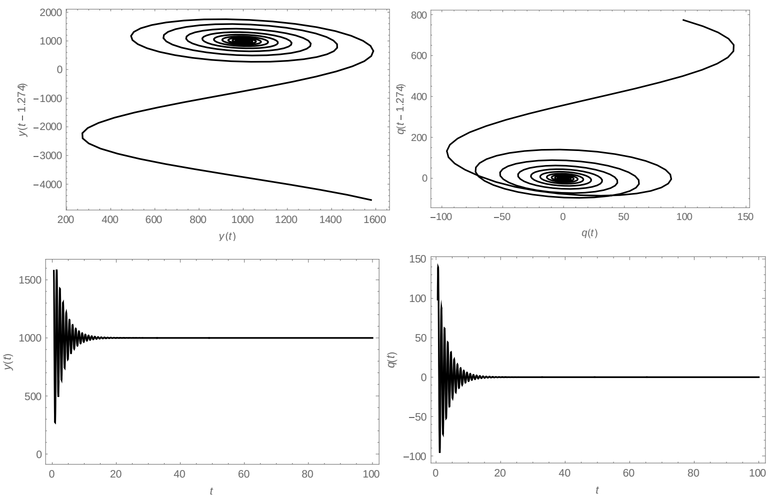


Fig. 7 State and Phase plot of $y(t)$ **and** $\mathrm{p}(t)$ **with** $\alpha = 1.274$

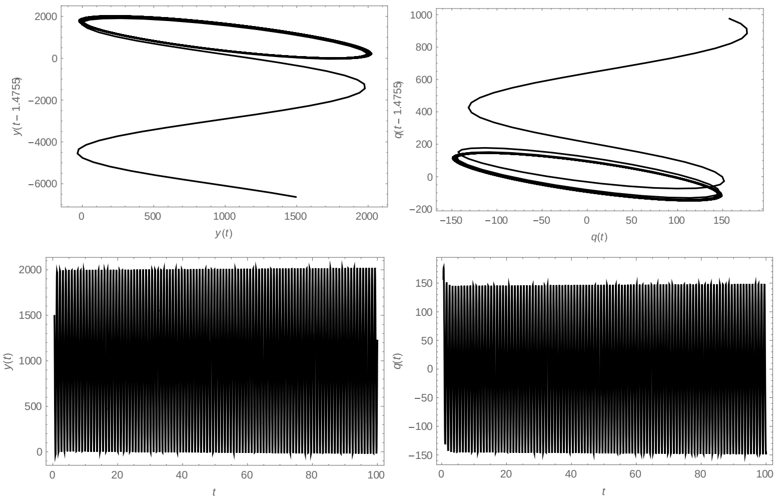


Fig. 8 State and Phase plot of $y(t)$ **and** $p(t)$ **with** $\alpha = 1.4755$

As can be seen from the above figure, the stability range of the controlled system with PD control added is larger than that of the non-controlled system.

V. CONCLUSIONS

Based on XCP network congestion power system, this paper studies a congestion model of XCP network with PD controller. On the basis of theoretical analysis, we first introduce the *Hopf* bifurcation behavior of the non-control system model. In order to delay this behavior, a PD controller is added. By selecting appropriate control parameters, we obtain the critical value to keep the controlled system stable, thus effectively alleviating the generation of *Hopf* branches. However, when the delay is increased to a certain value, the system will still be blocked or even crash. The numerical simulation results verify the correctness of the theoretical analysis. Therefore, although the bifurcation behavior is not eliminated through PD controller, we can effectively alleviate the generation of *Hopf* branches, expand the stable interval of wireless network, and achieve better service performance of wireless network.

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