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Square Sum Labelling for Lobster and Fan Graph

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***Abstract* -** Let $G = (V, E)$ be a (p, q)-graph and let $f: V(G) \to \{0,1,2,\ldots,p-1\}$be a bijection. We define $f*$ on $E(G)$ by $f*(uv) =.[f(u)]^2 + f(v)]^2$. If $f*$ is injective on $E(G)$, then f is called a square sum labelling. The graph G is said to be a square sum graph if G admits a square sum labelling.

Keywords - *Square Sum Labelling, Labelling of Graph, the lobster, full n-ary tree, and the amalgamation of a fan and a star graph.*

I. INTRODUCTION

Rosa introduced the notion of Graph labelling in 1967 [6]. A graph labelling is a mapping that carries a set of graph elements onto a set of numbers called labels (usually the set of integers). A dynamic survey on graph labelling is regularly updated by Gallian[4]. Germina introduced and proved some results of square sum labelling. Reena Sabastian etc.., all discussed the concepts of square sum labelling in 2014[5].

Next, we demonstrate that routes, the graph $R_p(n_1, n_2, \ldots, n_k)$, the lobster, full n-ary tree, and the amalgamation of a fan and a star allow square sum labelling.

II. PRELIMINARIES

Definition 2.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as a graph labelling.

Definition 2.2: A path in a graph G is a sequence of vertices such that from each of its vertex there is an edge to the next vertex in the sequence. The length of a path $\mathrm{P_n}: \mathrm{v_1 v_2 \ldots v_n} (n > 0)$ in G is $n-1$.

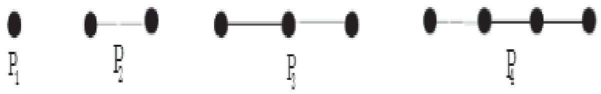

Fig 1. Path graphs P_1, P_2, P_3, and P_4

Definition 2.3: A path P_n: $v_1 v_2 \ldots v_n$ in graph G is called cycle C_n if: $v_1 = v_n$and $n \geq 3$.

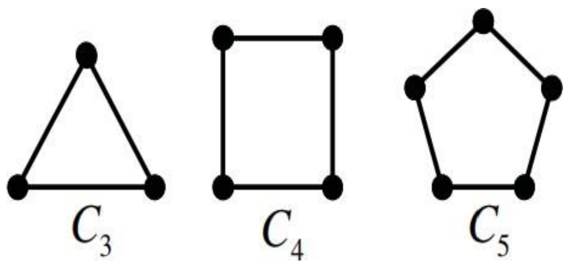

Fig 2. Cycle graphs C_3, C_4, dan C_5

Definition 2.4: A lobster graph, lobster tree, or simply "lobster," is a tree having the property that the removal of leaf nodes leaves a caterpillar graph

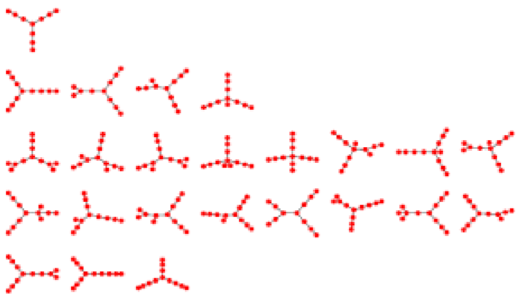

Fig 3. Lobster Graph

Definition 2.5 : A fan graph obtained by joining all vertices of F_n, $n \geq 2$ is a path P_n to a further vertex, called the centre. Thus F_n contains *n+1* vertices say C, $v_1, v_2, v_3 \ldots v_n$ and *(2n-1)* edges, say cv_i, $1 \leq i \leq n$ and $v_i v_{i+1} + 1$, $1 \leq i \leq n-1$.

Fig 4. Fan f_4

Definition 2.6 : The routing tree of D_n is the tree structure obtained from the (minimal) paths followed by a message when it is routed from every node to the identity node

Definition 2.7 : A star $S_n$is the complete bipartite graph $K_{1,n}$ is a tree with one internal node and *n* leaves.

III. MAIN RESULT

In this section, we investigate which classes of graphs admit square sum labelling.

Theorem 3.1: The route Pn is represented as a square sum graph.

Proof: Suppose $P_n = (v_1, v_2, \ldots, v_n)$ be a path. Define a function $f : V(P_n) \rightarrow N$ by $f(v_i) = T_i - 1$, $1 \leq i \leq n$. Since $T_i - 1 < T_i$, for $1 \leq i \leq n - 1$, we have $f(v_i) < f(v_i) + 1)$ and therefore f is one-one. Here we have $1 \leq i \leq n - 1$,

$$f^+(v_i v_{i+1}) = f(v_i) + f(v_{i+1})$$

$$= T_{i-1} + T_i = \frac{i(i-1)}{2} + \frac{i(i+1)}{2}$$

$$= i^2$$

$$= R_i$$

Thus, $f^+(E(P_n)) = \{R_1, R_2, \ldots, R_{n-1}\}$. Hence P_n is a square sum graph.

[The path P_n is a square sum graph.]

Theorem 3.2: The graph $R_p(n_1, n_2, \ldots, n_k)$ is a square sum graph.

Proof: Suppose $c_1, c_2, \ldots, c_k$ represents the centers of the k stars, where the star with centre c_i has n_i pendent edges which are given by

$\{c_i v_i j_i | N_{i-1} + 1 \leq j_i \leq N_i |\}$,

Here $1 \leq i \leq k, N_0 = 0, N_i = \sum_{j=1}^{i} n_j$

Define $f: V(G) \rightarrow N\ by,$

$f(c_i) = T_{i-1}$ &

$f\left(v_{i,j_i}\right) = R_{k+j_i-1} - T_{i-1}$

Here $T_{i-1} < T_i, for\ 1 \leq i \leq n-1$

then $f(c_i) < f(c_{i+1})$

$f\left(v_{i,1}\right) - f(v_k) = R_k - f(c_k)$

$= R_k - (T_{k-1}) > 0$

So $f\left(v_{i,1}\right) > f(c_k)$. And From definition it is clear that ,for all $j<i$ we have $f\left(v_{i,1}\right) < f\left(v_{i,j_i'}\right)$,since $R_{k+j_{i-1}} < R_{k+j_i'-1}$.

Following this we have ,

$f(v_{i+1}, N_{i+1}) - f(v_i, N_i) = S_{k+N_i} - T_i - S_{k+N_i-1} + T_{i-1} = S_{k+N_i} - S_{k+N_i-1} - (T_i - T_{i-1}) \geq 2k + 2N_i - 1 - k = k + 2N_i - 1 > 0,$

for every value of i with $1 \leq i \leq k-1$.

Theorem 3.3: The lobster T is a square sum graph.

Proof: Consider T be the lobster produced by connecting the centres of k copies of the same star $K_{1,n}$ to a new vertex w. Denote the centre vertex of the i^{th} star $K_{1,n}$ as w_i, $1 \leq i \leq k$ and the pendent vertices of the i^{th} star as $v_{i,j,1}, 1 \leq i \leq k, 1 \leq j \leq n$.

Note that T contains $(n+1)k$ edges. Define $f: V(T) \rightarrow N$ by

$f(w) = 0,\ f(w_i) = R_i,\ 1 \leq i \leq k$ and $f(v_{i,j}) = R_{k+j+m} - f(w_i),\ 1 \leq i \leq k,\ 1 \leq j \leq n,\ m = (i-1)n.$

Since $0 < R_1 < \dots < R_k$, we have, $f(w) < f(w_1) < \dots < f(w_k)$.

Also since, $R_{k+j+m} - f(w_i) < R_{k+j+m} - f(w_i)$, for $1 \leq i \leq k$ and $1 \leq j \leq n$, we have $f(v_{i,j}) < f(v_{i,j+1})$. For, $k \geq 1$, we have $f(w_k) = R_k$ and $f(v_{1,1}) = R_{k+1} - 1$. Since, $R_k < R_{k+1} - 1$, we have $f(w_k) < f(v_{1,1})$.

Further, we have, by the definition of f, $f(v_{i,n}) = R_{k+n+(i-1)n} - f(w_i)$ & $f(v_{i+1}, 1) = R_{k+1+in} - f(w_{i+1}),\ 1 \leq i \leq k-1$.

Clearly, $R_{k+n+(i-1)n} - f(w_i) < R_{k+1+in} - f(w_{i+1})$, since $k + i(n-1) > 0$. Therefore,

$f(v_{i,n}) < f(v_{i+1}, 1), for\ 1 \leq i \leq k-1.$

Thus f is one-one.

From the labelling, it follows that $f^+(E(T)) = \{R_1, R_2, \dots, R_{(n+1)K}\}$.

Hence the lobster T is a square sum graph.

Theorem 3.4: The graph G obtained by the amalgamation of the fan graph F_{n-1} with the centre of $K_{1,m}$ for a suitable m, is a square sum graph.

Proof: ***Step 1:*** Consider F_2. Let v, v_1, v_2, be the vertices of F_2. Label $v = 0$ and $v_1 = R_3$. We shall find b such that

$3^2 + b^2 = a^2$,.......(1)

for some integer a. The inequality (1) gives

$3^2 = (a+b)(a-b)$.

Therefore, we let $a + b = 3^2$ and $a - b = 1$. Solving for a and b we get, $a = 5$ and $b = 4$. Hence, we can label v_2 as R_4. Then the values of the edges vv_1, v_1v_2 and v_2v are respectively R_3, R_5 & R_4.

We now consider a star $K_{1,m}$ where $m \geq 2$. We amalgamate the centre of $K_{1,m}$ with the vertex v to get the graph G. Let the pendent vertices of G be u_j, $1 \leq j \leq m$. We define $f : V(G) \rightarrow N$ with,

$f(v) = 0, f(v_1) = R_3, f(v_2) = R_4,$

$f(u)_1 = R_1, f(u_2) = R_2$ and $f(u_j) = R_{j+3}$ for $3 \leq j \leq m.$

From the definition, it follows that f is one-one.

Then we have $f^+(E(T)) = \{R_1, R_2, \ldots, R_{m+3}\}$ and the graph is a square sum graph.

Step 2: Consider $F_{n-1}, n \geq 4$. Suppose $v, v_1, v_2, , \ldots, v_{n-1}$ be the vertices of F_{n-1}. Label $f(v) = 0$ and $f(v_1) = R_{b_1}$ where $b_1 \geq 5$. We shall find integers a_1 and b_2 such that

$b_1^2 + b_2^2 = a_1^2$ ....(2)

Here We have need to consider two scenario:

Case 1: If b_1 is odd, then we can write

$b_1 = p_1^{2\alpha 1} p_2^{2\alpha 2} \bullet\bullet\bullet p_k^{2\alpha k}$,

here $p_1 < p_2 < \cdots .. < p_k$, are all odd primes and α_i's are positive integers.

Then (2) yields

$(a_1 + b_2)(a_1 - b_2) = p_1^{2\alpha 1} p_2^{2\alpha 2} \ldots\ldots .. p_k^{2\alpha k}$

We consider

$(a_1 + b_2) = p_2^{2\alpha 2} \ldots .. p_k^{2\alpha k}$

$(a_1 - b_2) = p_1^{2\alpha 1}$

so that

$a_1 = \frac{p_2^{2\alpha 2} \bullet\bullet\bullet p_k^{2\alpha k} + p_1^{2\alpha 1}}{2}$ & $b_2 = \frac{p_2^{2\alpha 2} \bullet\bullet\bullet p_k^{2\alpha k} - p_1^{2\alpha 1}}{2}$

Case 2: If b_1 is even, then we can write

$b_1 = 2^{\alpha 1} p_2^{\alpha 2} \ldots . p_r^{\alpha r}$

Here $2 < p_2 < \cdots . < p_{r,}$ are all primes and $\alpha_i's$ are +ve integers. Then (2) yields

$(a_1 + b_2)(a_1 - b_2) = 2^{\alpha 1} p_2^{\alpha 2} \ldots . p_r^{\alpha r}$

We choose

$(a_1 + b_2) = 2^{\alpha 1} p_2^{2\alpha 2} \ldots . p_r^{2\alpha r}$

$(a_1 - b_2) = 2^{\alpha 1}$

Then we have,

$a_1 = \frac{2^{\alpha 1} p_2^{2\alpha 2} \bullet\bullet\bullet p_k^{2\alpha k} + 2^{\alpha 1}}{2}$ & $b_2 = \frac{2^{\alpha 1} p_2^{2\alpha 2} \bullet\bullet\bullet p_k^{2\alpha k} - 2^{\alpha 1}}{2}$

Having found b_2, we label $f(v_2) = R_{b_2}$. Then the value of the edge $f(v_1v_2)$ will be a_1^2. From the construction it follows that $b_1 < b_2$ and hence $R_{b_1} < R_{b_2}$. Proceeding like this we can label $f(v_k) = R_{b_k}$, $3 \leq k \leq n - 1$. Then from the construction it follows that the values of the edges will be perfect squares. Suppose the values of the $2n - 3$ edges are $R_{a_1}, R_{a_2}, \ldots R_{a_{n-2}}, \ldots., R_{b_1}, \ldots\ldots, R_{b_{n-1}}$

From the construction it follows that $R_{a_{n-2}}$ is the largest of these squares. Put

$A = \{R_1, R_2, \ldots, R_{a_{n-2}}\}$ and $B = \{R_{a1}, R_{a2}, \ldots\ldots, R_{an-2}, R_{b1}, R_{b2}, \ldots., R_{bn-1}\}$.

Now we amalgamate the center of a star $K_{1,m}$ where $m \geq (|A| - |B|)$, with vertex v of F_{n-1} to get the graph G. Label the $|A| - |B|$ pendent vertices of $K_{1,m}$ with the squares from the set $A - B$. Label the remaining $m - (|A| - |B|)$ pendent vertices with the consecutive squares

$R_{a_{n-2}}, R_{a_{n-2}}, \ldots., R_{a_{n-2}+m-(|A|-|B|)}$.

From the construction it follows that $f^+(E(G)) = \{R_1, R_2, \ldots R_{a+m-(|A|-|B|)}\}$ and the graph G is a square sum graph.

IV. CONCLUSION

We have discussed the concept of square sum labelling of graphs in this article. Following that, we show that routes, the graph, the lobster, the complete n-ary tree, and the combination of a fan and a star all enable square sum labelling

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