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ON WEAK COMPACTNESS IN $L^{\Phi}(\mu, X)$

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ABSTRACT. Here we characterize relative weak compact subset of Orlicz-Bochner space $L^{\Phi}(\mu, X)$, where X is a reflexive Banach space and Φ is a Young function, first using "convex compactness criterion" and then using regular method of summability. We also prove a convergence theorem in $L^{\Phi}(\mu, X)$ which generalization of similar result in $L^1(\mu, X)$ and $L^p(\mu, X)$ of A. Ülger[19] and S. Diaz [4].

1. INTRODUCTION

The problem of characterising relatively weakly compact subset of Orlicz space space $L^{\Phi}(\mu, X)$, where Φ is a Young function and X is any Banach space has been considered by many authers [1], [15] and [17]

In [8], J.Diestel, W. M. Ruess and W. Schachermayer characterize relatively weakly compact subset of $L^1(\mu, X)$ using 'convex compactness condition'. In the same paper they also extend this result to characterize relatively weakly compact subset of Köthe-Bochner space $E(X)$, where E is an ideal of L^0 such that $L^{\infty} \subset E \subset L^1$ and X is a Banach space. Here we first use this criteria to characterize relatively weakly compact subset of the Orlicz-Bochner space $L^{\Phi}(\mu, X)$.

In [4], S.Diaz describes the above 'convex compactness condition' in terms of regular method of summability in a Banach space X. This provides new characterization of weak compactness as well as weak conditional compactness in $L^1(\mu, X)$. In [14], M.Nowak uses regular method of summability to characterize relative $\sigma(E(X), \widetilde{E(X)}_n)$ compact and conditional $\sigma(E(X), \widetilde{E(X)}_n)$ compact subset of $E(X)$ where $\widetilde{E(X)}_n$ is the order continuous dual of $E(X)$. In [2], we use regular method of summabitility to characterize relatively weakly sequential compactness in $P_1(\mu, X)$, the space of all X-valued Pettis integrable functions where X is a seperable Banach space. Here in the section of our main results we use regular method of summabitility to characterize relativley weakly compact subset of the Orlicz-Bochner space $L^{\Phi}(\mu, X)$. We actually generalize [[4], Theorem 1, p.2686],

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[[4], Theorem 3, p.2688] and [[14], Theorem 2.3, p.420] to the Orlicz-Bochner space $L^{\Phi}(\mu, X)$.

2. Notations, Definitions and Preliminaries

Before we proceed to our main result we first recall some notations and preliminaries of Orlicz-Bochner space. Reader is referred to [7], [9] and [18] for further details.

Throughout the whole paper (Ω, Σ, μ) is a complete, positive and finite measure space and X is a reflexive Banach space with dual X^*. Φ is a Young function with Φ^* as its complementary function.

By a Young function here we mean a mapping $\Phi : [0, \infty) \to [0, \infty)$ that is convex, monotone non-decreasing, vanishing only at zero and $\lim\limits_{t\to 0} \Phi(t)/t = 0$ and $\lim\limits_{t\to\infty} \Phi(t)/t = \infty$.

For a Young function Φ, Φ^* denotes the complementary function to Φ in the sense of Young, i.e. $\Phi^*(u) = \sup\{ut - \Phi(t); t \geq 0\}$ for all $u \geq 0$. It is to be mentioned here that Φ^* is also a Young function and $\Phi^{**} = \Phi$. Also Φ and Φ^* satisfy Young inequality [20] i.e. for any $u \geq 0$, $v \geq 0$

$$uv \leq \Phi(u) + \Phi^*(v).$$

The Orlicz space L^{Φ} generated by the Young function Φ is an ideal of L^0 and is defined by

$$L^{\Phi} = \{u \in L^0 : \int_{\Omega} \Phi(\lambda|u(\omega)|)d\mu < \infty \quad \text{for some} \quad \lambda > 0\},$$

where L^0 is the equivalence classes of all measurable functions $f : \Omega \to \mathbb{R}$. The Orlicz norm and the Luxemberg norm or Gauge Norm [18] can be defined on L^{Φ} respectively by

Equation 2.0.1. $\|u\|_{\Phi} = \sup\{|\int_{\Omega} u(\omega)v(\omega)d\mu| : v \in L^{\Phi^*}, \int_{\Omega} \Phi^*(|v(\omega)|)d\mu \leq 1\}$.

Equation 2.0.2. $|u|_{\Phi} = \inf\{\lambda > 0 : |\int_{\Omega} \Phi(|u(\omega)|/\lambda)d\mu| \leq 1\}$.

The Orlicz norm and the Luxemberg norm on L^{Φ} given above by (2.0.1) and (2.0.2) respectively are equivalent.[[9], Theorem 3, p.52] and [18]. It is well known that both the norms on L^{Φ} satisfies the σ-Fatou property and σ-Levy property.[[6], Theorem 4.3.7]. Also $(L^{\Phi})' = L^{\Phi^*}$ and

$$E^{\Phi} = \{u \in L^{\Phi} : \int_{\Omega} \Phi(\lambda \cdot |u(\omega)|)d\mu \leq \infty, \text{ for all } \lambda > 0\}.$$

It is known that $E^{\Phi} = (L^{\Phi})_a$ where $(L^{\Phi})_a$ is an ideal of L^{Φ} consisting of all $g \in L^{\Phi}$ such that the seminorm $\rho_{L^{\Phi^*}}$ on L^{Φ^*} defined by
$\rho_{L^{\Phi^*}}(\chi_{A_n} g) = \sup\{\int_{A_n} |\langle f(\omega), g(\omega)\rangle| d\mu; \quad f \in L^{\Phi^*}\} \to 0$, for all $A_n \in \Sigma$ with $A_n \searrow \varphi$.

The Köthe-Bochner space $L^{\Phi}(\mu, X) = \{f \in L^0(\mu, X) : \tilde{f} \in L^{\Phi}\}$, where $L^0(\mu, X)$ is the equivalence classes of strongly measurable functions $f : \Omega \to X$. The norms in $L^{\Phi}(\mu, X)$ corresponding to the Orlicz Norm given by (2.0.1) and the Luxemberg Norm given by (2.0.2) in L^{Φ} is respectively given by

$$\|f\|_{L^{\Phi}(\mu,X)} = \|\tilde{f}\|_{\Phi} \text{ and } |f|_{L^{\Phi}(\mu,X)} = |\tilde{f}|_{\Phi}.$$

A subset H of $L^{\Phi}(\mu, X)$ is said to be solid whenever f_1 and f_2 be two elements in $L^{\Phi}(\mu, X)$ such that $\|f_1(\omega)\|_X \leq \|f_2(\omega)\|_X$ for ω-a.e., then $f_2 \in H$ implies that $f_1 \in H$.

A Young function Ψ is said to be completely weaker than another Φ, in symbol $\Psi \lhd \Phi$, If for an arbitrary $c > 1$ there exists $d > 1$ such that $\Psi(ct) \leq d\Phi(t)$ for $t \geq 0$. It is known that the relation $\Psi \lhd \Phi \to L^{\Phi} \subset E^{\Psi}$.[[18], Theorem 5.3.1].

A Young function $\Phi : [0, \infty) \to [0, \infty)$ is said to satisfy Δ_2 condition (globally), denoted by $\Phi \in \Delta_2$(globally), if $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$ for some constant $K \geq 0$. It is to be noted that a Young function Φ satisfies Δ_2 condition if $\Phi \lhd \Phi$.

A Young Function $\Phi : [0, \infty) \to [0, \infty)$ is said to satisfy ∇_2 condition(globally), denoted by $\Phi \in \nabla_2$(globally), if $\Phi(x) \leq (1/2l)\Phi(lx)$ for all $x \geq 0$ for some constant $l \geq 1$. A Young function Φ more rapidly than another Young function Ψ, denoted by $\Psi \leftarrow \Phi$, if for $c > 0$ there exists a $d > 0$ such that $c\Psi(t) \leq (1/d)\Phi(d \cdot t)$ for all $t \geq 0$. It is to be noted that Φ satisfies ∇_2 condition iff $\Phi \leftarrow \Phi$.

It can be verified that for the Young function Φ and Ψ the relation $\Psi \lhd \Phi$ holds if $\Psi^* \leftarrow \Phi^*$ holds.[[18], Proposition 2.2.4]. It can be shown that a Young function Φ satisfies ∇_2 condition if $\Phi \leftarrow \Phi$.

If Φ satisfies Δ_2 condition then by [[16], Theorem 2, p.2], we have

Equation 2.0.3. $L^{\Phi}(\mu, X)^* = L^{\Phi^*}(X^*, X)$.

If X^* has RNP, by [[16], p.2] or by [[17], p.114], we have

Equation 2.0.4. $L^{\Phi^*}(X^*, X) = L^{\Phi^*}(\mu, X^*)$.

An infinite matrix $T = (t_{n,m})$ of scalars is said to be a regular method of summability in a Banach space X if, for every convergent sequence $\{x_n\}$ in X, the series $x_n^T = \Sigma_{m=1}^{\infty} t_{n,m} x_m$ exists for each $n \in \mathbb{N}$ and the sequence $\{x_n^T\}$ is convergent to the same limit as $\{x_n\}$. It is to be mentioned further that regular matrix T is independent of the particular Banach space X. Regular matrices are characterized by the vectorial version of the classical Silverman-Toeplitz theorem.[4]

Theorem A (Silverman-Toeplitz) A scalar infinite matrix $T = (t_{n,m})$ is a regular method of summability in X iff it satisfies the following three conditions:

(a) $\sup_n \Sigma_{m=1}^{\infty} |t_{n,m}| < \infty$.
(b) $\lim_{n \to \infty} t_{n,m} = 0$ for all $m \in \mathbb{N}$.
(c) $\Sigma_{m-1}^{\infty} t_{n,m} = 1$.

3. Main Results

3.1. Relative Weak compactness in $L^{\Phi}(\mu, X)$ using convex combination criteria.

Theorem 3.1.1. *Let H be a norm bounded solid subset of $L^{\Phi}(\mu, X)$ where X is reflexive and Φ satisfies Δ_2 condition. Then the following are equivalent:*
1) H is relatively weakly compact.
2) $\{\tilde{f}\cdot\tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$ and given any sequence $\{f_n\}_n$ in H, there exists a sequence $\{g_n\}_n$ with $g_n \in co\{f_k;\ k \geq n\}$ such that $\{g_n(\omega)\}$ is norm convergent for a.e. ω in Ω.
3)$\{\tilde{f}\cdot\tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$ and given any sequence $\{f_n\}$ in H, there exists a sequence $\{g_n\}$ with $g_n \in co\{f_k;\ k \geq n\}$ such that $\{g_n(\omega)\}$ is weakly convergent for a.e. ω in Ω.

Proof. 1) $\Longrightarrow$ 2)
Let H is relatively weakly compact.
Since Φ satisfies Δ_2 condition, by the equation (2.0.3), $L^{\Phi}(\mu, X)^* = L^{\Phi^*}(X^*, X)$ and since X is reflexive, by [[5], cor.13, p.76], X^* has RNP, so by the equation (2.0.4), $L^{\Phi^*}(X^*, X) = L^{\Phi^*}(\mu, X^*)$.
Consider the seminorm $\rho_H(\cdot)$ defined on $L^{\Phi^*}(\mu, X^*)$ by

$$\rho_H(g) = \sup_{f\in H}\int_{\Omega}|\langle f(\omega), g(\omega)\rangle| d\mu.$$

It is absolutely continuous by [[15], Theorem 2.2, p.79].
Since H is a solid subset of $L^{\Phi}(\mu, X)$ by [[13], Theorem 1.3, p.199], it follows that,

$$\rho_H(g) = \sup_{f\in H}\int_{\Omega}|\langle f(\omega), g(\omega)\rangle| d\mu = \sup_{f\in H}\int_{\Omega}\|f(\omega)\|_X \cdot \|g(\omega)\|_{X^*} d\mu$$

Therefore for every $h \in L^{\Phi^*}(\mu, X^*)$ and for every $\epsilon > 0$ there exists a $\delta > 0$ such that for $A_0 \in \Sigma$ with $\mu(A_0) < \infty$ such that $\rho_H(\chi_A h) \leq \epsilon$ for all $A \in \Sigma$ such that $\mu(A) < \delta$ and $\rho_H(\chi_{\Omega\setminus A_0} g) \leq \epsilon$.
Hence $\{\tilde{f}\cdot\tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$.
Again since H is relatively weakly compact, for any sequence $\{f_n\}$ in H there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges weakly to some function $f \in L^{\Phi}(\mu, X)$. Now by Mazur's Theorem there exists a sequence $\{g_k\}$ with $g_k \in co\{f_m;\ m \geq k\}$ such that $\|g_k - f\|_{\Phi} \to 0$ as $k \to \infty$ i.e. $\{g_k\}$ converges to f in measure and hence there exists a subsequence $\{g_n\}$ of $\{g_k\}$ which converges in norm to f a.e. $\omega \in \Omega$.
The implication 2)$\Rightarrow$ 3) is obvious.
We now prove 3)$\Rightarrow$ 1). For this let us suppose that the condition 3)holds.

Let $\{f_n\}$ be a sequence in H. Then by the hypothesis there exists a sequence $\{g_k\}$ with $g_k \in co\{f_m;\ m \geq k\}$ such that $\{g_n(\omega)\}$ is weakly convergent for a.e. ω in Ω. Let $E \in \Sigma$ be the exceptional set with $\mu(E) = 0$ such that $\{g_k\}$ converges weakly for all $\omega \in \Omega \setminus E$. Put $f(\omega) =$ weak-limit $g_k(\omega)$ for all $\omega \in \Omega \setminus E$.
Then clearly f, being the weak-limit of a sequence of measurable functions g_k, is measurable.
Since by the hypothesis, $\{\tilde{f} \cdot \tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$, it is easy to check that the set $\{\tilde{g_k} \cdot \tilde{h};\ k \in \mathbb{N}\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$.
As H is norm-bounded and solid, by Fatou's Lemma, Hölder's inequality and by [[13], Theorem 1.3, p.199], we have for any h in $L^{\Phi^*}(\mu, X^*)$,

$$\begin{aligned} \textstyle\int_\Omega |\langle f(\omega), h(\omega)\rangle| d\mu &= \textstyle\int_\Omega \|f(\omega)\|_X \cdot \|h(\omega)\|_{X^*} d\mu \\ &\leq \textstyle\int_\Omega \liminf \|g_k(\omega)\|_X \cdot \|h(\omega)\|_{X^*} d\mu \\ \leq \liminf_k \int_\Omega \|g_k(\omega)\|_X \cdot \|h(\omega)\|_{X^*} d\mu &\leq \sup_k \int_\Omega \|g_k(\omega)\|_X \cdot \|h(\omega)\|_{X^*} d\mu \\ &\leq \sup_k \|g_k\|_\Phi \cdot \|h\|_{\Phi^*} < \infty. \end{aligned}$$

Therefore $f \in L^\Phi(\mu, X)$.
Since Φ satisfies Δ_2 condition, by the equation (2.0.3), $L^\Phi(\mu, X)^* = L^{\Phi^*}(X^*, X)$ and since X is reflexive, by [[5], cor.13, p.76], X^* has RNP, so by the equation (2.0.4), $L^{\Phi^*}(X^*, X) = L^{\Phi^*}(\mu, X^*)$.
Therefore for any h in $L^{\Phi^*}(\mu, X^*)$,

$$\lim_{k\to\infty} \langle h(\omega), g_k(\omega)\rangle = \langle h(\omega), f(\omega)\rangle \quad \text{for a.e. } \omega \in \Omega.$$

Also $\langle h(\cdot), g_k(\cdot)\rangle$ and $\langle h(\cdot), f(\cdot)\rangle$ belong to L^1 and for all $h \in L^{\Phi^*}(\mu, X^*)$, $\{\tilde{g_k} \cdot \tilde{h};\ k \in \mathbb{N}\}$ is uniformly integrable .
It follows from Vitali Convergence Theorem in L^1,

$$\lim_{k\to\infty} \int_\Omega \langle h(\omega), g_k(\omega)\rangle = \int_\Omega \langle h(\omega), f(\omega)\rangle.$$

Hence $\{f_n\}$ has a subsequence $\{g_k\}$, with $g_k \in co\{f_m;\ m \geq k\}$, which converges weakly to f in $L^\Phi(\mu, X)$.
Hence by [[8], Corollary 2.2, p.449], H is relatively weakly compact. □

3.2. **Relative Weak compactness in $L^\Phi(\mu, X)$ using Regular Method of Summability.**

Theorem 3.2.1. *Let H be a norm bounded solid subset in $L^\Phi(\mu, X)$ where X is reflexive and Φ satisfies Δ_2 condition. The the following are equivalent:*
1) H is relatively weakly compact.
2) $\{\tilde{f} \cdot \tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$ and given any sequence $\{f_n\}_n$ in H, there exists regular method of summability such that $\{f_n^T(\omega)\}$

is weakly convergent for a.e. ω *in* Ω.
3) $\{\tilde{f}\cdot\tilde{h};\ f\in H\}$ *is uniformly integrable for all* $h\in L^{\Phi^*}(\mu,X^*)$ *and given any sequence* $\{f_n\}_n$ *in* H*, there exists regular method of summability such that* $\{f_n^T(\omega)\}$ *is norm convergent for a.e.* ω *in* Ω.

Proof. 1) $\Longrightarrow$ 2)
Let H is relatively weakly compact.
Since Φ satisfies Δ_2 condition, by the equation (2.0.3), $L^{\Phi}(\mu,X)^*=L^{\Phi^*}(X^*,X)$ and since X is reflexive, by [[5], cor.13, p.76], X^* has RNP, so by the equation (2.0.4), $L^{\Phi^*}(X^*,X)=L^{\Phi^*}(\mu,X^*)$.
Consider the seminorm $\rho_H(\cdot)$ defined on $L^{\Phi^*}(\mu,X^*)$ by

$$\rho_H(g)=\sup_{f\in H}\int_\Omega|\langle f(\omega),g(\omega)\rangle|d\mu.$$

By [[15], Theorem 2.2, p.79,], it is absolutely continuous.
Since H is a solid subset of $L^{\Phi}(\mu,X)$ from [[13], Theorem 1.3, p.199], it follows that,

$$\rho_H(g)=\sup_{f\in H}\int_\Omega|\langle f(\omega),g(\omega)\rangle|d\mu=\sup_{f\in H}\int_\Omega\|f(\omega)\|_X\cdot\|g(\omega)\|_{X^*}d\mu$$

Therefore for every $g\in L^{\Phi^*}(\mu,X^*)$ and $\epsilon>0$ there exists a $\delta>0$ such that for $A_0\in\Sigma$ with $\mu(A_0)<\infty$ such that $\rho_H(\chi_A g)\le\epsilon$ for all $A\in\Sigma$ such that $\mu(A)<\delta$ and $\rho_H(\chi_{\Omega\setminus A_0}g)\le\epsilon$. Therefore $\{\tilde{f}\cdot\tilde{h};\ f\in H\}$ is uniformly integrable for all $h\in L^{\Phi^*}(\mu,X^*)$.
Let $\{f_n\}\subset H$. By [[4], Theorem 1, p.2686], there exists a regular method of summability T such that $\{f_n^T\}$ converges in norm to some f in $L^{\Phi}(\mu,X)$. Therefore a subsequence of $\{f_n^T\}$ converges to f pointwise a.e. $\omega\in\Omega$. Dropping suitable rows from T, we finally obtain a regular method of summability $T^{'}$ such that $\{f_n^{T^{'}}\}$ converges in norm convergent a.e. $\omega\in\Omega$.
The implication 2)$\Longrightarrow$ 3) is obvious.
we now prove 3)$\Longrightarrow$ 1). For this we assume that condition 3) holds.
Let $\{f_n\}$ be a sequence in H. Then by the hypothesis there exists a regular methods of summability T such that $\{f_n^T\}$ converges a.e.ω in Ω. Let $E\in\Sigma$ be the exceptional set with $\mu(E)=0$ such that $\{f_n^T\}$ converges weakly for all $\omega\in\Omega\setminus E$.
Put $f(\omega)=$ weak-limit $f_n^T(\omega)$, $\omega\in\Omega\setminus E$.
Since $f_n^T(\omega)=\Sigma_{m=1}^{\infty}t_{n,m}f_m(\omega)$, each f_n^T, being the pointwise limit of a sequence of strongly measurable functions $\{f_m\}$, is strongly measurable.
Now by[[13], Theorem 1.3, p.199], norm-boundedness of H in $L^{\Phi}(\mu,X)$, Hölder inequality and condition (a) of Silvermann-Toeplitz Theorem(A), we have

$$\int_\Omega\|h(\omega)\|_{X^*}\cdot\|f_n^T(\omega)\|_X d\mu=\int_\Omega|\langle h(\omega),f_n^T(\omega)\rangle|d\mu=$$

$$\int_{\Omega} |\Sigma_{m=1}^{\infty} t_{n,m} \langle h(\omega), f_m(\omega)\rangle| d\mu$$

$$\leq \int_{\Omega} \Sigma_{m=1}^{\infty} |t_{n,m}| |\langle h(\omega), f_m(\omega)\rangle| d\mu \leq$$

$$\Sigma_{m=1}^{\infty} |t_{n,m}| \int_{\Omega} |\langle h(\omega), f_m(\omega)\rangle| d\mu \leq \|h\|_{\Phi^*} \cdot \|f_n\|_{\Phi} < \infty.$$

So $\langle f_n^T(\cdot), h(\cdot)\rangle \in L_1(\mu)$ for all $h \in L^{\Phi^*}(\mu, X^*)$ and for all $n \in \mathbb{N}$
Moreover, the set $\{\tilde{f_n^T} \cdot \tilde{h};\ n \in \mathbb{N}\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$ as for all $h \in L^{\Phi^*}(\mu, X^*)$ and $n \in \mathbb{N}$, we have

Equation 3.2.2. $\int_A \| \| f_n^T(\omega) \cdot h(\omega) \|_{X^*} \|_X d\mu = \int_A |\langle f_n^T(\omega), h(\omega)\rangle| d\mu =$
$\int_A |\Sigma_{m=1}^{\infty} t_{n,m} \langle f_m(\omega), h(\omega)\rangle| d\mu$
$\leq \int_A \Sigma_{m=1}^{\infty} |t_{n,m}| |\langle f_m(\omega), h(\omega)\rangle| d\mu \leq \Sigma_{m=1}^{\infty} |t_{n,m}| \sup_m \int_A |\ \langle f_m(\omega), h(\omega)\rangle| d\mu.$

So by condition (a) of Theorem A and by the hypothesis the set $\{\tilde{f} \cdot \tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$, it follows that $\Sigma_{m=1}^{\infty} |t_{n,m}| \sup_m \int_A |\langle h(\omega), f_m(\omega)\rangle| d\mu$ tends to zero as $\mu(A)$ tends to zero for all $A \in \Sigma$.
We now show that $f \in L^{\Phi}(\mu, X)$.
Let h be any element in $L^{\Phi^*}(\mu, X^*)$.
Since $\langle f_n^T(\omega), h(\omega)\rangle \to \langle f(\omega), h(\omega)\rangle$ as $n \to \infty$, a.e. ω in Ω and the set $\{\tilde{f_n^T} \cdot \tilde{h};\ n \in \mathbb{N}\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$, by Vitali convergence theorem

Equation 3.2.3. $\lim_{n\to\infty} \int_{\Omega} \langle f_n^T(\omega), h(\omega)\rangle d\mu = \int_{\Omega} \langle f(\omega), h(\omega)\rangle d\mu.$

Also $\langle f(\omega), h(\omega)\rangle \in L^1$. Hence $f \in L^{\Phi}(\mu, X)$.
Finally we show that $\{f_n^T\}$ converges weakly to f in $L^{\Phi}(\mu, X)$.
Since Φ satisfies Δ_2 condition, by the equation (2.0.3), $L^{\Phi}(\mu, X)^* = L^{\Phi^*}(X^*, X)$ and since X^* has RNP, by the equation (2.0.4), $L^{\Phi^*}(X^*, X) = L^{\Phi^*}(\mu, X^*)$ and from 3.2.3, we see that $\{f_n^T\}$ converges weakly to f in $L^{\Phi}(\mu, X)$.
Therefore by [[4], Theorem 1, p.2686], H is relatively weakly compact.

□

3.3. A Convergence Theorem in $L^{\Phi}(\mu, X)$.

Lemma 3.3.1. *Assume that X is reflexive and Φ satisfies Δ_2 condition. If $\{f_n\}$ be a norm bounded sequence in $L^{\Phi}(\mu, X)$ such that*
a) $\{\tilde{f} \cdot \tilde{h};\ f \in H\}$ is uniformly integrable for all $h \in L^{\Phi^*}(\mu, X^*)$
b) $\{f_n\}$ be weakly convergent almost surely to a function $f \in L^{\Phi}(\mu, X)$.
Then $\{f_n\}$ converges weakly to f in $L^{\Phi}(\mu, X)$.

Proof. Since Φ satisfies Δ_2 condition, by the equation (2.0.3), $L^{\Phi}(\mu, X)^* = L^{\Phi^*}(X^*, X)$ and since X is reflexive, by [[5], cor.13, p.76], X^* has RNP, so by the equation (2.0.4), $L^{\Phi^*}(X^*, X) = L^{\Phi^*}(\mu, X^*)$.
Let $h \in L^{\Phi^*}(\mu, X^*)$ be arbitrary. Therefore by given hypothesis a)of the lemma the set $\{\tilde{f_n} \cdot \tilde{h};\ n \in \mathbb{N}\}$ is uniformly integrable and the sequence $\{\langle f_n(\cdot), h(\cdot)\rangle$ in L^1 converges pointwise almost surely to the function $\{\langle f(\cdot), h(\cdot)\rangle$. By Vitali's convergence theorem in L^1, we have,

$$\lim_{n\to\infty} \int_\Omega \langle f_n(\omega), h(\omega)\rangle d\mu = \int_\Omega \langle f(\cdot), h(\cdot)\rangle d\mu.$$

Hence $\{f_n\}$ converges weakly to $f \in L^{\Phi}(\mu, X)$. □

Lemma 3.3.2. *Let $\{f_n\}$ be a sequence in $L^{\Phi}(\mu, X)$. If $\{f_n\}$ be weakly convergent to $f \in L^{\Phi}(\mu, X)$ and $\{f_n(\omega)\}$ be weakly Cauchy almost surely a.e. ω in Ω, then $\{f_n\}$ converges almost surely to f.*

Proof. Since Φ satisfies Δ_2 condition, by the equation (2.0.3), $L^{\Phi}(\mu, X)^* = L^{\Phi^*}(X^*, X)$ and since X^* has RNP, by the equation (2.0.4), $L^{\Phi^*}(X^*, X) = L^{\Phi^*}(\mu, X^*)$.
Let E_1 be the negligible set in Σ such that $\mu(E) = 0$ and $\{f_n(\omega)\}$ is weakly Cauchy for all $\omega \in \Omega \setminus E$. For $x^* \in X^*$ and $\omega \in \Omega \setminus E_1$, let

Equation 3.3.3. $g_{x^*}(\omega) = \lim_{n\to\infty} \langle x^*, f_n(\omega)\rangle.$

Since the $\{f_n\}$ converges weakly to $f \in L^{\Phi}(\mu, X)$, by Mazur's Theorem, there exists a sequence $\{f_n^{'}\}$, with $f_n^{'} \in co\{f_k;\ k \geq n\}$, such that $\{f_n^{'}\}$ converges in norm to f in $L^{\Phi}(\mu, X)$.

Equation 3.3.4. $\lim_{n\to\infty} \|f_n^{'} - f\|_\Phi = 0.$

Now every $\{f_n^{'}\}$ is of the form

$$f_n^{'} = \Sigma_{l=0}^{m_n} \alpha_l f_{n+l}$$

, with $\alpha_l \geq 0$ and $\Sigma_{l=0}^{m_n} \alpha_l = 1$.
Using(3.3.3) and expression of each $\{f_n^{'}\}$, it can be easily shown that for each $\omega \in \Omega \setminus E_1$ and $x^* \in X^*$,

Equation 3.3.5. $\lim_{n\to\infty} \langle x^*, f_n^{'}(\omega)\rangle = g_{x^*}(\omega).$

Now by (3.3.4), there exists a set $E_2 \in \Sigma$ and integers $n_1 < n_2 \cdots < n_k < \cdots$ such that, for each $\omega \in \Omega \setminus E_2$, $\|f_{n_k}^{'}(\omega) - f(\omega)\|_X \to 0$ as $k \to \infty$ which further implies that for each $\omega \in \Omega \setminus E_2$, $f_{n_k}^{'}(\omega)$ converges weakly to $f(\omega)$ in X i.e. for each $\omega \in \Omega \setminus E_2$ and for each $x^* \in X^*$,

Equation 3.3.6. $\lim_{n\to\infty} \langle x^*, f_n^{'}(\omega)\rangle = \langle x^*, f(\omega)\rangle.$

Therefore by (3.3.3) and (3.3.5) and (3.3.6), we have for all $\omega \in \Omega \setminus (E_1 \cup E_2)$

$$\lim_{n\to\infty} \langle x^*, f_n(\omega)\rangle = g_{x^*}(\omega) = \lim_{n\to\infty} \langle x^*, f_n^{'}(\omega)\rangle = \langle x^*, f(\omega)\rangle.$$

That is $f_n(\omega)$ converges weakly to $f(\omega)$ in X for all $\omega \in \Omega \setminus (E_1 \cup E_2)$. Again $\mu(E_1 \cup E_2) = 0$. Hence $f_n(\omega)$ converges weakly to $f(\omega)$ in X a.e.$\omega \in \Omega$. □

References

[1] F.G.Bombal, Sobre los espacios de Orlicz de functioes vectoriales, Collect. Math., 32(1981),1-42.

[2] N.D. Chakraborty, T.Choudhury, Convergence Theorems for Pettis Integrable Functions and Regular Methods of Summability. J. Math. Anal. Appl., 359 (2009), 95-105.

[3] N.D. Chakraborty, T.Choudhury, On Some Properties of Pettis Integrable Multifunction,J. of Convex Analysis, 19(2012),No.3, 671-683.

[4] S.Diaz, Weak compactness in $L^1(\mu, X)$, Proc. Amer. Math. Soc. 124(9)(1996), 2685-2693.

[5] J.Diestel, J.J.Uhl Jr., Vector Measures, Math. Surveys Monogr. vol. 15, Amer. Math. Soc. Providence, RI, 1977 -116.

[6] L.V. Kantorovich, G.P.Akilov, Functional Analysis, Nauka, Moscow(3rd ed. in Russian)(1984).

[7] M.A. Krasnoselškii, Ya. Ruteckii, Convex functions and orlicz spaces, Noordhoff, Groningen(1961).

[8] J.Diestel, W. M. Ruess, W. Schachermayer, Weak compactness in $L^1(\mu, X)$, Proc. Amer. Math. Soc.118(2)(1993), 447-453.

[9] W.A.Luxemberg, Function spaces, Delft(1955)

[10] W.A.Luxemberg, A.C.Zaanen, Compactness of integral operators in Banach function spaces, Math. Annalen, 149(1963),150-180.

[11] M.Nowak, Mixed topology on normed function spaces I, Bull. Pol. Acad. Sci. 36(5-6)(1988), 251-262.

[12] M.Nowak, Order continuous seminorms and weak compactness in Orliczs spaces, Collect. Math., 44(1993), 217-236.

[13] M.Nowak, Duality theory of vector-valued function spaces I, Bull. Pol. Acad. Sci. 37(1997), 195-215.

[14] M.Nowak, Weak compactness in Köthe-Bochner spaces and regular mettods of summability, Bull. Pol. Acad. Sci. 50(4)(2002), 417-425.

[15] M.Nowak, Weak compactness in Köthe-Bochner spaces and Orlicz-Bochner spaces. Indag. Mathem., N.S., 10(1), 73-76.

[16] M.Nowak, Integration in Orlicz-Bochner Spaces, J.Function spaces, Volume 2018, Article ID 9380350

[17] M.Nowak, Continuous linear operators on Orlicz-Bochner spaces, Open Mathematics, 2019; 17: 1147-1155

[18] M.M.Rao, Z.D.Ren, Theory of Orlicz spaces, Marcel Dekker, New York, Basel, Hong Kong(1991).

[19] A. Ülger, Weak Compactness in $L^1(\mu, X)$, Proc. Amer. Math. Soc. ,113 (1991), 143-149.

[20] H.W.Young, On classes of summable functions and their Fourier series, Procceedings of the Royal Society of London, 87(1912), 225-229.

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