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GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF THE $2k$-VARIABLE ADDITIVE FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES AND BANACH SPACES

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Abstract. In this paper we use the direct method to proved two the generalized additive functional inequalities with 2k-variables and their Hyers-Ulam-Rassias stability. First are investigated in Banach spaces and the last are investigated in non-Archimedean Banach spaces. We will show that the solutions of the inequalities are additive mappings. These are the main results of this paper.

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1. Introduction

Let $\mathbf{X_1}$ and $\mathbf{X_2}$ be a normed spaces on the same field $\mathbb{K}$, and $F:\mathbf{X_1}\to\mathbf{X_2}$. We use the notation $\Big\|\cdot\Big\|$ for all the norm on both $\mathbf{X_1}$ and $\mathbf{X_2}$. In this paper, we investisgate some additive functional inequality when $\mathbf{X_1}$ and $\mathbf{X_2}$ is a Banach spaces or $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach space.
In fact, when $\mathbf{X_1}$ and $\mathbf{X_2}$ is Banach spaces we solve and prove the Hyers-Ulam-Rassias type stability of forllowing additive functional inequality.

$$\Bigg\|F\bigg(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\bigg)-\sum_{j=1}^{k}F\bigg(\frac{x_{k+j}}{k}\bigg)-\sum_{j=1}^{k}F\Big(x_j\Big)\Bigg\|_{\mathbf{X_2}}$$
$$\leq\Bigg\|F\bigg(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\bigg)-\frac{1}{k}\sum_{j=1}^{k}F\bigg(\frac{x_{k+j}}{k}\bigg)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\bigg)\Bigg\|_{\mathbf{X_2}} \quad (1.1)$$

and when $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach spaces we solve and prove the Hyers-Ulam stability of forllowing additive functional inequality.

$$\Bigg\|F\bigg(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\bigg)-\frac{1}{k}\sum_{j=1}^{k}F\bigg(\frac{x_{k+j}}{k}\bigg)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\Bigg\|_{\mathbf{X_2}}$$
$$\leq\Bigg\|F\bigg(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\bigg)-\sum_{j=1}^{k}F\bigg(\frac{x_{k+1}}{k}\bigg)-\sum_{j=1}^{k}F\Big(x_j\Big)\Bigg\|_{\mathbf{X_2}}, \quad (1.2)$$

The study of the functional equation stability originated from a question of S.M. Ulam, concerning the stability of group homomorphisms. Let $\left(\mathbf{G}, *\right)$ be a group and let $\left(\mathbf{G}', \circ, d\right)$ be a metric group with metric $d\left(\cdot,\cdot\right)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbf{G} \to \mathbf{G}'$ satisfies

$$d\left(f\left(x * y\right), f\left(x\right) \circ f\left(y\right)\right) < \delta$$

for all $x, y \in \mathbf{G}$ then there is a homomorphism $h : \mathbf{G} \to \mathbf{G}'$ with

$$d\left(f\left(x\right), h\left(x\right)\right) < \epsilon$$

for all $x \in \mathbf{G}$?, if the answer, is affirmative, we would say that equation of homomophism $h\left(x * y\right) = h\left(y\right) \circ h\left(y\right)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers [10] gave a first affirmative answes the question of Ulam as follows:

Theorem 1.1. *(D. H. Hyers 1941) Let $\epsilon \geq 0$ and let f be a function where $\mathbf{X}$ and $\mathbf{Y}$ are Banach space, such that*

$$\left\|f\left(x+y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \epsilon,$$

for all $x, y \in \mathbf{X}$ and some $\epsilon \geq 0$. Then there exists a unique additive mapping $T : \mathbf{X} \to \mathbf{Y}$, satisfying

$$\left\|f\left(x\right) - T\left(x\right)\right\| \leq \epsilon, \forall x \in \mathbf{X}.$$

Next Th. M. Rassias [19] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded:

Theorem 1.2. *(Th. M. Rassias.) Consider $\mathbf{E}, \mathbf{E}'$ to be two Banach spaces, and let $f : \mathbf{E} \to \mathbf{E}'$ be a mapping such that $f\left(tx\right)$ is continous in t for each fixed x. Assume that there exist $\theta > 0$ and $p \in [0, 1]$ such that*

$$\left\|f\left(x+y\right) - f\left(x\right) - f\left(y\right)\right\| \leq \epsilon\left(\left\|x\right\|^p + \left\|y\right\|^p\right), \forall x, y \in \mathbb{E}.$$

then there exists a unique linear $L : \mathbf{E} \to \mathbf{E}'$ satifies

$$\left\|f\left(x\right) - L\left(x\right)\right\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p, x \in \mathbf{E}.$$

After that, Hyers' Theorem was generalized by Aoki[1] additive mappings and by Rassias [19] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The Hyers-Ulam stability for functional inequalities have been investigated such as in [6, 20], Gilány showed that is if satisfies the functional inequality

$$\left\|2f\left(x\right) + 2f\left(y\right) - f\left(xy^{-1}\right)\right\| \leq \left\|f\left(xy\right)\right\|$$

then f satisfies the Jordan-von Newman functional equation

$$2f\Big(x\Big)+2f\Big(y\Big)=f\Big(xy\Big)+f\Big(xy^{-1}\Big). \tag{1.3}$$

Gil$\acute{a}$nyi [9] and Fechner [6] proved the Hyers-Ulam-Rassia stability of the functional inequality.

Choonkil $Park$ [15] obtained the solutions of the additive functional inequality. Recently, in [2, 5, 15] the authors studied the Hyers-Ulam-Rassia stability for the following functional inequalities in Banach space and non-Archimedean Banach space:

$$\left\|f\Big(x+y\Big)-f\Big(x\Big)-f(y)\right\|\le\left\|f\left(\frac{x+y}{2}\right)-\frac{1}{2}f\Big(x\Big)-\frac{1}{2}f\Big(y\Big)\right\| \tag{1.4}$$

and

$$\left\|f\left(\frac{x+y}{2}\right)-\frac{1}{2}f\Big(x\Big)-\frac{1}{2}f\Big(y\Big)\right\|\le\left\|f\Big(x+y\Big)-f\Big(x\Big)-f\Big(y\Big)\right\|. \tag{1.5}$$

Next

$$\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f\Big(z\Big)\right\|\le\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f\Big(z\Big)\right\| \tag{1.6}$$

and

$$\left\|f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{2}f\Big(z\Big)\right\|\le\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f\Big(z\Big)\right\|. \tag{1.7}$$

Final

$$\left\|f\left(\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}f\Big(x_i\Big)\right\|\le\left\|f\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)-\frac{1}{n}\sum_{i=1}^{n}f\Big(x_i\Big)\right\| \tag{1.8}$$

and

$$\left\|f\left(\frac{1}{n}\sum_{i=1}^{k}x_i\right)-\frac{1}{n}\sum_{i=1}^{n}f\Big(x_n\Big)\right\|\le\left\|f\Big(\sum_{i=1}^{n}x_i\Big)-\sum_{i=1}^{n}f\Big(x_i\Big)\right\| \tag{1.9}$$

In this paper, we solve and proved the Hyers-Ulam-Rassias type stability for two additive functional inequalities (1.1)-(1.2), ie the additive functional inequalities with $2k-variables$. Under suitable assumptions on spaces $\mathbf{X_1}$ and $\mathbf{X_2}$, we will prove that the mappings satisfying the additive functional inequatilies (1.1) or (1.2). Thus, the results in this paper are generalization of those in [2, 5, 15, 16, 17, 18] for functional inequatilies with $2k-variables$.

The paper is organized as followns:

In section preliminaries we remind some basic notations in [11, 13, 15, 16, 17, 18] such as We only redefine the solution definition of the equation of the additive function.

Section 3: is devoted to prove the Hyers-Ulam stability of the additive functional inequalities (1.1) when we assume that $\mathbf{X_1}$ and $\mathbf{X_2}$ is a Banach spaces.

Section 4: is devoted to prove the Hyers-Ulam stability of the addive additive functional inequalities (1.2) when $\mathbf{X_1}$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach space.

2. PRELIMINARIES

2.1. **non-Archimedean normed spaces.** In this subsection we recall some basic notations from [12, 15] such as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces.

A valuation is a function $\Big|\cdot\Big|$ from a field $\mathbb{K}$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation,

$$\Big|r\Big| = 0 \Leftrightarrow r = 0$$

$$\Big|r\cdot s\Big| := \Big|r\Big|\ |s|, \forall r,s \in \mathbb{K}$$

and the triangle inequality holds, i.e.,

$$\Big|r+s\Big| \leq \Big|r\Big| + \Big|s\Big|, \forall r,s \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the strong triangle inequality is replaced by

$$\Big|r+s\Big| \leq max\Big\{\Big|r\Big|, \Big|s\Big|\Big\}, \forall r,s \in \mathbb{K},$$

then the function $\Big|\cdot\Big|$ is called a non-Archimedean valuation. Clearly, $\Big|1\Big| = \Big|-1\Big| = 1$ and $\Big|n\Big| \leq 1, \forall n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $\Big|\cdot\Big|$ talking everything except for 0 into 1 and $\Big|0\Big| = 0$. In this paper, we assume that the base field is a non-Archimedean field with $\Big|2\Big| \neq 1$, hence call it simply a field.

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Definition 2.1. Let be a vecter space over a filed K with a non -Archimedean $\Big|\cdot\Big|$. A function $\Big\|\cdot\Big\| : X \to \Big[0,\infty\Big)$ is said a non -Archimedean norm if it satisfies the follwing conditions:

(1) $\Big\|x\Big\| = 0$ if and only if $x = 0$;

(2) $\Big\|rx\Big\| = \Big|r\Big|\Big\|x\Big\|(r \in \mathbb{K}, x \in X)$;

(3) $\Big\|x+y\Big\| \leq max\Big\{\Big\|x\Big\|, \Big\|y\Big\|\Big\}x, y \in X$ hold.

Then $\Big(X, \Big\|\cdot\Big\|\Big)$ is called a norm -Archimedean norm space.

Definition 2.2.

A sequence $\Big\{x_n\Big\}$ in a norm -Archimedean $\Big(n,\beta\Big)$-normed space **X** is a Cauchy sequence if and only if $\Big\{x_n - x_m\Big\} \to 0$.

Definition 2.3. Let $\Big\{x_n\Big\}$ be a sequence in a norm -Archimedean normed space X.

(1) A sequence $\Big\{x_n\Big\}_{n=1}^{\infty}$ in a non -Archimedean space is a Cauchy sequence iff the sequence $\Big\{x_{n+1} - x_n\Big\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\left\{x_n\right\}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in X$ such that

$$\left\|x_n - x\right\| \leq \epsilon. \forall n \geq N,$$

for all n,m $\geq N$. The we call x$\in X$ a limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in X converger, then the norm -Archimedean normed space X is called a norm -Archimedean Bnanch space.

2.2. **Solutions of the equation.** The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an *additive mapping*.

3. Additiver functional inequality in Banach space

Now, we study the solutions of (1.1). Note that for these inequalitie, $\mathbf{X}_1$and $\mathbf{X_2}$ is a Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Lemma 3.1. *A mapping $F : \mathbf{X}_1 \to \mathbf{X}_2$ satilies*

$$\left\|F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X}_2}$$
$$\leq \left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(x_j\right)\right)\right\|_{\mathbf{X_2}} \tag{3.1}$$

for all $x_j, x_{k+j} \in \mathbf{X}_1$ for all $j = 1 \to k$ if and only if $F : \mathbf{X}_1 \to \mathbf{X}_2$ is additive.

Proof. Assume that $F : \mathbf{X}_1 \to \mathbf{X}_2$ satisfies (3.1).
Letting $x_j = x_{k+j} = 0, j = 1 \to k$ in (3.1), we get

$$\left(\left|2k-1\right| - 1\right)\left\|F\left(0\right)\right\|_{\mathbf{X}_2} \leq 0.$$

So $F\left(0\right) = 0$.
Thus

Letting $x_{k+j} = 0$ and $x_j = x$ for all $j = 1 \to k$ in (3.1), we get

$$\left\|F\left(kx\right) - kF\left(x\right)\right\|_{\mathbf{X_2}} \leq 0 \tag{3.2}$$

and so F$\left(\text{kx}\right)$ = kF$\left(\text{x}\right)$ for all $x \in \mathbf{X_1}$.
Thus

$$F\left(\frac{x}{k}\right) = \frac{1}{k}F(x) \tag{3.3}$$

for all $x \in \mathbf{X_1}$ It follows from (3.1) and (3.3) that:

$$\begin{aligned} &\left\|F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F(x_j)\right\|_{\mathbf{X_2}} \\ &\qquad \leq \left\|\frac{1}{k}F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+1}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} F(x_j)\right)\right\|_{\mathbf{X_2}} \\ &\qquad = \frac{1}{k}\left\|F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+1} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F(x_j)\right)\right\|_{\mathbf{X_2}} \end{aligned} \tag{3.4}$$

and so

$$F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) = \sum_{j=1}^{k} F\left(\frac{x_{k+1}}{k}\right) + \sum_{j=1}^{k} F\left(x_j\right)$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$ for all $j = 1 \to k$. Hence $F : \mathbf{X_1} \to \mathbf{X_2}$ is additive.
The coverse is obviously true. □

Theorem 3.2. *Let $\varphi : \mathbf{X_1^{2k}} \to [0, \infty)$ be a function and let $F : \mathbf{X}_1 \to \mathbf{X}_2$ be mapping such that*

$$\varphi\left(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\right) = \sum_{j=1}^{\infty} k^j \psi\left(\frac{x_1}{k^j}, \frac{x_2}{k^j}, ..., \frac{x_k}{k^j}, \frac{x_{k+1}}{k^j}, \frac{x_{k+2}}{k^j}, ..., \frac{x_{2k}}{k^j}\right) < \infty \tag{3.5}$$

$$\begin{aligned} &\left\|F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X_2}} \\ &\qquad \leq \left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X_2}} \\ &\qquad + \psi\left(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\right) \end{aligned} \tag{3.6}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : \mathbf{X}_1 \to \mathbf{X}_2$ such that

$$\left\|F\left(x\right) - Q\left(x\right)\right\|_{\mathbf{X_2}} \leq \frac{1}{k}\varphi\left(x, x, ..., x, 0, 0..., 0\right) \tag{3.7}$$

for all $x \in \mathbf{X_1}$.

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (3.6), we get

$$\left(\left|2k - 1\right| - 1\right)\left\|F\left(0\right)\right\|_{\mathbf{X_2}} \leq 0. \tag{3.8}$$

So

$$F\left(0\right) = 0.$$

Letting $x_{k+j} = 0$, $x_j = x$ for all $j = 1 \to k$ in (3.6), we get

$$\left\|F\left(kx\right) - kF\left(x\right)\right\|_{\mathbf{X_2}} \leq \frac{1}{k}\psi\left(x, x, ..., x, 0, 0, ..., 0\right) \tag{3.9}$$

$$\left\|F\left(x\right)-kF\left(\frac{x}{k}\right)\right\|_{\mathbf{X_2}}\le\frac{1}{k}\psi\Big(\frac{x}{k},\frac{x}{k},...,\frac{x}{k},0,0,...,0\Big)$$

Hence

$$\begin{aligned}&\left\|k^lF\left(\frac{x}{k^l}\right)-k^mF\left(\frac{x}{k^m}\right)\right\|_{\mathbf{X_2}}\\&\le\sum_{j=l}^{m-1}\left\|k^jF\left(\frac{x}{k^j}\right)-k^{j+1}F\left(\frac{x}{k^{j+1}}\right)\right\|_{\mathbf{X_2}}\\&\le\frac{1}{k}\sum_{j=l+1}^{m}k^j\psi\Big(\frac{x}{k^j},\frac{x}{k^j},...,\frac{x}{k^j},0,0,...,0\Big)\end{aligned}\tag{3.10}$$

for all nonnegative integers m and l with $m>l$ and all $x\in\mathbf{X_1}$. It follows from (3.10) that the sequence $\left\{k^nF\left(\frac{x}{k^n}\right)\right\}$ is a cauchy sequence for all $x\in\mathbf{X_1}$. Since $\mathbf{X_2}$ is complete space, the sequence $\left\{k^nF\left(\frac{x}{k^n}\right)\right\}$ coverges.
So one can define the mapping $Q:\mathbf{X}_1\to\mathbf{X}_2$ by

$$Q\big(x\big):=\lim_{n\to\infty}k^nf\left(\frac{x}{k^n}\right)$$

for all $x\in\mathbf{X_1}$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (3.10), we get (3.7).
Now, It follows from (3.5) and (3.6) that

$$\begin{aligned}&\left\|Q\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}Q\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}Q\big(x_j\big)\right\|_{\mathbf{X_2}}\\&=\lim_{n\to\infty}k^n\left\|F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+1}}+\frac{1}{k^n}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k^{n+1}}\right)-\sum_{j=1}^{k}F\Big(\frac{x_j}{k^n}\Big)\right\|_{\mathbf{X_2}}\\&\le\lim_{n\to\infty}k^n\left\|F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{n+2}}+\frac{1}{k^n}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k^{n+1}}\right)-\frac{1}{k}\sum_{j=1}^{k}F\Big(\frac{x_j}{k^n}\Big)\right\|_{\mathbf{X_2}}\\&+\lim_{n\to\infty}k^n\psi\Big(\frac{x_1}{k^j},\frac{x_2}{k^j},...,\frac{x_k}{k^j},\frac{x_{k+1}}{k^j},\frac{x_{k+2}}{k^j},...,\frac{x_{2k}}{k^j}\Big)\\&=\left\|F\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\big(x_j\big)\right\|_{\mathbf{X_2}}\end{aligned}\tag{3.11}$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$, for all $j=1\to k$.
So

$$\begin{aligned}&\left\|Q\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}Q\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}Q\big(x_j\big)\right\|_{\mathbf{X_2}}\\&\le\left\|Q\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}Q\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}Q\big(x_j\big)\right\|_{\mathbf{X_2}}\end{aligned}\tag{3.12}$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$, for all $j=1\to k$. By Lemma (3.1), the mapping $Q:\mathbf{X_1}\to\mathbf{X_2}$ is additive.

Next, suppose that $T : \mathbf{X_1} \to \mathbf{X_2}$ be another additive mapping satisfying (3.7). Then we have

$$\begin{aligned}
\left\|Q\left(x\right) - T\left(x\right)\right\|_{\mathbb{Y}} &= k^n\left\|Q\left(\frac{x}{k^n}\right) - T\left(\frac{x}{k^n}\right)\right\|_{\mathbb{Y}} \\
&\leq k^n\left(\left\|Q\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|_{\mathbb{Y}} + \left\|T\left(\frac{x}{k^n}\right) - f\left(\frac{x}{k^n}\right)\right\|_{\mathbb{Y}}\right) \\
&\leq k^n\Big(\frac{1}{k^n}\varphi(\frac{x}{k^n}, \frac{x}{k^n}, ..., \frac{x}{k^n}, 0, 0, ..., 0) + \frac{1}{k^n}\varphi(\frac{x}{k^n}, \frac{x}{k^n}, ..., \frac{x}{k^n}, 0, 0, ..., 0)\Big) \\
&= k^n \cdot \frac{2}{k}\varphi(\frac{x}{k^n}, \frac{x}{k^n}, ..., \frac{x}{k^n}, 0, 0, ..., 0) \\
&\leq k^n\varphi(\frac{x}{k^n}, \frac{x}{k^n}, ..., \frac{x}{k^n}, 0, 0, ..., 0) \qquad (3.13)
\end{aligned}$$

which tends to zero as$n \to \infty$ for all $x \in \mathbf{X_1}$. So we can conclude that $Q\left(x\right) = T\left(x\right)$ for all $x \in \mathbf{X_1}$. This proves the uniqueness of Q. Thus the mapping $Q : \mathbf{X_1} \to \mathbf{X_2}$ is a unique additive mapping satisfying (3.7). □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers and $F : \mathbf{X_1} \to \mathbf{X_2}$ be a mapping satisfying*

$$\begin{aligned}
&\left\|F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X_2}} \\
&\leq \left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X_2}} \\
&+ \theta\left(\sum_{j=1}^{k}\left\|x_i\right\|_{\mathbf{X}_1}^r + \sum_{j=1}^{k}\left\|x_{k+i}\right\|_{\mathbf{X}_1}^r\right) \qquad (3.14)
\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : \mathbf{X}_1 \to \mathbf{X}_2$ such that

$$\left\|F\left(x\right) - Q\left(x\right)\right\|_{\mathbf{X}_2} \leq \frac{2k\theta}{k^r - k}\left\|x\right\|^r \qquad (3.15)$$

for all $x \in \mathbf{X_1}$

Theorem 3.4. *Let $\varphi : \mathbf{X_1^{2k}} \to [0, \infty)$ be a function and let $F : \mathbf{X}_1 \to \mathbf{X}_2$ be mapping such that*

$$\begin{aligned}
&\varphi\Big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\Big) \\
&= \sum_{j=1}^{\infty}\frac{1}{k^j}\psi\Big(k^j x_1, k^j x_2, ..., k^j x_k, k^j x_{k+1}, k^j x_{k+2}, ..., k^j x_{2k}\Big) < \infty \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}&\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X_2}}\\&\qquad\le\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X_2}}\\&\qquad+\psi\left(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\right)\end{aligned}\tag{3.17}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $Q : \mathbf{X}_1 \to \mathbf{X}_2$ *such that*

$$\left\|F\left(x\right)-Q\left(x\right)\right\|_{\mathbf{X_2}}\le\frac{1}{k}\varphi\left(x,x,...,x,0,0,...,0\right)\tag{3.18}$$

, for all $x \in \mathbf{X_1}$.

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (3.17), we get

$$\left(\left|2k-1\right|-1\right)\left\|F\left(0\right)\right\|_{\mathbf{X_2}}\le 0.\tag{3.19}$$

So

$$F\left(0\right)=0.$$

Letting $x_{k+j} = 0$, $x_j = x$ for all $j = 1 \to k$ in (3.17), we get

$$\left\|F\left(kx\right)-kF\left(x\right)\right\|_{\mathbf{X_2}}\le\psi\left(x,x,...,x,0,0,...,0\right)\tag{3.20}$$

thus

$$\left\|F\left(x\right)-\frac{1}{k}F\left(kx\right)\right\|_{\mathbf{X_2}}\le\frac{1}{k}\psi\left(x,x,...,x,0,0,...,0\right)$$

Hence

$$\begin{aligned}&\left\|\frac{1}{k^l}F\left(k^lx\right)-\frac{1}{k^m}F\left(k^mx\right)\right\|_{\mathbf{X_2}}\\&\qquad\le\sum_{j=l}^{m-1}\left\|\frac{1}{k^j}F\left(k^jx\right)-\frac{1}{k^{j+1}}F\left(k^{j+1}x\right)\right\|_{\mathbf{X_2}}\\&\qquad\le\frac{1}{k}\sum_{j=l}^{m}\frac{1}{k^{j+1}}\psi\left(k^jx,k^jx,...,k^jx,0,0,...,0\right)\end{aligned}\tag{3.21}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X_1}$. It follows from (3.21) that the sequence $\left\{\frac{1}{k^n}F\left(k^nx\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X_1}$. Since $\mathbf{X_2}$ is complete space, the sequence $\left\{\frac{1}{k^n}F\left(k^nx\right)\right\}$ coverges.
So one can define the mapping $Q : \mathbf{X}_1 \to \mathbf{X}_2$ by

$$Q\left(x\right):=\lim_{n\to\infty}\frac{1}{k^n}F\left(k^nx\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.21), we get (3.18).
We use the similar manner to the proof of Theorem 3.2 for the rest of the proof.

$\square$

Corollary 3.5. *Let $r < 1$ and θ be nonnegative real numbers and $F : \mathbf{X_1} \to \mathbf{X_2}$ be a mapping satisfying*

$$\begin{aligned}\Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\Big)-\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\sum_{j=1}^{k}F\Big(x_j\Big)\Big\|_{\mathbf{X_2}}\\ \leq\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\Big)-\frac{1}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\Big\|_{\mathbf{X_2}}\\ +\theta\Big(\sum_{j=1}^{k}\Big\|x_i\Big\|^r+\sum_{j=1}^{k}\Big\|x_{k+i}\Big\|^r\Big)\end{aligned}\tag{3.22}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : \mathbf{X}_1 \to \mathbf{X}_2$ such that

$$\Big\|F\Big(x\Big)-Q\Big(x\Big)\Big\|_{\mathbf{X_2}}\leq\frac{2k\theta}{k-k^r}\|x\|^r\tag{3.23}$$

for all $x \in \mathbf{X_1}$

4. Additive functional inequality in non-Archimedean Banach space

Now, we study the solutions of (1.2). Note that for these inequalitie, $\mathbf{X}_1$ is a non-Archimedean normed space and $\mathbf{X_2}$ is a non-Archimedean Banach spaces. Under this setting, we can show that the mapping satisfying (1.2) is additive. These results are give in the following. Assume that where k is a fixed positive integer with $|k| \neq 1$.

Lemma 4.1. *A mapping $F : \mathbf{X}_1 \to \mathbf{X}_2$ satilies $F(0) = 0$*

$$\begin{aligned}\Big\|F\Big(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\Big)-\frac{1}{k}\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\Big)\Big\|_{\mathbf{X_2}}\\ \leq\Big\|F\Big(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\Big)-\sum_{j=1}^{k}F\Big(\frac{x_{k+j}}{k}\Big)-\sum_{j=1}^{k}F\Big(x_j\Big)\Big\|_{\mathbf{X_2}}\end{aligned}\tag{4.1}$$

for all $x_j, x_{k+j} \in \mathbf{X}_1$ for all $j = 1 \to k$ if and only if $F : \mathbf{X}_1 \to \mathbf{X}_2$ is additive.

Proof. Assume that $F : \mathbf{X}_1 \to \mathbf{X}_2$ satisfies (4.1).
Letting $x_1 = x$, $x_{j+1} = x_{k+j} = 0, j = 1 \to k$ in (3.1), we obtain

$$\Big\|F\Big(\frac{x}{k}\Big)-\frac{1}{k}F\Big(x\Big)\Big\|_{\mathbf{X_2}}\leq 0$$

and so $F\Big(\frac{x}{k}\Big)=\frac{1}{k}F\Big(x\Big)$Thus

$$\begin{aligned}&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\right)\right\|_{\mathbf{X_2}}\\
&\qquad=\left\|F\left(\frac{1}{k}\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\right)\right\|_{\mathbf{X_2}}\\
&\qquad=\Big|\frac{1}{k}\Big|\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\Big(x_j\Big)\right)\right\|_{\mathbf{X_2}}\\
&\qquad\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\Big(x_j\Big)\right\|_{\mathbf{X_2}}\end{aligned}\tag{4.2}$$

for all $x_j, x_{k+j}\in\mathbf{X}_1$ for all $j=1\to k$. Since $\Big|k\Big|<1$

$$F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\Big(x_j\Big)=0$$

for all $x_j, x_{k+j}\in\mathbf{X}_1$ for all $j=1\to k$. On the other hand the converse is obviously true. □

Theorem 4.2. *Let $\varphi:\mathbf{X_1^{2k}}\to[0,\infty\Big)$ be a function and let $F:\mathbf{X}_1\to\mathbf{X}_2$ be mapping with $F(0)=0$ satisfying*

$$\varphi\Big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\Big)=\sum_{j=1}^{\infty}\big|k^j\big|\psi\big(\frac{x_1}{k^j},\frac{x_2}{k^j},...,\frac{x_k}{k^j},\frac{x_{k+1}}{k^j},\frac{x_{k+2}}{k^j},...,\frac{x_{2k}}{k^j}\big)<\infty\tag{4.3}$$

$$\begin{aligned}&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\Big(x_j\Big)\right\|_{\mathbf{X_2}}\\
&\qquad\leq\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\Big(x_j\Big)\right\|_{\mathbf{X_2}}\\
&\qquad+\psi\Big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\Big)\end{aligned}\tag{4.4}$$

for all $x_j, x_{k+j}\in\mathbf{X_1}$, for all $j=1\to k$. Then there exists a unique additive mapping $H:\mathbf{X}_1\to\mathbf{X}_2$ such that

$$\Big\|F\Big(x\Big)-H\Big(x\Big)\Big\|_{\mathbf{X_2}}\leq\big|k\big|\varphi\Big(x,0,...,0,0,..,0\Big)\tag{4.5}$$

for all $x\in\mathbf{X_1}$.

Proof. Letting $x_1=x$, $x_{j+1}=x_{k+j}=0$ for all $j=1\to k$ in (4.4), we get

$$\left\|F\Big(x\Big)-kF\Big(\frac{x}{k}\Big)\right\|_{\mathbf{X_2}}\leq\big|k\big|\psi\Big(x,0,...,0,0,0,...,0\Big)\tag{4.6}$$

for all $x \in \mathbf{X_1}$ Hence

$$\begin{aligned}
&\left\|k^l F\left(\frac{x}{k^l}\right) - k^m F\left(\frac{x}{k^m}\right)\right\|_{\mathbf{X_2}} \\
&\leq max\left\{\left\|k^l F\left(\frac{x}{k^l}\right) - k^{l+1} F\left(\frac{x}{k^{l+1}}\right)\right\|_{\mathbf{X_2}}, \cdots, \left\|k^{m-1} F\left(\frac{x}{k^{m-1}}\right) - k^m F\left(\frac{x}{k^m}\right)\right\|_{\mathbf{X_2}} \\
&\leq max\left\{|k|^l \left\|F\left(\frac{x}{k^j}\right) - kF\left(\frac{x}{k^{j+1}}\right)\right\|_{\mathbf{X_2}}, \cdots, |k|^{m-1}\left\|F\left(\frac{x}{k^{m-1}}\right) - kF\left(\frac{x}{k^m}\right)\right\|_{\mathbf{X_2}} \\
&\leq \sum_{j=l}^{\infty} |k|^{j+1} \psi\left(\frac{x}{k^j}, 0, ..., 0, 0, 0, ..., 0\right) \qquad (4.7)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbf{X_1}$. It follows from (4.7) that the sequence $\left\{k^n F\left(\frac{x}{k^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbf{X_1}$. Since $\mathbf{X_2}$ is complete space, the sequence $\left\{k^n F\left(\frac{x}{k^n}\right)\right\}$ coverges.
So one can define the mapping $H : \mathbf{X}_1 \to \mathbf{X}_2$ by

$$H(x) := \lim_{n\to\infty} k^n F\left(\frac{x}{k^n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.7), we get (4.5). Now, It follows from (4.3) and (4.4) that

$$\begin{aligned}
&\left\|H\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} H\left(x_j\right)\right\|_{\mathbf{X_2}} \\
&= \lim_{n\to\infty} |k|^n \left\|F\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^{n+2}} + \frac{1}{k^{n+1}}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k^{n+1}}\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_j}{k^n}\right)\right\|_{\mathbf{X_2}} \\
&\leq \lim_{n\to\infty} |k|^n \left\|F\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^{n+1}} + \frac{1}{k^n}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k^{n+1}}\right) - \sum_{j=1}^{k} F\left(\frac{x_j}{k^n}\right)\right\|_{\mathbf{X_2}} \\
&+ |\lim_{n\to\infty} |k|^n \psi\left(\frac{x_1}{k^n}, \frac{x_2}{k^n}, ..., \frac{x_k}{k^n}, \frac{x_{k+1}}{k^n}, \frac{x_{k+2}}{k^n}, ..., \frac{x_{2k}}{k^n}\right) \\
&= \left\|F\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X_2}} \qquad (4.8)
\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$.
So

$$\begin{aligned}
&\left\|H\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} H\left(x_j\right)\right\|_{\mathbf{X_2}} \leq \\
&\qquad \left\|H\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} H\left(x_j\right)\right\|_{\mathbf{X_2}} \qquad (4.9)
\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, for all $j = 1 \to k$. By Lemma 4.1, the mapping $H : \mathbf{X_1} \to \mathbf{X_2}$ is additive.
Next, suppose that $T : \mathbf{X_1} \to \mathbf{X_2}$ be another additive mapping satisfying (4.5). Then we

have

$$\begin{aligned}\left\|H\left(x\right)-T\left(x\right)\right\|_{\mathbf{X_2}} &= \left\|k^nH\left(\frac{x}{k^n}\right)-k^nT\left(\frac{x}{k^n}\right)\right\|_{\mathbf{X_2}}\\ &\le max\left\{\left\|k^nH\left(\frac{x}{k^n}\right)-k^nF\left(\frac{x}{k^n}\right)\right\|_{\mathbf{X_2}},\left\|k^nT\left(\frac{x}{k^n}\right)-k^nF\left(\frac{x}{k^n}\right)\right\|_{\mathbf{X_2}}\right\}\\ &\le \frac{1}{|k|}|k|^n\varphi(\frac{x}{k^n},0,...,0,0,...,0) \qquad (4.10)\end{aligned}$$

which tends to zero as$n\to\infty$ for all $x\in\mathbf{X_1}$. So we can conclude that $H\left(x\right)=T\left(x\right)$ for all $x\in\mathbf{X_1}$. This proves the uniqueness of H. Thus the mapping $H:\mathbf{X_1}\to\mathbf{X_2}$ is a unique additive mapping satisfying (4.5). □

Corollary 4.3. *Let $r<1$ and θ be nonnegative real numbers and $F:\mathbf{X_1}\to\mathbf{X_2}$ be a mapping with $F(0)=0$ satisfying*

$$\begin{aligned}&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X_2}}\\ &\le\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X_2}}\\ &+\theta\left(\sum_{j=1}^{k}\left\|x_i\right\|_{\mathbf{X}_1}^r+\sum_{j=1}^{k}\left\|x_{k+i}\right\|_{\mathbf{X}_1}^r\right) \qquad (4.11)\end{aligned}$$

for all $x_j,x_{k+j}\in\mathbf{X_1}$, for all $j=1\to k$. Then there exists a unique additive mapping $H:\mathbf{X}_1\to\mathbf{X}_2$ such that

$$\left\|F\left(x\right)-H\left(x\right)\right\|_{\mathbf{X}_2}\le\frac{|k|^{r+1}\theta}{|k|^r-|k|}\|x\|^r \qquad (4.12)$$

for all $x\in\mathbf{X_1}$

Theorem 4.4. *Let $\varphi:\mathbf{X_1^{2k}}\to[0,\infty)$ be a function and let $F:\mathbf{X}_1\to\mathbf{X}_2$ be mapping with $F(0)=0$ satisfying*

$$\begin{aligned}&\varphi\left(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\right)\\ &=\sum_{j=1}^{\infty}\frac{1}{|k^j|}\psi\left(k^jx_1,k^jx_2,...,k^jx_k,k^jx_{k+1},k^jx_{k+2},...,k^jx_{2k}\right)<\infty \qquad (4.13)\end{aligned}$$

$$\begin{aligned}&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X_2}}\\ &\le\left\|F\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}F\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}F\left(x_j\right)\right\|_{\mathbf{X_2}}\\ &+\psi\left(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\right) \qquad (4.14)\end{aligned}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $H : \mathbf{X}_1 \to \mathbf{X}_2$ *such that*

$$\left\|F\left(x\right) - H\left(x\right)\right\|_{\mathbf{X_2}} \leq |k|\psi\left(x, 0, ..., 0, 0, .., 0\right) \tag{4.15}$$

for all $x \in \mathbf{X_1}$.

Proof. Letting $x_1 = x$, $x_{j+1} = x_{k+j} = 0$ for all $j = 1 \to k$ in (4.14), we get

$$\left\|F\left(x\right) - \frac{1}{k}F\left(kx\right)\right\|_{\mathbf{X}_2} \leq \psi\left(kx, 0, ..., 0, 0, 0, ..., 0\right) \tag{4.16}$$

for all $x \in \mathbf{X_1}$ We use the similar manner to th □

Corollary 4.5. *Let* $r > 1$ *and* θ *be nonnegative real numbers and* $F : \mathbf{X_1} \to \mathbf{X_2}$ *be a mapping with* $F(0) = 0$ *satisfying*

$$\begin{aligned}
&\left\|F\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \frac{1}{k}\sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X}_2} \\
&\leq \left\|F\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} F\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} F\left(x_j\right)\right\|_{\mathbf{X}_2} \\
&+ \theta\left(\sum_{j=1}^{k}\left\|x_i\right\|_{\mathbf{X}_1}^{r} + \sum_{j=1}^{k}\left\|x_{k+i}\right\|_{\mathbf{X}_1}^{r}\right)
\end{aligned} \tag{4.17}$$

for all $x_j, x_{k+j} \in \mathbf{X_1}$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $H : \mathbf{X}_1 \to \mathbf{X}_2$ *such that*

$$\left\|F\left(x\right) - H\left(x\right)\right\|_{\mathbf{X}_2} \leq \frac{\left|k\right|^{r+1}\theta}{\left|k\right| - \left|k\right|^{r}}\left\|x\right\|^{r} \tag{4.18}$$

for all $x \in \mathbf{X_1}$

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