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On a Subclass of Analytic Functions Defined By Differential operator

D.Madhava Krishna, K.Narender, P.Thirupathi Reddy

[1]*Department of Mathematics, Government Degree College (W) (N), Asifabad - 504295, Kumuram Bheem, Dist.,Telangana, India.*
[2]*Department of Mathematics, Governament Junior College, Bela- 504001, Adilabad Dist., Telangana, India.*
[3]*Department of Mathematics, School of Engineering, - NNRESGI-500 088, Medichal Dist.,Telangana, India.*

Abstract - *In this paper , we introduce and study a new subclass of analytic functions which are defined by means of a differential operator. Some results connected to coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity and integral means inequalities related to the subclass is obtained.*

Keywords - *Analytic, coefficient, distortion, convexity, starlike, subordination.*

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I. INTRODUCTION

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $E = \{z \in C : |z| < 1\}$.

Denote by T the subclass of A consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0, \tag{1.2}$$

This subclass was introduced and extensively studied by Silverman [6].
Let f be a function in the class A. We define the following differential operator introduced by Denizand Ozkan [1].

$$D_\lambda^0 f(z) = f(z)$$
$$D_\lambda^1 f(z) = D_\lambda f(z) = \lambda z^3 (f(z))''' + (2\lambda + 1) z^2 (f(z))'' + z f'(z)$$
$$D_\lambda^2 f(z) = D_\lambda (D_\lambda^1 f(z))$$
.
.
.
$$D_\lambda^m f(z) = D_\lambda \ (D_\lambda^{m-1} f(z)) \tag{1.3}$$

where $\lambda \geq 0$ and $m \in N_0 = N \cup \{0\}$..

If f is given by (1.1) then from the definition of the operator $D_\lambda^m f(z)$ it is to see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \phi^m(\lambda, n) a_n z^n \tag{1.4}$$

where

$$\phi^m(\lambda, n) = n^{2m}[\lambda(n-1)+1]^m \tag{1.5}$$

If $f \in T$ is given by (1.2) then we have

$$D_\lambda^m f(z) = z - \sum_{n=2}^{\infty} \phi^m(\lambda, n) a_n z^n \tag{1.6}$$

Where $\phi^m(\lambda, n)$ is given by (1.5).

In this paper, using the differential operator $D_\lambda^m f(z)$, we define the following new class motivated by Niranjan et al [3].

Definition 1.1: The function f of the form (1.1) is in the class $S_\lambda^m(\gamma, k)$ if it satisfies the inequality

$$\operatorname{Re}\left\{ \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - \gamma \right\} > k \left| \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1 \right| \qquad \text{for} \quad 0 \le \gamma \le 1,\ k \ge 0.$$

Further, we define $TS_\lambda^m(\gamma, k) = S_\lambda^m(\gamma, k) \cap T$.

II. MAIN RESULTS

Theorem 2.1: A function f form (1.1) is in $S_\lambda^m(\gamma, k)$ if

$$\sum_{n=2}^{\infty} [n(1+k) - (\gamma + k)]\ \phi^m(\lambda, n) |a_n| \le 1 - \gamma, \tag{2.1}$$

where $0 \le \gamma < 1,\ k \ge 0$ and $\phi^m(\lambda, n)$ is given by (1.5).

Proof: It suffices to show that

$$k \left| \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1 \right\} \le 1 - \gamma.$$

We have

$$k \left| \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1 \right| - \operatorname{Re}\left\{ \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1 \right\}$$

$$\le (1+k) \left| \frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)} - 1 \right|$$

$$\leq \frac{(1+k)\sum_{n=2}^{\infty}(n-1)\phi^m(\lambda,n)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}\phi^m(\lambda,n)|a_n||z|^{n-1}}$$

$$\leq \frac{(1+k)\sum_{n=2}^{\infty}(n-1)\phi^m(\lambda,n)|a_n|}{1-\sum_{n=2}^{\infty}\phi^m(\lambda,n)|a_n|}$$

The last expression is bounded above by $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\ \phi^m(\lambda,n)|a_n| \leq 1-\gamma$$

and the proof is complete.

Theorem 2.2: Let $0 \leq \gamma < 1,\ k \geq 0$ then a function *f* of the form (1.2) to be in the class $TS_{\lambda}^{m}(\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\ \phi^m(\lambda,n) \leq 1-\gamma \tag{2.2}$$

where $\phi^m(\lambda,n)$ are given by (1.5).

Proof: In view of Theorem 2.1, we need only to prove the necessity. If $f \in TS_{\lambda}^{m}(\gamma,k)$ and *z* is real then

$$\mathrm{Re}\left\{\frac{1-\sum_{n=2}^{\infty}n\phi^m(\lambda,n)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\phi^m(\lambda,n)a_n z^{n-1}}-\gamma\right\} > k\left|\frac{\sum_{n=2}^{\infty}(n-1)\phi^m(\lambda,n)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\phi^m(\lambda,n)a_n z^{n-1}}\right|$$

Letting z$\rightarrow$ 1 along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\ \phi^m(\lambda,n)|a_n| \leq 1-\gamma,$$

where $0 \leq \gamma < 1,\ k \geq 0$ and $\phi^m(\lambda,n)$ are given by (1.5).

Corollary 2.3: If $f(z) \in T\underset{\lambda}{\overset{m}{S}}(\gamma,k)$, then

$$|a_n| \leq \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)} \tag{2.3}$$

where $0 \leq \gamma < 1,\ k \geq 0$ and $\phi^m(\lambda,n)$ are given by (1.5).Equality holds for the function

$$f(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)} z^n \tag{2.4}$$

Theorem 2.4: Let $f_1(z) = z$ and

$$f_n(z) = z - \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)} z^n\ ,\quad n \geq 2. \tag{2.5}$$

then $f(z) \in TS_{\lambda}^{m}(\gamma, k)$, if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} w_n f_n(z) \ , w_n \geq 0, \sum_{n=1}^{\infty} w_n = 1 \tag{2.6}$$

Proof: Suppose $f(z)$ can be written as in (2.6).Then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)} z^n$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{(1-\gamma)[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.$$

Thus $f(z) \in TS_{\lambda}^{m}(\gamma, k)$. Conversely, let us have $f(z) \in TS_{\lambda}^{m}(\gamma, k)$.
Then by using (2.3), we get

$$w_n = \frac{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{(1-\gamma)} a_n \ , \ n \geq 2$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$. then we have $f(z) = \sum_{n=1}^{\infty} w_n f_n(z)$ and hence this completes the proof of Theorem.

Theorem 2.5: The class $TS_{\lambda}^{m}(\gamma, k)$ is a convex set.

Proof: Let the function

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \ , \quad a_{n,j} \geq 0, \ \text{j=1, 2} \tag{2.7}$$

be in the class $TS_{\lambda}^{m}(\gamma, k)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi f_1(z) + (1-\xi) f_2(z) \ , \ 0 \leq \xi < 1,$$

is in the class $TS_{\lambda}^{m}(\gamma, k)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} \left[\xi a_{n,1} + (1-\xi) a_{n,2} \right] z^n,$$

an easy computation with the aid of Theorem 2.2 gives

$$\sum_{n=2}^{\infty} [n(1+k)-(\gamma+k)] \xi \phi^m(\lambda,n) a_{n,1} + \sum_{n=2}^{\infty} [n(1+k)-(\gamma+k)] (1-\xi) \phi^m(\lambda,n) a_{n,2} \leq \xi(1-\gamma) + (1-\xi)(1-\gamma)$$

$$\leq (1-\gamma),$$

which implies that $h \in TS_{\lambda}^{m}(\gamma, k)$.

Hence $TS_{\lambda}^{m}(\gamma,k)$ is convex.

Theorem 2.6: Let the function $f(z)$ defined by (1.2) belong to the class $TS_{\lambda}^{m}(\gamma,k)$ .Then is close-to-convex of order δ $(0\le\delta<1)$ in the disc $|z|<r_1$, where

$$r_1=\inf_{n\ge2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{n(1-\gamma)}\right]^{1/n-1}, \quad n\ge2. \tag{2.8}$$

The result is sharp, with the extremal function $f(z)$ is given by (1.2)

Proof: Given $f\in T$, and f is close-to-convex of order δ, we have

$$|f'(z)-1|<1-\delta. \tag{2.9}$$

For the left hand side of (2.9) we have

$$|f'(z)-1|\le\sum_{n=2}^{\infty}na_n|z|^{n-1}$$

The last expression is less than $1-\delta$

$$\sum_{n=2}^{\infty}\frac{n}{1-\delta}a_n|z|^{n-1}\le1.$$

Using the fact, that $f(z)\in TS_{\lambda}^{m}(\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{(1-\gamma)}a_n\le1,$$

Using the fact that (2.9) is true if

$$\frac{n}{1-\delta}|z|^{n-1}\le\frac{[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{(1-\gamma)}$$

or, equivalently,

$$|z|\le\left\{\frac{(1-\delta)[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{n(1-\gamma)}\right\}^{1/n-1}$$

which completes the proof.

Theorem 2.7: Let the function $f(z)$ defined by (1.2) belong to the class $TS_{\lambda}^{m}(\gamma,k)$. Then $f(z)$ is starlike of order δ $(0\le\delta<1)$ in the disc $|z|<r_2$, where

$$r_2=\inf_{n\ge2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)}{(n-\delta)(1-\gamma)}\right]^{1/n-1} \tag{2.10}$$

The result is sharp, with extremal function f is given by (2.5).

Proof: Given $f \in T$, and f is starlike of order δ , we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\delta \tag{2.11}$$

For the left hand side of (2.11) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \sum_{n=2}^{\infty} \frac{(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n |z|^{n-1}}$$

The last expression is less than 1-δ, if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n |z|^{n-1} < 1.$$

Using the fact that $f(z) \in TS_{\lambda}^{m}(\gamma, k)$ if and if

$$\sum_{n=2}^{\infty} \frac{[n(1+k)-(\gamma+k)]\phi^m(\lambda, n)}{(1-\gamma)} a_n \le 1,$$

It follows that (2.11) is true if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} |z|^{n-1} \le \frac{[n(1+k)-(\gamma+k)]\phi^m(\lambda, n)}{(1-\gamma)}$$

or equivalently

$$|z|^{n-1} \le \frac{(1-\delta)[n(1+k)-(\gamma+k)]\phi^m(\lambda, n)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family.

In [4], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T . He applied this function to resolve his integral means inequality conjunctured [5] and settled in [6], that

$$\int_0^{2\pi} \left| f(re^{i\varphi}) \right|^{\eta} d\varphi \le \int_0^{2\pi} \left| f_2(re^{i\varphi})^{\eta} \right| d\varphi ,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [6] , he also proved his conjuncture for the

subclasses $T^*(\alpha)$ and $C(\alpha)$ of T .

Now, we prove Silverman's conjecture for the class of functions $TS_{\lambda}^{m}(\gamma, k)$.

We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2].

Two functions f and g , which are analytic in E , the function f is said to be subordinate to g in E if there exists a function w analytic in E with $w(0)=0$, $|w(z)|<1$, $(z\in E)$ such that $f(z)=g(w(z))$, $(z\in E)$.

We denote this subordination by $f(z)\prec g(z)$.

Lemma 2.8: If the functions f and g are analytic in E with $f(z)\prec g(z)$,

then for $\eta>0$ and $z=re^{i\varphi}$, $0<r<1$

$$\int_0^{2\pi}\left|g(re^{i\varphi})\right|^{\eta}d\varphi\le\int_0^{2\pi}\left|f\ (re^{i\varphi})\right|^{\eta}d\varphi$$

Now, we discuss the integral means inequalities for functions f in $TS_{\lambda}^{m}(\gamma,k)$.

Theorem 2.9: Let $f\left(z\right)\in TS_{\lambda}^{m}(\gamma,k)$, , $0\le\gamma<1,\ k\ge 0$ and $f_2(z)$ be defined by $f_2(z)=z-\dfrac{1-\gamma}{\phi_2(\gamma,k)}z^2$

(2.12)

Proof: For $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$, (2.12) is equivalent to

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|^{\eta}d\varphi\le\int_0^{2\pi}\left|1-\frac{1-\gamma}{\varphi_2(\gamma,k)}z\right|^{\eta}d\varphi$$

By Lemma 2.8, it is enough to prove that

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec 1-\frac{1-\gamma}{\varphi_2(\gamma,k)}z$$

Assuming

$$1-\sum_{n=2}^{\infty}a_nz^{n-1}\prec 1-\frac{1-\gamma}{\varphi_2(\gamma,k)}w(z)\ ,$$

and using (2.2) we obtain

$$|w(z)|=\left|\sum_{n=2}^{\infty}\ \frac{\varphi_2(\gamma,k)}{1-\gamma}a_nz^{n-1}\right|\le|z|\sum_{n=2}^{\infty}\frac{\varphi_n(\gamma,k)}{1-\gamma}a_n\le|z|$$

where $\varphi_n(\gamma,k)=[n(1+k)-(\gamma+k)]\phi^m(\lambda,n)$

This completes the proof.

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