

On Modified Heston Model for Forecasting Stock Market Prices

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Abstract

Heston stochastic volatility model is the most popular stochastic volatility model used in pricing options despite its inefficiencies with Short term maturities. In this paper, we improved the Heston Model by incorporating the log-normal jump diffusions in the model. The derivation of the closed form solution of the Heston model and the full derivation of the Heston Model with jump are presented. We applied the models to National Association of Securities Dealers Automatic Quotation System (NASDAQ April 2021) stock data and back-tested them against the Black Scholes model since it predicts better option prices in short term maturities as compared to the Heston Model. Using Mean Squared Error, Heston Model with jump performs better than the Black scholes Model by 47.3% and Heston model by 58.08% error reduction.

Keywords: Heston stochastic volatility model; Heston Model with jump; Black scholes Model; PDE; Mean Squared Error.

1 Introduction

In the fields of financial economics and quantitative finance, stochastic volatility (SV) is the core term used to deal with the endemic time-varying volatility and co dependence observed in financial markets[10]. Theoretical and empirical research into modeling and predicting stock market volatility and prices has been extensive. Many market price and volatility applications necessitate the prediction or forecasting of a volatility parameter. Volatility is also a key parameter for pricing financial assets and derivatives.

Many fundamental options pricing models such as Black Scholes assumes constant volatility, which creates inefficiencies and bias in pricing [6]. Stochastic models that let volatility vary randomly such as the Heston model attempt to correct for this blind spot.

The conceptual framework introduced here identifies structural and economic behavior of historical stock prices. The prediction of stock market behavior is very important for investors who are seekers for capital appreciation. The application of stochastic volatility models for forecasting the future states of stock is based on the strong feature of volatility of its prices.

Forecasts of financial market volatility play a crucial role in financial decision making and the need for accurate forecasts is apparent. All investors face the decision whether or not to hedge the risks associated with their investments [4]. Investors will then base their hedging decisions on their risk perception over the remaining investment horizon; the more volatile the market the more inclined investors will be to hedge their exposures.

The issue of accurate volatility forecast is therefore firmly positioned at the centre of financial decision making [11]. It is against this background that this study employs the Heston model under the Stochastic volatility models which builds on the simple Black Scholes model to produce robust predictive prices. Our major contribution is that we take into account the existence of a specific cause of instability that drives the volatility of the underlying asset. [9] suggested one of the first stochastic volatility models, in which the underlying stock price and volatility mechanism is modelled as "Ito diffusions" induced by individual Brownian Motions. Following that, several models were suggested, including influential affine stochastic volatility model [7],stochastic volatility [2] and stochastic interest rates [1].

In our model, the volatility is described by a stochastic process underlying the stock which can take into account the asymmetry and excess kurtosis that are typically observed in financial assets [5]. Therefore, with our modified fitted Heston model, the short-term, middle-term and long-term asset price predictions minimizes bias as against the existing models.

2 Call Option Pricing techniques

In this section, we introduce the call option pricing techniques employed. The Heston model, Heston model with the jump and Black Scholes are the techniques employed in this paper.

2.1 Heston Model

Among stochastic volatility models, the Heston model presents two main advantages. First, it models an evolution of the underlying asset which can take into account the asymmetry and excess kurtosis that are typically observed (and expected) in financial asset returns. Second, it provides closed-form solutions for the forecasting of stock prices [12].

The Heston Model is given by:

$$dS_t = rS_t dt + \sqrt{V_t} S_t * dW_t^s \quad (1)$$

Where, r is the risk-free rate, V_t is the instantaneous variance and

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} * dW_t^v \quad (2)$$

σ is the volatility of volatility. k is the rate at which V_t returns to 0. θ is the long-run price variance.

Note: W_t^s is the Brownian motion corresponding to the stock's life, W_t^v is the Brownian motion corresponding to the variance.

2.2 Heston Model with Jump

A jump process is a stochastic process that makes transitions between discrete states at times that can be fixed or random, modeled as a simple or compound poisson process. In order to model the dynamics of the jump process, we have the equation:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t + \sum_{i=1}^{N_t} Y_i \tag{3}$$

where σ is the volatility and r is the drift, W_t is a standard wiener process, N_t is a Poisson process with jump intensity α and $Y_i \sim N(\theta, \delta^2)$ is the jump size. According to [13], Brownian motion(the diffusion part) and Poisson process are the two essential building blocks of any jump diffusion model(the jump part). Every option is familiar with the Brownian motion since the introduction of the Black Scholes model.

Combining the Heston model with the jump gives:

$$dS_t = \mu S_t dt + S_t \sqrt{V_t} dW_t^s + R_t S_t dN_t \tag{4}$$

$$dV_t = k(\theta - V_t)dt + \sigma \sqrt{V_t} dW_t^v \tag{5}$$

where r is the risk free rate, αR is the expected jump size, $r - \alpha R = \mu$ is the drift term. N_t is a Poisson process under the risk neutral measure with jump intensity $\alpha > 0$.

2.3 Black Scholes Model

[3] introduced the Black Scholes model under the assumptions that an asset’s price follows a Brownian motion but the asset’s volatility is constant. The stock price under risk neutral measure, has the dynamic,

$$dS_t = rS_t dt + \sigma S_t dW_t$$

The Black Scholes call price is:

$$C = S_t N(d_1) - K e^{-rT} N(d_2)$$

where: $N(d) = \frac{1}{2} \int_{-\infty}^d e^{-\frac{x^2}{2}} dx$

$$d_1 = \frac{\ln(S_0/k) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

And: C is the asset price. N is the CDF of normal distribution. S_t is the spot price of the asset. K is the strike price. r is the risk free interest rate. T is the time of maturity or forecasted time. t is the current time σ is volatility of the asset.

3 Methodology

3.1 Heston Model

The Heston Model follows the assumptions;

1. Stock price follows stochastic process related to current volatility.
2. Short selling is allowed in stock market without transaction.
3. There is no stock dividend within the term of validity of stock.

In Heston model, the stock is modeled as:

$$dS_t = rS_t dt + \sqrt{V_t} S_t * dW_t^s \tag{6}$$

$$dV_t = k(\theta - V_t)dt + \sigma \sqrt{V_t} * dW_t^v \tag{7}$$

W_t^s is the Brownian motion corresponding to the stock's life, W_t^v is the Brownian motion corresponding to the variance. We also assume that both Brownian motions are negatively correlated. This is evident from the fact that a sharp drop in equity prices increases volatility. $dW_t^s * dW_t^v = \rho dt$ where ρ is the correlation coefficient, $\sqrt{V_t}$ is the volatility of the asset price, dt is the indefinitely small positive time increment. ρ accounts for the correlation between the stocks price and its instantaneous volatility, often interpreted as leverage effect. The process V_t is purely positive if the parameters satisfy the following condition (known as the Feller condition)

$$2k\theta > \varepsilon^2$$

3.2 Heston Model Characteristic function and Call option equation

Each stochastic volatility model has its own characteristic function that describes the model's probability density function. [8] proposed the following characteristic function for equation 6.

$$C = \frac{1}{2}S_t + \frac{e^{-r(T-t)}}{\pi} \int_0^\infty Re[\frac{k^{-i\phi}f(i\phi + 1)}{i\phi}]\delta\phi - ke^{-r(T-t)}(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re[\frac{k^{-i\phi}f(i\phi)}{i\phi}]\delta\phi)$$

Where $f(i\phi) = \varepsilon(e^{i\phi x})$ defines the characteristic function for a random variable x .

Fixing for the explicit form of the characteristic function in order to measure the Heston model's option pricing success, We employed the Ito's Lemma, a stochastic calculus version of the chain law, to find the explicit characteristic function for equation 6. That is, for a two variable case involving a time dependent stochastic process of two variables, t and X_t , Ito's Lemma states

Assume that X_t satisfies the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dW_t$$

If $f(t, X_t)$ is a twice differentiable scalar function , then,

$$\delta f(t, X_t) = (\frac{\delta f}{\delta t} + \mu_t \frac{\delta f}{\delta x} + \frac{\sigma_t^2}{2} \frac{\delta^2 f}{\delta x^2})\delta t + \sigma_t \frac{\delta f}{\delta x} \delta W_t$$

We applied Ito's Lemma to three variables since Heston's stochastic volatility model considers t , X_t , and V_t as variables. Assuming we have the following system of two standard stochastic differential equations, where $f(X_t; V_t; t)$ is a scalar function that is continuous and twice differentiable:

$$\begin{aligned} dX_t &= \mu_x dt + \sigma_x dW_t^s \\ dV_t &= \mu_v dt + \sigma_v dW_t^v \end{aligned}$$

Further, let W_t^s and W_t^v have correlation ρ , where $-1 \leq \rho \leq 1$. For a function $f(X_t, V_t, t)$, we wish to find $\delta f(X_t, V_t, t)$ We discover that the derivative of a three-variable equation involving two stochastic processes equals the following expression using multivariable Taylor series expansion and Ito Calculus properties.

$$\delta f(X_t, V_t, t) = [\mu_x f_x + \mu_v f_v + f_t + f_{xv} \sigma_x \sigma_v \rho + \frac{1}{2}(f_{xx} \sigma_x^2 + f_{vv} \sigma_v^2)]\delta t + [\sigma_x f_x] \delta W_t^s + [\sigma_v f_v] \delta W_t^v$$

We know the form of the derivative of any function of X_t , V_t , and t , where X_t and V_t are governed by stochastic differential equations, owing to Ito's Lemma in three variables. Since the characteristic function of the Heston model is a function of X_t , V_t , and t , Ito's Lemma defines the structure of the characteristic function's derivative. We also know that the characteristic function for a three-variable stochastic process has the exponential affine form

$$f(X_t, V_t, t) = e^{A(T-t)+B(T-t)X_t+C(T-t)V_t+i\phi X_t}$$

let $T - t = \lambda$, the explicit form of the Heston model's characteristic function will be:

$$f(i\phi) = e^{A(\lambda)+B(\lambda)X_t+C(\lambda)V_t+i\phi X_t}$$

where:

$$A(\lambda) = r i \phi \lambda + \frac{k \theta}{\sigma^2} [-(\rho \sigma i \phi - k - d_j) \lambda - 2 \ln \left(\frac{1 - g_j e^{d_j \lambda}}{1 - g_j} \right)]$$

$$B(\lambda) = 0$$

$$C(\lambda) = \frac{(e^{d_j \lambda} - 1)(\rho \sigma i \phi - k - d_j)}{\sigma^2 (1 - g_j e^{d_j \lambda})}$$

where:

$$d_j = \sqrt{(\rho \sigma i \phi - k)^2 + \sigma^2 (i \phi + \phi^2)}$$

$$g_j = \frac{\rho \sigma i \phi - k - d_j}{\rho \sigma i \phi - k + d_j}$$

The variables r, σ, k, ϕ and θ in the Heston model's characteristic function need numerical values in order to be used in the option pricing formula.

3.3 Closed form solution for the Heston Model.

From the Heston model in Equations 6 and 7, The stock price variance drift is given by:

$$dV = (k(\theta - V) - \lambda \sigma \sqrt{V}) dt + \sigma \sqrt{V} dW_t^v$$

Abbreviating the drift term of the V process,

$$dV = U_v dt + \sigma \sqrt{V} dW_t^s$$

By delta and sigma hedging, the price of a derivative satisfies the partial derivative equation:

$$\frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2} \sigma^2 V \frac{\delta^2 V}{\delta v^2} + \rho \sigma v S \frac{\delta^2 V}{\delta v \delta S} + r S \frac{\delta V}{\delta S} - r V = 0$$

The price of a derivative can also be written as the expected value of the discounted payoff. H represents the pay-off function, so for a European call option, it is given by the max function

$$V_0 = E^1 [e^{-rT} h(S_T) | S_0, V_0]$$

$$h(S_T) = \max(S_T - K, 0)$$

Transforming the stock price to the log scale and applying Ito's lemma, the multiplicative S disappears and the drift term is adjusted by $(-\frac{1}{2})$ the variance:

For; $x = \ln S$

$$dx = d \ln S = (r - \frac{1}{2} V) dt + \sqrt{V} dW_t^s$$

Transforming the PDE in terms of x ,

$$\frac{\delta V}{\delta t} + \frac{1}{2} v \frac{\delta^2 V}{\delta x^2} + \frac{1}{2} \sigma^2 V \frac{\delta^2 V}{\delta v^2} + \rho \sigma v \frac{\delta^2 V}{\delta v \delta x} + (r - \frac{1}{2} v) \frac{\delta V}{\delta x} + \mu_v \frac{\delta V}{\delta v} - r V = 0$$

It's known that $\log S = \frac{1}{S}$ and the derivative of $\frac{1}{S} = \frac{-1}{S^2}$

Given three variables t, x, V , We infer to the general form of the solution.

$$V_0 = E^1 [e^{-rT} h(S_T) | S_0, V_0]$$

$$h(S_T) = \max(S_T - K, 0)$$

Writing the pay-off in terms of the indicator function;

$$h(S_T) = \max(S_T - K, 0) = S_T I_{S_T > K} - K I_{S_T > K} \tag{8}$$

Splitting the pay-off into two terms, we substitute 8 into the valuation formula:

$$\begin{aligned} V_0 &= E^1[e^{-rT}h(S_T)|S_0, V_0] \\ &= E^1[e^{-rT}(S_T I_{S_T > K} - K I_{S_T > K})] \end{aligned} \quad (9)$$

The asset scaled by the stock price is a martingale.

$$\frac{V_0}{S_0} = E^1\left[\frac{V_T}{S_T} | \mathcal{F}_0\right]$$

Having $S_0 = 1$ and growing to e^{rT} by time T .

$$\begin{aligned} V_0 &= E^1 e^{-rT} [S_0 I_{S_T > K}] - K e^{-rT} E^1 [I_{S_T > K}] \\ &= S_0 P_1 - K e^{-rT} P_2 \end{aligned}$$

P_1 represents the probability that the stock price is greater than K under the stock measure and P_2 represents the probability that the stock price is greater than K but under the risk neutral measure

$$V_0 = E^1[e^{-rT}h(S_T)|S_0, V_0] = S_0 P_1 - K e^{-rT} P_2$$

For maturity τ , $V_\tau = S_0 P_1 - K e^{-r\tau} P_2$ Since our PDE is in terms of x ; $x = \ln S$,

$$V_\tau = e^x P_1 - K e^{-r\tau} P_2 \quad (10)$$

Deducing PDEs for P_1 and P_2 , and because price is a linear combination of the two terms, so if V_τ solves the PDE, then the two terms solve this PDE.

Let the first term be V_1 from which we shall deduce P_1 . $V_1 = e^x P_1$; $\frac{\delta V_1}{\delta \tau} = e^x \frac{\delta P_1}{\delta \tau}$; $\frac{\delta V_1}{\delta v} = e^x \frac{\delta P_1}{\delta v}$; $\frac{\delta^2 V_1}{\delta v^2} = e^x \frac{\delta^2 P_1}{\delta v^2}$

For the derivatives with respect to x , we use the product rule:

$$\begin{aligned} \frac{\delta V_1}{\delta x} &= e^x P_1 + e^x \frac{\delta P_1}{\delta x} \\ \frac{\delta^2 V_1}{\delta x^2} &= e^x P_1 + e^x \frac{\delta P_1}{\delta x} + e^x \frac{\delta P_1}{\delta x} + e^x \frac{\delta^2 P_1}{\delta x^2} \\ \frac{\delta^2 V_1}{\delta x \delta v} &= e^x \frac{\delta P_1}{\delta v} + e^x \frac{\delta^2 P_1}{\delta v \delta x} \end{aligned}$$

From the general PDE of stochastic equations,

$$\begin{aligned} \frac{\delta V}{\delta t} &= \frac{1}{2} v \frac{\delta^2 V}{\delta x^2} + \frac{1}{2} \sigma^2 V \frac{\delta^2 V}{\delta v^2} + \rho \sigma v \frac{\delta^2 V}{\delta v \delta x} + (r - \frac{1}{2} v) \frac{\delta V}{\delta x} + \mu_v \frac{\delta V}{\delta v} - rV \\ \frac{\delta P_1}{\delta \tau} &= \frac{1}{2} v (P_1 + 2 \frac{\delta P_1}{\delta x} + \frac{\delta^2 P_1}{\delta x^2}) + \frac{1}{2} \sigma^2 v \frac{\delta^2 P_1}{\delta v^2} + \rho \sigma v (\frac{\delta P_1}{\delta v} + \frac{\delta^2 P_1}{\delta x \delta v}) \\ &\quad + (r - \frac{1}{2} v) (P_1 + \frac{\delta P_1}{\delta x}) + \mu_v \frac{\delta P_1}{\delta v} - r P_1 \\ \frac{\delta P_1}{\delta \tau} &= \frac{1}{2} v \frac{\delta^2 P_1}{\delta x^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 P_1}{\delta v^2} + \rho \sigma v \frac{\delta^2 P_1}{\delta v \delta x} + (r + \frac{1}{2} v) \frac{\delta P_1}{\delta x} + (\mu_v + \rho \sigma v) \frac{\delta P_1}{\delta v} \end{aligned}$$

The second term $V_2 = e^{-r\tau} P_2$ must also satisfy the PDE $\frac{\delta V_2}{\delta \tau} = e^{-r\tau} \frac{\delta P_2}{\delta \tau} - r e^{-r\tau} P_2$; $\frac{\delta V_2}{\delta x} = e^{-r\tau} \frac{\delta P_2}{\delta x}$; $\frac{\delta V_2}{\delta v} = e^{-r\tau} \frac{\delta P_2}{\delta v}$

From the stochastic PDE, we shall have:

$$\frac{\delta P_2}{\delta \tau} = \frac{1}{2} v \frac{\delta^2 P_2}{\delta x^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 P_2}{\delta v^2} + \rho \sigma v \frac{\delta^2 P_2}{\delta v \delta x} + (r - \frac{1}{2} v) \frac{\delta P_2}{\delta x} + \mu_v \frac{\delta P_2}{\delta v}$$

Letting $u_i = \frac{1}{2}$, $u_j = -\frac{1}{2}$,

$$\frac{\delta P_j}{\delta \tau} = \frac{1}{2} v \frac{\delta^2 P_j}{\delta x^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 P_j}{\delta v^2} + \rho \sigma v \frac{\delta^2 P_j}{\delta v \delta x} + (r + u_j v) \frac{\delta P_j}{\delta x} + \mu_v \frac{\delta P_j}{\delta v}$$

We deduce the initial condition of equation . From Feynman-Kac theorem,

$$\begin{aligned} P_j &= E^1[I_{X_T > \ln K} | S_0, V_0] \\ f_j &= E^1[e^{i\phi X_T} | S_0, V_0] \\ f_j(x, v, \tau) &= \int e^{i\phi x} P_j(x, v, \tau) dx \end{aligned} \tag{11}$$

Writing 11 as a normal probability density,

$$E(e^{i\phi x}) = \int_{-\infty}^{\infty} e^{i\phi x} P(x) dx$$

We know a characteristic function $X \sim N(m, \sigma^2)$ is:

$$E[e^{i\phi x}] = e^{(i\phi m + \frac{1}{2}(i\phi)^2 \sigma^2)}$$

Let $i\phi = u$; where u is a complex number because $i\phi$ represents complex numbers,

$$E[e^{ux}] = e^{um + \frac{1}{2}u^2 \sigma^2} \tag{12}$$

For a geometric Brownian, $\log S$ is normally distributed ie

$$\begin{aligned} \ln S_T &\sim N[x + (r - \frac{1}{2}\sigma^2)\tau, \sigma^2\tau] \\ x &\rightarrow \log x_0 \end{aligned}$$

X_τ^x is the value of the Markov process at time t. From 30,

$$E[e^{uX_\tau^x}] = e^{\psi(t,u)x + \phi(t,u)} \tag{13}$$

ϕ and ψ are some generic functions of U and τ with U being a complex number.

$$E[e^{uX_\tau^x} | F_1] = e^{ux + u(r - \frac{1}{2}\sigma^2)\tau + \frac{1}{2}u^2\sigma^2\tau} \tag{14}$$

13 is the affine exponential characteristic function.

Deducing the form of the solution under the Heston model,

$$f(x, v, \tau) = e^{C(\tau) + D(\tau)v + i\phi x}$$

Substituting the solution in PDE and obtaining the relevant derivatives using the chain rule;

$$\frac{\delta f}{\delta \tau} = (\frac{\delta C}{\delta \tau} + v\frac{\delta D}{\delta \tau})f; \frac{\delta f}{\delta x} = i\phi f; \frac{\delta f}{\delta v} = Df; \frac{\delta^2 f}{\delta v^2} = D^2 f; \frac{\delta^2 f}{\delta v \delta x} = i\phi Df; \frac{\delta^2 f}{\delta x^2} = (i\phi)^2 f$$

Substituting the derivatives into the PDE;

$$\begin{aligned} (\frac{\delta C}{\delta \tau} + v\frac{\delta D}{\delta \tau})f &= -\frac{1}{2}v\phi^2 f + \frac{1}{2}\sigma^2 v D^2 f + \rho\sigma v i\phi Df + (r + u_j v)i\phi f + (a - b_j v)Df \\ (-\frac{\delta D}{\delta \tau} - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\phi D + u_j i\phi - b_j D)v - \frac{\delta C}{\delta \tau} + ri\phi + aD &= 0 \end{aligned}$$

Letting the coefficient of $v = 0$. (If the equation is to be zero for all values of v ; then the coefficient of v must be equal to zero and the sum of the other terms must also equal to zero.) This results into the Riccati system:

$$\begin{aligned} -\frac{\delta D}{\delta \tau} - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D^2 + \rho\sigma i\phi D + u_j i\phi - b_j D &= 0 \\ -\frac{\delta C}{\delta \tau} + ri\phi + aD &= 0 \\ \frac{\delta C}{\delta \tau} &= ri\phi + aD \end{aligned}$$

Deducing the initial conditions, we know that if $\tau = 0$, the characteristic function will equal to $e^{i\phi x}$
 $f(s, v, 0) = e^{i\phi x}$. Because x will be known, D must equal to zero and C must equal to zero; $D=0$; $C=0$
 Solving for the equation, we know the Riccati equation is in the form:

$$\frac{dy}{d\tau} = a + by + cy^2$$

where; $a = i\mu_j\phi - \frac{1}{2}\phi^2$, $b = i\rho\sigma\phi - b_j$ and $C = \frac{1}{2}\sigma^2$ Solving the Riccati equation basing on the initial condition of Riccati system,

$$d = \sqrt{b^2 - 4ac}$$

$$y = -\frac{1}{2c} \frac{-(b-d)e^{d\tau} + b-d}{-\frac{b-d}{b+d}e^{d\tau} + 1}; y(0) = 0$$

$$= \frac{d-b}{2c} \frac{1 - e^{d\tau}}{1 - \frac{b-d}{b+d}e^{d\tau}}$$

Substituting the expressions for a,b and C in the square root term;

$$d = \sqrt{b^2 - 4ac}$$

$$d_j = \sqrt{(i\rho\sigma\phi - b_j)^2 - 4(i\mu_j\phi - \frac{1}{2}\phi^2)\frac{1}{2}\sigma^2}$$

Substituting for b and c in the solution,

$$D(\tau) = \frac{d-b}{2c} \frac{1 - e^{d\tau}}{1 - \frac{b-d}{b+d}e^{d\tau}}$$

$$= \frac{d_j + b_j - i\rho\sigma\phi}{\sigma^2} \frac{1 - e^{d_j\tau}}{1 - \frac{i\rho\sigma\phi - b_j - d_j}{i\rho\sigma\phi - b_j + d_j}e^{d_j\tau}}$$

Solving for C; $C(0, \phi) = 0$

$\frac{\delta C}{\delta \tau} = ir\phi + aD$ Writing in differential form and integrating,

$$\int_0^\tau dC = \int_0^\tau ir\phi dt + a \int_0^\tau D(t)dt$$

$$C(\tau) - C(0) = ir\phi\tau + a \int_0^\tau D(\tau)dt$$

Since i, r, ϕ are constants, substituting for D,

$$C(\tau) = ir\phi\tau + a \frac{d_j + b_j - i\rho\sigma\phi}{\sigma^2} \int_0^\tau \frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} dt$$

for $g_j = \frac{i\rho\sigma\phi - b_j - d_j}{i\rho\sigma\phi - b_j + d_j}$

$$y = e^{d_j\tau} : At, t = 0, y = 1$$

$$dy = e^{d_j\tau} d_j d\tau : At, t = \tau, y = e^{d_j\tau}$$

$$dy = y d_j d\tau$$

$$d\tau = \frac{1}{y d_j} dy$$

Substituting for the integral,

$$\int_0^\tau \frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}} dt = \int_1^{e^{d_j\tau}} \frac{1 - y}{1 - g_j y} \frac{1}{y d_j} dy$$

Since d_j is constant,

$$C(\tau) = ir\phi\tau + a \frac{d_j + b_j - i\rho\sigma\phi}{\sigma^2} \frac{1}{d_j} \int_1^{e^{d_j\tau}} \frac{1-y}{1-g_jy} \frac{1}{y} dy$$

Let $\frac{1-y}{1-g_jy} \frac{1}{y} = \frac{D}{1-gy} + \frac{E}{y}$ but $E=1$

$$1-y = Dy + E(1-gy)$$

$$1-y = 1 + y(D-g)$$

$$D = g-1$$

$$\frac{1-y}{(1-gy)y} = \frac{g-1}{1-gy} + \frac{1}{y}$$

$$\begin{aligned} \int_1^{e^{d\tau}} \frac{1-y}{1-gy} \frac{1}{y} dy &= \int_1^{e^{d\tau}} \frac{g-1}{1-gy} dy + \int_1^{e^{d\tau}} \frac{1}{y} dy \\ &= \frac{1-g}{g} \int_1^{e^{d\tau}} \frac{g}{1-gy} dy + \int_1^{e^{d\tau}} \frac{1}{y} dy \\ &= \frac{1-g}{g} \int_1^{e^{d\tau}} d\ln(1-gy) + \int_1^{e^{d\tau}} d\ln y \\ &= \frac{1-g_j}{g_j} \ln \frac{1-g_j e^{d_j\tau}}{1-g_j} + d_j\tau \end{aligned}$$

$$C(\tau) = ir\phi\tau + a \frac{d_j + b_j - i\rho\sigma\phi}{\sigma^2} \frac{1}{d_j} \left[\frac{1-g_j}{g_j} \ln \frac{1-g_j e^{d_j\tau}}{1-g_j} + d_j\tau \right]$$

Substituting for g ($\frac{i\rho\sigma\phi - b_j - d_j}{i\rho\sigma\phi - b_j + d_j}$) from g_j ,

$$\begin{aligned} \frac{1-g}{g} &= \frac{1}{g} - 1 = \frac{b - i\rho\sigma\phi - d}{b - i\rho\sigma\phi + d} - 1 \\ &= \frac{-2d}{b - i\rho\sigma\phi + d} \end{aligned}$$

Substituting back in the main expression;

$$C(\tau) = ir\phi\tau + a \frac{d_j + b_j - i\rho\sigma\phi}{\sigma^2} \frac{1}{d_j} \left[\frac{-2d}{b - i\rho\sigma\phi + d} \ln \frac{1-g_j e^{d_j\tau}}{1-g_j} + d_j\tau \right]$$

$$C(\tau) = ir\phi\tau + \frac{a}{\sigma^2} \left[-2d \ln \frac{1-g_j e^{d_j\tau}}{1-g_j} + (b - i\rho\sigma\phi + d)\tau \right]$$

$$D(\tau) = \frac{b - i\rho\sigma\phi + d}{\sigma^2} \frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}$$

These are the two probabilities that appear in the European option price formula.

$$V_t = e^x P_1 - K e^{-r\tau} P_2; P_j(x > \ln K)$$

3.4 Heston Model with Jump

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t + \sum_{i=1}^{N_t} Y_i \tag{15}$$

where σ is the volatility and r is the drift, W_t is a standard wiener process, N_t is a Poisson process with jump intensity α and $Y_i \sim N(\theta, \delta^2)$ is the jump size.

3.5 Deriving a combined Heston model with jump equation

Given that the stock price and variance satisfy equations 6 and 7, the combination of Heston model and jump process is derived from a partial differential equation with a riskless portfolio. Letting a portfolio Π that contains the option with its value to be $V = V(S, v, t)$, Δ units of stock S , p units of another option $U = U(S, v, t)$ that hedges the

volatility, we have:

$$\Pi = V - \Delta S - pU \quad (16)$$

The small change in the portfolio is given by:

$$d\Pi = dV - \Delta dS - p dU \quad (17)$$

Applying Ito's lemma to dV and dU and differentiating with respect to the variables S, v, t We know that Ito's formula for the jump process is given by:

$$df(X_t, t) = \frac{\delta f(X_t, t)}{\delta t} dt + b_t \frac{\delta f(X_t, t)}{\delta X} + \frac{1}{2} \sigma^2 \frac{\delta^2 f(X_t, t)}{\delta X^2} + [f(X_{t-} + cX_t) + f(cX_{t-})]$$

Applying it to our 3-variable option price function $H(S, v, t)$ we have :

$$\begin{aligned} dV = & \frac{\delta V}{\delta t} dt + \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta v} dv + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} dt + \frac{1}{2} \sigma^2 v \frac{\delta^2 V}{\delta v^2} dt \\ & + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} dt + [V(RS, t) - V(S, t)] dN_t \end{aligned} \quad (18)$$

When a jump occurs, $[V(RS, t) - V(S, t)] dN_t$ describes the change in option value.

Applying Ito's lemma to dU , we have:

$$\begin{aligned} dU = & \frac{\delta U}{\delta t} dt + \frac{\delta U}{\delta S} dS + \frac{\delta U}{\delta v} dv + \frac{1}{2} v S^2 \frac{\delta^2 U}{\delta S^2} dt + \frac{1}{2} \sigma^2 v \frac{\delta^2 U}{\delta v^2} dt \\ & + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} dt + [U(RS, t) - U(S, t)] dN_t \end{aligned} \quad (19)$$

Inserting equations 18 and 19 into 17, we have:

$$\begin{aligned} d\Pi = & \frac{\delta V}{\delta t} dt + \frac{\delta V}{\delta S} dS + \frac{\delta V}{\delta v} dv + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} dt + \frac{1}{2} \sigma^2 v \frac{\delta^2 V}{\delta v^2} dt + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} dt \\ & + [V(RS, t) - V(S, t)] dN_t - \Delta dS \\ & - p \left[\frac{\delta U}{\delta t} dt + \frac{\delta U}{\delta S} dS + \frac{\delta U}{\delta v} dv + \frac{1}{2} v S^2 \frac{\delta^2 U}{\delta S^2} dt + \frac{1}{2} \sigma^2 v \frac{\delta^2 U}{\delta v^2} dt + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} dt \right. \\ & \left. + [U(RS, t) - U(S, t)] dN_t \right] \end{aligned} \quad (20)$$

We re-arranged 20 to have same-variable derivatives together

$$\begin{aligned} d\Pi = & \left[\frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - p \left(\frac{\delta U}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 U}{\delta S^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 U}{\delta v^2} + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} \right) \right] dt \\ & + \left[\frac{\delta V}{\delta S} - \Delta - \frac{\delta U}{\delta S} \right] dS + \left[\frac{\delta V}{\delta v} - p \frac{\delta U}{\delta v} \right] dv + [(V(RS, t) - V(S, t)) - p(U(RS, t) - U(S, t))] dN_t \end{aligned}$$

In order to have a risk free portfolio, δS and δv must be eliminated and their coefficients equated to zero since they contribute to the risk according to 7.

$$\begin{aligned} d\Pi = & \left[\frac{\delta V}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - p \left(\frac{\delta U}{\delta t} + \frac{1}{2} v S^2 \frac{\delta^2 U}{\delta S^2} + \frac{1}{2} \sigma^2 v \frac{\delta^2 U}{\delta v^2} + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} \right) \right] dt \\ & + [(V(RS, t) - V(S, t)) - p(U(RS, t) - U(S, t))] dN_t \end{aligned} \quad (21)$$

The hedge parameters will now be:

$$p = \frac{\frac{\delta V}{\delta v}}{\frac{\delta U}{\delta v}}, \Delta = \frac{\delta V}{\delta S} - p \frac{\delta U}{\delta S}$$

A risk free rate should be earned by the portfolio. Hence:

$$d\Pi = r(V - \Delta S - pU) dt \quad (22)$$

Equating equations 21 and 22 and dividing by dt yields:

$$\left[\frac{\delta V}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - p \left(\frac{\delta U}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 U}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 U}{\delta v^2} + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} \right) \right] + [(V(RS, t) - V(S, t)) - p(U(RS, t) - U(S, t))]dN_t = r(V - \Delta S - PU) \tag{23}$$

Inserting the values of P and Δ , we have:

$$\left[\frac{\delta V}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - \frac{\delta V}{\delta v} \left(\frac{\delta U}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 U}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 U}{\delta v^2} + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} \right) \right] + [(V(RS, t) - V(S, t)) - \frac{\delta V}{\delta v} (U(RS, t) - U(S, t))]dN_t = r(V - \left(\frac{\delta V}{\delta S} - \frac{\delta U}{\delta S} \frac{\delta V}{\delta v} \right) S - \frac{\delta V}{\delta v} U) \tag{24}$$

Rearranging 24 and taking expectations over the probability distribution of jumps, we have:

$$\frac{\left[\frac{\delta V}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - rV + rS \frac{\delta V}{\delta S} + \alpha E(V(R_t S_t, t) - V(S_t, t)) \right] dN_t}{\frac{\delta V}{\delta v}} = \frac{\left[\frac{\delta U}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 U}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 U}{\delta v^2} + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} + rS \frac{\delta U}{\delta S} - rU + \alpha E[(U(R_t S_t, t) - U(S_t, t))] \right] dN_t}{\frac{\delta U}{\delta v}} \tag{25}$$

We know that :

$$E[V(R_t S_t, t) - V(S_t, t)] = \int_0^\infty [V(R_t S_t, t) - V(S_t, t)]V(R)dR \tag{26}$$

In regard to the jump probability distribution function, the expected value in the change in the option price is given by equation 26 We therefore have equation 25 expressed as:

$$\frac{\left[\frac{\delta V}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - rV + rS \frac{\delta V}{\delta S} + \alpha \int_0^\infty [V(R_t S_t, t) - V(S_t, t)]V(R)dR \right] dN_t}{\frac{\delta V}{\delta v}} = \frac{\left[\frac{\delta U}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 U}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 U}{\delta v^2} + \sigma v \rho S \frac{\delta^2 U}{\delta v \delta S} + rS \frac{\delta U}{\delta S} - rU + \alpha \int_0^\infty [U(R_t S_t, t) - U(S_t, t)]U(R)dR \right] dN_t}{\frac{\delta U}{\delta v}} \tag{27}$$

V and U are similar expressions but representing different options. Writing both expressions as a function of S, v and t , we shall have:

$$V(S, v, t) = -k(\theta - v) + \alpha(S, v, t)$$

that is,

$$\frac{\left[\frac{\delta V}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - rV + rS \frac{\delta V}{\delta S} + \alpha \int_0^\infty [V(R_t S_t, t) - V(S_t, t)]V(R)dR \right] dN_t}{\frac{\delta V}{\delta v}} = -k(\theta - v) + \alpha(S, v, t) \tag{28}$$

multiplying 28 by $\frac{\delta V}{\delta v}$ both sides, we have:

$$\frac{\delta V}{\delta t} + \frac{1}{2}vS^2 \frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2 v \frac{\delta^2 V}{\delta v^2} + \sigma v \rho S \frac{\delta^2 V}{\delta v \delta S} - rV + rS \frac{\delta V}{\delta S} + \alpha \int_0^\infty [V(R_t S_t, t) - V(S_t, t)]V(R)dR = -k(\theta - v) \frac{\delta V}{\delta v} + \alpha(S, v, t) \frac{\delta V}{\delta v} \tag{29}$$

Rearranging the equation, we have:

$$\frac{\delta V}{\delta t} + \frac{1}{2}vS^2\frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2v\frac{\delta^2 V}{\delta v^2} + \sigma v\rho S\frac{\delta^2 V}{\delta v\delta S} - rV + rS\frac{\delta V}{\delta S} + \alpha \int_0^\infty [V(R_t S_t, t) - V(S_t, t)]V(R)dR + k(\theta - v)\frac{\delta V}{\delta v} - \alpha(S, v, t)\frac{\delta}{\delta v} = 0 \tag{30}$$

Since market risk is priced as a linear function of volatility, for

$$\alpha(S, t, v) = \alpha v \tag{31}$$

We have equation 30 as:

$$\frac{\delta V}{\delta t} + \frac{1}{2}vS^2\frac{\delta^2 V}{\delta S^2} + \frac{1}{2}\sigma^2v\frac{\delta^2 V}{\delta v^2} + \sigma v\rho S\frac{\delta^2 V}{\delta v\delta S} - rV + rS\frac{\delta V}{\delta S} + \alpha \int_0^\infty [V(R_t S_t, t) - V(S_t, t)]V(R)dR + k(\theta - v)\frac{\delta V}{\delta v} - \alpha v\frac{\delta V}{\delta v} = 0 \tag{32}$$

Following Nimalin(2005) and cited by [13], the boundary and initial conditions imposed are: $V(S, v, t) = \max(S, -k, 0)$. $V(S, v, t) = 0$ implies that the call price will be 0 when the stock price is 0. $\frac{\delta V}{\delta S}(\infty, v, t) = 1$ implies that delta tends to 1 with increase in stock price. $V(S, \infty, t) = S$ implies that as the volatility increases, the call price equals to stock price.

4 Results and Discussion

In order to test the accuracy of the models, NASDAQ 2021 options data was employed. The model that returned the closest prices to the market prices was considered the best model. Parameters were estimated for all the models

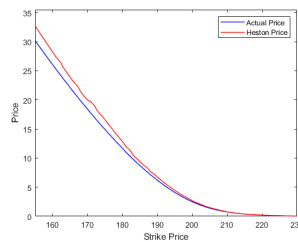


Figure 1: Heston Model

The following parameter values were used: $\sigma = 1.5$, $k = 2$, $\rho = -0.65$ and $r = 0.229677$

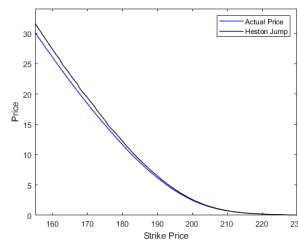


Figure 2: Heston Model with jump

The following parameter values were used: $\sigma = 1.5$, $k = 2$, $\rho = -0.65$, $r = 0.229677$, $\sigma_j = 0.15529$, $\alpha_j = 0.025663$ and $\mu_j = -0.0005175$.

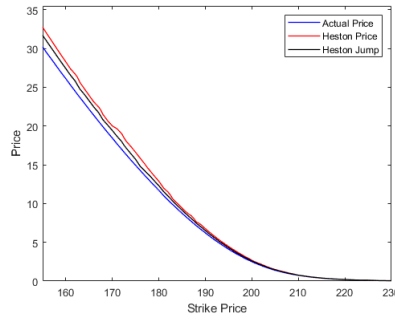


Figure 3: Comparison figure

The graph 3 is the combination of graph 1 and graph 2. It represents how close the estimated prices from both Heston Model and Heston Model with jump are to the actual market prices.

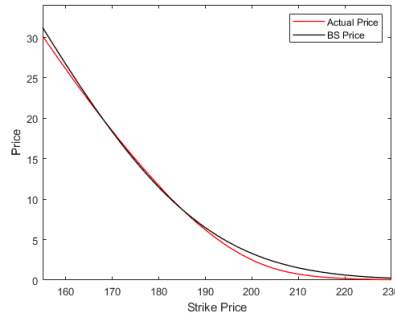


Figure 4: Black Scholes Model

Parameters used: $\sigma = 0.378915$ which is the average of the standard deviation on returns for the 30days from April 2nd 2021, $\rho = -0.65$, $k = 2$, $r = 0.229677$

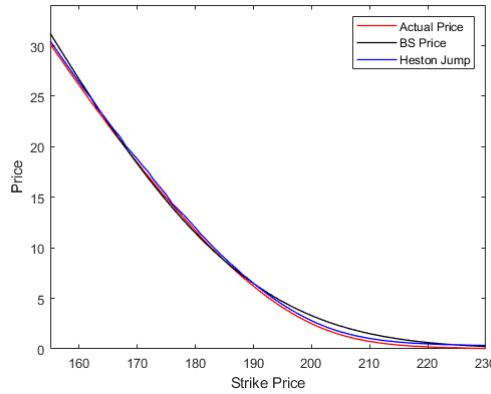


Figure 5: Black Scholes-Jump

The graph 4 indicates the option prices estimates from the Black Scholes Model in comparison to the actual market prices while the graph 5 presents a comparison between the market prices and estimates from both Black Scholes Model and Heston Model with jump. Table 1 represents the numerical short term option price estimates generated by the applied models.

Table 1: Short term maturities: Comparison between prices

Strike Price	Option price	BSM	Heston with Jump
155	30.18	31.18	30.77
157	28.52	29.32	29.08
161	25.26	25.7	25.77
165	22.13	22.23	22.58
167	20.61	20.57	21.12
171	17.68	17.41	18.12
175	14.91	14.5	15.24
179	12.31	11.88	12.56
183	9.85	9.561	10.00
187	7.675	7.563	7.82
189	6.68	6.681	6.804
193	4.89	5.144	4.974
197	3.40	3.89	3.46
201	2.235	2.89	2.268
205	1.395	2.109	1.415
209	0.845	1.512	0.8565
215	0.38	0.8898	0.382
220	0.205	0.556	0.2053
225	0.115	0.339	0.1145
230	0.045	0.2018	0.0436

4.1 Error Measurement Comparison

For testing the accuracy of the models, we employed the Mean Squared Error. The model with the smallest error is considered the better model for the estimation.

$$MSE = \frac{1}{N} \sum_{i=1}^N (Y_i - \hat{Y}_i)^2$$

In the equation above, N is the number of data points, Y_i represents the market prices and \hat{Y}_i represents the model prices.

Table 2: Error Measurement Comparison

Error Measure	Heston Model	BSM	Heston with Jump
MSE	0.331311	0.2634	0.138897

In short term maturities, Black Scholes Model is a better estimator than the Heston Model whereas Heston model with jump offers better estimates than the Black Scholes. From the actual prices, Heston model, Black Scholes and Heston with jump deviate by 0.331311, 0.2634 and 0.138897 respectively. Heston with jump reduces both the Black scholes error by 47.3% and Heston model error by 58.08%

5 Conclusion

Due to inconsistencies in the Heston model prices for short maturities, poisson jumps were accounted for in the application of the Heston Model. The new model (Heston model with jump) was compared to the Original Heston model and Black Scholes model for short maturities. With application of the mean squared error, the new model performed better than both the Heston model and Black Scholes model by 58.08% and 47.3% respectively

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