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Weighted Sharing and Uniqueness of Entire Functions whose Difference Polynomials Sharing a Polynomial of Certain Degree

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Abstract. ***In this paper, we deal with the uniqueness problem of an entire functions whose difference polynomials sharing a polynomial of certain degree with finite weight. This paper extends and improves some classical results obtained by A. Banerjee and S. Majumder*** [20]***.***

Keywords: Difference polynomials, Entire function, Weighted Sharing etc.,

I. Introduction and Main Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let $a \in \mathbb{C}$. We say that $f(z)$ and $g(z)$ share a CM, provided that $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities. Similarly, we say that $f(z)$ and $g(z)$ share a IM, provided that $f(z)-a$ and $g(z)-a$ have the same zeros with ignoring multiplicities. In addition we say that $f(z)$ and $g(z)$ share ∞ CM if $\frac{1}{f(z)}$ and $\frac{1}{g(z)}$ share 0 CM, and we say that f and g share ∞ IM, if $\frac{1}{f(z)}$ and $\frac{1}{g(z)}$ share 0 IM. We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function $f(z)$, we denote by $T(r,f)$ the Nevanlinna characteristic of $f(z)$ and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$. The order of $f(z)$ is defined by

$$\sigma(f) = 1 - \overline{\lim_{r\to\infty}} \frac{log\ T(r,f)}{log\ r}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities. We say that a finite value z_0 is called a fixed point of $f(z)$ if $f(z_0) = z_0$ or z_0 is a zero of $f(z) - z$. For the sake of simplicity we also use the notation

$$m^* := \begin{cases} 0, \ if \ \ m = 0 \\ m, \ if \ \ m \in \mathbb{N}. \end{cases}$$

Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. During the last few decades many authors investigated the value distributions of $f^n f'$.

In 1959, W. K. Hayman (see [5]) proved the following theorem.

Theorem 1. [5] *Let f be a transcendental meromorphic function and n (≥ 3) is an integer. Then $f^n f' = 1$ has infinitely many solutions.*

The case $n = 2$ was settled by Mues [14] in 1979. Bergweiler and Eremenko [1] showed that $f^n f' - 1$ has infinitely many zeros. For an analogue of the above results Laine and Yang investigated the value distribution of difference products of entire functions in the following manner.

Theorem 2. [10] *Let f be a transcendental entire function of finite order, and c be a non-zero complex constants. Then, for $n \geq 2$, $f^n f(z+c)$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

Afterwards, Liu and Yang [13] improved Theorem 2 and obtained next result.

Theorem 3. *Let f be a transcendental entire function of finite order, and c be a non-zero complex constant. Then, for $n \geq 2$, $f^n f(z+c) - p(z)$ has infinitely many zeros, where $p(z)$ is a non-zero polynomial.*

Next we recall the uniqueness result corresponding to Theorem 1, obtained by Yang and Hua [17] which may be considered a gateway to a new research in the direction of sharing values of differential polynomials.

Theorem 4. *Let f and g be two non-constant entire functions, $n \in \mathbb{N}$ such that $n \geq 6$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ where $c_1,\ c_2,\ c \in \mathbb{C}$ satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2001, Fang and Hong [4] studied the uniqueness of differential polynomials of the form $f^n(f-1)f'$ and proved the following result.

Theorem 5. [4] *Let f and g be two non-constant entire functions, and let $n \geq 11$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.*

In 2004, Lin and Yi [12] extended the above result in view of the fixed point and they proved the following.

Theorem 6. [12] *Let f and g be two non-constant entire functions, and let $n \geq 7$ be a positive integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value z CM, then $f \equiv g$.*

In 2010, Zhang [19] got a analogue result in difference

Theorem 7. [19] *Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small functon with respect to both $f(z)$ and $g(z)$. Suppose that c is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(f-1)f(z+c)$ and $g^n(g-1)g(z+c)$ share $\alpha(z)$ CM, then $f(z) \equiv g(z)$.*

In 2010, Qi, Yang and Liu [15] obtained the difference counterpart of Theorem 4 by proving the following theorem.

Theorem 8. [15] *Let f and g be two transcendental entire functions of finite order, and c be a non-zero complex constant, let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share z CM, then $f \equiv t_1 g$ for a constant t_1 satisfies $t_1^{n+1} = 1$.*

Theorem 9. [15] *Let f and g be two transcendental entire functions of finite order, and c be a non-zero complex constant, let $n \geq 6$ be an integer. If $f^n f(z+c)$ and $g^n g(z+c)$ share 1 CM, then $fg = t_2$ or $f \equiv t_3 g$ for some constants t_2 and t_3 that satisfy $t_3^{n+1} = 1$.*

In 2020, A. Banerjee and S. Majumder [20] proved the following result.

Theorem 10. *Let f and g be two transcendental entire functions of finite order, c be a non-zero complex constant and let $p(z)$ be a non-zero polynomial with $deg(p) \leq n-1$, $n(\geq 1)$, $m^*(\geq 0)$ be two integers such that $n > m^* + 5$. Let $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ be a non-zero polynomial. If $f^n P(f) f(z+c) - p$ and $g^n P(g) g(z+c) - p$ share $(0, 2)$ then*

(I) when $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ is a non-zero polynomial, one of the following three cases holds.

(I1) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$ and $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(I2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1(a_m\omega_1^n + \ldots + a_0) - \omega_2(a_m\omega_2^n + \ldots + a_0)$,

(I3) $P(\omega)$ reduces to a non-zero monomial, viz., $P(\omega) = a_i\omega^i \not\equiv 0$, for $i \in \{0, 1, 2, \ldots, m\}$ if $p(z)$ is a non-zero constant b, then we have $f = e^{\alpha(z)}$ and $g = e^{\beta(z)}$ where α, β are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n_i+1)d} = b^2$;

(II) when $P(\omega) = \omega^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;

(III) when $P(\omega) = (\omega - 1)^m$, $(m \geq 2)$ one of the following two cases holds:

(III1) $f \equiv g$,

(III2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1(a_m\omega_1^n + \ldots + a_0) - \omega_2(a_m\omega_2^n + \ldots + a_0)$;

(IV) when $P(\omega) \equiv c_0$ one of the following two cases holds:

(IV1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,

(IV2) $f = e^{\alpha(z)}$ and $g = e^{\beta(z)}$ where α, β are two non-constant polynomials such that $\alpha + \beta = d \in \mathbb{C}$ and $c_0^2 e^{(n+1)d} = b^2$.

In this paper we are replacing $f(z+c)$ by $\sum_{j=1}^{p} a_j f(z+c_j)$ and obtained the following result.

Theorem 11. *Let f and g be two transcendental entire functions of finite order, c be a non-zero complex constant and let $p(z)$ be a non-zero polynomial with $deg(p) \leq n-1$, $n(\geq 1)$, $m^*(\geq 0)$ be two integers such that $n > m^* + p + 4$. Let $P(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ be a non-zero polynomial. If $f^nP(f)\sum_{j=1}^{p} a_j f(z+c_j) - p$ and $g^nP(g)\sum_{j=1}^{p} a_j g(z+c_j) - p$ share $(0,2)$ then*

(I) when $p(\omega) = a_m\omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$ is a non-zero polynomial, one of the following three cases holds.

(I1) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m+p, \ldots, n+m+p-i, \ldots, n)$ and $a_{m-i} \neq 0$ for some $i = 1, 2, \ldots, m$,

(I2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(\omega_1, \omega_2) = \omega_1(a_m\omega_1^n + \ldots + a_0) - \omega_2(a_m\omega_2^n + \ldots + a_0),$$

(I3) $P(\omega)$ reduces to a non-zero monomial, viz., $P(\omega) = a_i\omega^i \not\equiv 0$, for $i \in \{0, 1, 2, \ldots, m\}$ if $p(z)$ is a non-zero constant b, then we have $f = e^{\alpha(z)}$ and $g = e^{\beta(z)}$ where α, β are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n_i+p)d} = b^2$;

(II) when $P(\omega) = \omega^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;

(III) when $p(\omega) = (\omega - 1)^m$, $(m \geq 2)$ one of the following two cases holds:

(III1) $f \equiv g$,

(III2) f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(\omega_1, \omega_2) = \omega_1(a_m\omega_1^n + \ldots + a_0) - \omega_2(a_m\omega_2^n + \ldots + a_0);$$

(IV) when $P(\omega) \equiv c_0$ one of the following two cases holds:

(IV1) $f \equiv tg$ for some constant t such that $t^{n+1} = 1$,

(IV2) $f = e^{\alpha(z)}$ and $g = e^{\beta(z)}$ where α, β are two non-constant polynomials such that $\alpha + \beta = d \in \mathbb{C}$ and $c_0^2 e^{(n+p)d} = b^2$.

II. Auxiliary Definitions

Definition 1. [7] *Let $a \in \mathbb{C} \cup \{\infty\}$. For a positive integer p we denote by $N(r, a; f| \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f| \leq p)$ we denote the corresponding reduced counting function. In an analogous manner we can define $N(r, a; f| \geq p)$ and $\overline{N}(r, a; f| \geq p)$.*

Definition 2. [9] *Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a- points of f where an a- point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}_{(2}(r, a; f) + \ldots + \overline{N}_{(k}(r, a; f).$$

Clearly, $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 3. [8], [9] *Let k be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$ we say that $f,\ g$ share the value a with weight k. The definition implies that if $f,\ g$ share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n. We write $f,\ g$ share (a, k) to mean that $f,\ g$ share the value a with weight k. Clearly, if $f,\ g$ share (a, k) then $f,\ g$ share (a, p) for any integer p, $0 \leq p < k$. Also we note that $f,\ g$ share a value a IM or CM if and only if $f,\ g$ share $(a, 0)$ or (a, ∞) respectively.*

III. Lemmas

Lemma 1. [16] *Let f be a non-constant meromorphic function, and let $a_n(\not\equiv 0),\ a_{n-1}, \ldots, a_0$ be meromorphic functions such that $T(r, a_i) = S(r, f)$ for $i = 0,\ 1, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_0) = nT(r, f) + S(r, f).$$

Lemma 2. [2] *Let f be a meromorphic function function of inite order σ, and let c be fixed non-zero complex constant. Then for each $\epsilon > 0$, we have*

$$m\Big(r, \frac{f(z+c)}{f(z)}\Big) + m\Big(r, \frac{f(z)}{f(z+c)}\Big) = O(r^{\sigma-1+\epsilon}).$$

Lemma 3. [2] *Let f be a meromorphic function function of inite order σ, $c \neq 0$ be fixed. Then for each $\epsilon > 0$, we have*

$$T\Big(r, f(z+c)\Big) = O(r^{\sigma-1+\epsilon}).$$

Lemma 4. *Let f be an entire function of finite order σ, c be a fixed non-zero complex constant and let $n \in \mathbb{N}$ and $P(\omega)$ be defined as in Theorem 10. Then for each $\epsilon > 0$, we have*

$$T(r, f^n P(f) \sum_{j=0}^{p} a_j f(z+c_j)) = T(r, f^{n+p} P(f)) + O(r^{\sigma-1+\epsilon}).$$

Proof. By Lemma 2 we have

$$\begin{aligned} T(r, f^n P(f) f(z+c)) &= m(r, f^n P(f) \sum_{j=0}^{p} a_j f(z+c_j)) \\ &\leq m(r, f^n P(f) f) + m\left(r, \frac{\sum_{j=0}^{p} a_j f(z+c_j)}{f(z)}\right) \\ &\leq m(r, f^{n+p} P(f)) + O(r^{\sigma-1+\epsilon}) \\ &\leq T(r, f^{n+p} P(f)) + O(r^{\sigma-1+\epsilon}). \end{aligned}$$

Also, we have

$$T(r, f^{n+p}P(f)) = m(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j))$$

$$\leq m\left(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)\right) + m\left(r, \frac{\sum_{j=0}^{p} a_jf(z+c_j)}{f(z)}\right)$$

$$\leq m\left(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)\right) + O(r^{\sigma-1+\epsilon})$$

$$\leq m\left(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)\right) + O(r^{\sigma-1+\epsilon}) \leq T\left(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)\right) + O(r^{\sigma-1+\epsilon}).$$

Hence

$$T(r, f^{n+p}P(f)) = T\left(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)\right) + O(r^{\sigma-1+\epsilon}).$$

□

Remark 1. *Under the conditions of Lemma 4, by Lemma 1 we have* $S\left(r, f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)\right) = S(r, f)$.

Lemma 5. *([3]) Let f be a non-constant meromorphic function of finite order and $c \in \mathbb{C}$. Then*

$$N(r, 0; f(z+c)) = N(r, 0; f) + S(r, f).$$

$$\overline{N}(r, 0; f(z+c)) = \overline{N}(r, 0; f) + S(r, f).$$

$$N(r, \infty; f(z+c)) = N(r, \infty; f) + S(r, f).$$

$$\overline{N}(r, \infty; f(z+c)) = \overline{N}(r, \infty; f) + S(r, f).$$

Lemma 6. *Let f be transcendental entire function of finite order σ, c be a fixed non-zero complex constant, $n(\geq 1)$, $m^*(\geq 0)$ be two integers and let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to f. If $n > 1$, then $f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j) - \alpha(z)$ has infinitely many zeros.*

Proof. Let $\phi = f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)$. Now in view of Lemma 5 and second fundamental theorem of small functions (see [18]) we get

$$T(r, \phi) = \overline{N}(r, 0; \phi) + \overline{N}(r, \infty; \phi) + \overline{N}(r, a; \phi) + (\epsilon + o(1)) + T(r, f)$$

$$\leq \overline{N}(r, 0; f^nP(f)) + \overline{N}\left(r, 0; \sum_{j=0}^{p} a_jf(z+c_j)\right) + \overline{N}(r, a; \phi) + (\epsilon + o(1)) + T(r, f)$$

$$\leq 2\overline{N}(r, 0; f) + \overline{N}(r, 0; P(f)) + \overline{N}(r, a; \phi) + (\epsilon + o(1)) + T(r, f)$$

$$\leq (p + m^* + 1)T(r, f) + \overline{N}(r, a; \phi) + (\epsilon + o(1)) + T(r, f).$$

for all $\epsilon > 0$. From Lemmas 1 and 4 we get

$$(n+m^*+p)T(r,f) \leq (p+m^*+1)T(r,f) + \overline{N}(r,a;\phi) + (\epsilon + o(1)) + T(r,f).$$

Take $\epsilon < 1$. Since $n > 1$ from the above one can easily say that $\phi - a(z)$ has infinitely many zeros. This completes the Lemma. □

Lemma 7. [9] *Let f and g be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following holds.*
$(i) T(r,f) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r,f) + S(r,g)$.
$(ii) fg \equiv 1$.
$(iii) f \equiv g$.

Lemma 8. *[Hadamard Factorization Theorem]. Let f be an entire function of finite order ρ with zeros $a_1, \ldots$ each zeros is counted as often as its multiplicity. Then f can be expressed in the form*

$$f(z) = Q(z)e^{\alpha(z)},$$

where $\alpha(z)$ is a polynomial of degree not exceeding ρ and $Q(z)$ is the canonical product formed with the zeros of f.

Lemma 9. *Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $p(z)$ be a non-zero polynomial such that $deg(p) \leq n-1$, where $n \in \mathbb{N}$. Let $P(\omega)$ be a non-zero polynomial defined as in Theorem 10. Suppose*

$$f^n P(f) \sum_{j=0}^{p} a_j f(z+c_j) g^n P(g) \sum_{j=0}^{p} a_j g(z+c_j) \equiv p^2.$$

Then $P(\omega)$ reduces to a non-zero monomial, namely $P(\omega) = a_i \omega^i \not\equiv 0$, for $i \in \{0,\ 1, \ldots, m\}$. If $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are two non-constant polynomials such that $\alpha + \beta \equiv d \in \mathbb{C}$ and $a_i^2 e^{(n+i+p)d} = b^2$.

Proof. Suppose

$$f^n P(f) \sum_{j=0}^{p} a_j f(z+c_j) g^n P(g) \sum_{j=0}^{p} a_j g(z+c_j) \equiv p^2. \tag{0.1}$$

We consider the following cases:

Case 1. Let $deg(p(z)) = l\ (\geq 1)$. From the assumption that f and g are two transcendental entire functions, we deduce by (0.1) that $N(r,0;f^n P(f)) = O(log\ r)$ and $N(r,0;g^n P(g)) = O(log\ r)$. First we suppose that $P(\omega)$ is not a non-zero monomial. for the sake of simplicity let $P(\omega) = \omega - a$ where $a \in \mathbb{C} \setminus \{0\}$ clearly

$\Theta(0; f) + \Theta(a; f) = 2$ which is impossible for an entire function. Thus $P(\omega)$ reduces to a non-zero monomial, namely $P(\omega) \not\equiv a_i\omega^i$ for some $i \in \{0,\ 1, \dots, m\}$ and so (0.1) reduces to

$$a_1^2 f^{n+i} \sum_{j=0}^{p} a_j f(z + c_j) g^{n+i} \sum_{j=0}^{p} a_j g(z + c_j) \equiv p^2. \tag{0.2}$$

From (0.2) it follows that $N(r, 0; f) = O(log\ r)$ and $N(r, 0; g) = O(log\ r)$. Now by Lemma 8 we obtain that $f = h_1 e^{\alpha_1}$ and $g = h_2 e^{\beta_1}$, where $h_1,\ h_2$ are two non-zero polynomials. By virtue of the polynomials $p(z)$, from (0.2) we arrive at a contradiction.

Case 2. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Then from (0.1) we ahve

$$f^n P(f) \sum_{j=0}^{p} a_j f(z + c_j) = g^n P(g) \sum_{j=0}^{p} a_j g(z + c_j) \equiv b^2. \tag{0.3}$$

Now from the assumption that f and g are two non-constant entire functions, we deduce by (0.3) that $f^n P(f) \neq 0$ and $g^n P(g) \neq 0$. By Picard's Theorem, we clain that $P(\omega) = a_i\omega^i$ for $i \in \{0, 1, \dots, m\}$, otherwise the Picard's exceptional values are atleast three, which is a contradiction. Then (0.3) reduces to

$$a_i^2 f^{n+i} \sum_{j=0}^{p} a_j f(z + c_j) g^{n+i} \sum_{j=0}^{p} a_j g(z + c_j) \equiv b^2. \tag{0.4}$$

Hence by Lemma 8 we obtain that

$$f = e^{\alpha(z)}, \quad g = e^{\beta(z)}, \tag{0.5}$$

where $\alpha(z),\ \beta(z)$ are twow non-constant polynomials. Now from (0.4) and (0.5) we obtain

$$(n + i)(\alpha(z) + \beta(z)) + \sum_{j=0}^{p} a_j \alpha(z + c_j) + \sum_{j=0}^{p} a_j \beta(z + c_j) \equiv d_1.$$

where $d_1 \in \mathbb{C}$, i.e.,

$$(n + i)(\alpha'(z) + \beta'(z)) + \sum_{j=0}^{p} a_j \alpha'(z + c_j) + \sum_{j=0}^{p} a_j \beta'(z + c_j) \equiv 0. \tag{0.6}$$

Let $\gamma(z) = \alpha'(z) + \beta'(z)$. Then from (0.6) we have

$$(n + i)\gamma(z) + \sum_{j=0}^{p} a_j \gamma(z + c_j) \equiv 0. \tag{0.7}$$

We assert that $\gamma(z) \equiv 0$. It is not suppose $\gamma \not\equiv 0$. Note that if $\gamma(z) \equiv d_2 \in \mathbb{C}$, from (0.7) we must have $d_2 = 0$. Suppose that $deg(\gamma) \geq 1$. Let $\gamma(z) = \sum_{j=1}^{m} b_i z^i$, where $b_m \neq 0$. Therefore the co-efficient of z^m in $(n + i)\gamma(z) + \sum_{j=0}^{p} a_j \gamma(z + c_j)$ is $(n + p + i)b_m \neq 0$. Thus we arrive at a contradiction from (0.7). Hence $\gamma(z) \equiv 0$, i.e., $\alpha + \beta \equiv d \in \mathbb{C}$. Also from (0.4) we have $a_i^2 e^{(n+i+p)d} = b^2$. This completes the proof. □

Lemma 10. *Let f and g be two transcendental entire functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $p(z)$ be a non-zero polynomial such that $deg(p) \leq n-1$, where $n \in \mathbb{N}$. Let $P(\omega)$ be defined as in Theorem 10 with atleast two of $a_i,\ i = 0, 1, \ldots, m$ are non-zero. Then*

$$f^n P(f) \sum_{j=0}^{p} a_j f(z + c_j) g^n P(g) \sum_{j=0}^{p} a_j g(z + c_j) \not\equiv p^2.$$

Proof. Proof of the Lemma follows from Lemma 9. □

Lemma 11. *Let f, g be two transcendental entire functions of finite order, $c \in \mathbb{C} \setminus \{0\}$ and $n \in \mathbb{N}$ with $n > 1$. If $f^n P(f) \sum_{j=0}^{p} a_j f(z + c_j) \equiv g^n P(g) \sum_{j=0}^{p} a_j g(z + c_j)$ where $P(\omega)$ is defined as in Theorem 10 then*
(I) when $p(\omega) = a_m \omega^m + a_{m-1}\omega^{m-1} + \ldots + a_1\omega + a_0$, one of the following two cases holds:
(I1) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m, \ldots, n+m-i, \ldots, n)$, $a_{m-i} \neq 0$ for some $i = 1,\ 2, \ldots, m$,
(I2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(\omega_1, \omega_2) = \omega_1 P(\omega_1) \sum_{j=0}^{p} a_j \omega_1(z + c_j) - \omega_2 P(\omega_2) \sum_{j=0}^{p} a_j \omega_2(z + c_j).$$

(II) when $P(\omega) = \omega^m - 1$, then $f \equiv tg$ for some constant t such that $t^m = 1$;
(III) when $p(\omega) = (\omega - 1)^m,\ (m \geq 2)$ one of the following two cases holds:
(III1) $f \equiv g$,
(III2) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(\omega_1, \omega_2) = \omega_1^n (\omega_1 - 1)^m \sum_{j=0}^{p} a_j \omega_1(z + c_j) - \omega_2^n (\omega_2 - 1)^m \sum_{j=0}^{p} a_j \omega_2(z + c_j);$$

(IV) when $P(\omega) \equiv c_0$ then $f \equiv tg$ for some constant t such that $t^{n+1} = 1$.

Proof. Suppose

$$f^n P(f) \sum_{j=0}^{p} a_j f(z + c_j) \equiv g^n P(g) \sum_{j=0}^{p} a_j f(z + c_j). \tag{0.8}$$

Since g is transcendental entire function, hence $g(z)$, $\sum_{j=0}^{p} a_j g(z + c_j) \not\equiv 0$. We consider following two cases.

Case 1. $P(\omega) \equiv c_0$. Let $h = \frac{f}{g}$. If h is a constant, by putting $f = hg$ in (0.8) we get

$$a_m g^m (h^{m+n+p} - 1) + a_{m-1} g^{m-1} (h^{m+n} - 1) + \ldots + a_0 (h^{n+1} - 1) \equiv 0,$$

which implies that $h^d = 1$, where $d = GCD(n+m+p, \ldots, n+m+p-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0, 1, \ldots, m\}$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n+m+p, \ldots, n+m+$

$p-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i \in \{0,1,\ldots,m\}$. If h is not a constant, we know by (0.8) that f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where

$$R(\omega_1,\omega_2) = \omega_1^n P(\omega_1) \sum_{j=0}^{p} a_j \omega_1(z+c_j) - \omega_2^n P(\omega_2) \sum_{j=0}^{p} a_j \omega_2(z+c_j).$$

We now discuss the following subcases.

Subcase 1. $P(\omega) = \omega - 1$. Then from (0.8) we have

$$f^n(f^m-1) \sum_{j=0}^{p} a_j f(z+c_j) \equiv g^n(g^m-1) \sum_{j=0}^{p} a_j g(z+c_j). \tag{0.9}$$

Let $h = \frac{f}{g}$. clearly from (0.9) we get

$$g^m[h^{m+n} \sum_{j=0}^{p} a_j h(z+c_j) - 1] \equiv h^n \sum_{j=0}^{p} a_j h(z+c_j) - 1. \tag{0.10}$$

First we suppose that h is non-constant. We assert that $h^{m+n} \sum_{j=0}^{p} a_j h(z+c_j)$ is a non-constant. If not let $h^{m+n} \sum_{j=0}^{p} a_j h(z+c_j) \equiv c_1 \in \mathbb{C} \setminus \{0\}$. Then we have

$$h^{n+m} \equiv \frac{c_1}{\sum_{j=0}^{p} a_j h(z+c_j)}.$$

Now by Lemmas 1 and refL3 we get

$$(n+m)T(r,h) \leq T(r, \sum_{j=0}^{p} a_j h(z+c_j)) + S(r,h),$$

which contradicts with $n > m+p+4$. Thus from (0.10) we have

$$g^m \equiv \frac{h^n \sum_{j=0}^{p} a_j h(z+c_j) - 1}{h^{m+n} \sum_{j=0}^{p} a_j h(z+c_j) - 1}. \tag{0.11}$$

Let z_0 be a zero of $h^{m+n} \sum_{j=0}^{p} a_j h(z+c_j) - 1$. Since g is an entire function, it follows that z_0 is also a zero of $h^n \sum_{j=0}^{p} a_j h(z+c_j) - 1$. Consequently z_0 is a zero of $h^m - 1$ and so

$$\overline{N}(r,0;h^{m+n} \sum_{j=0}^{p} a_j h(z+c_j)) \leq \overline{N}(r,0;h^m) \leq mT(r,h) + O(1).$$

So in view of Lemmas 1, 4, 5 and second fundamental theorem we get

$$\begin{aligned}(n+m+p)T(r,h) &= T(r,h^{m+n}\sum_{j=0}^{p}a_jh(z+c_j))+S(r,h)\\ &\leq \overline{N}(r,0;h^{m+n}\sum_{j=0}^{p}a_jh(z+c_j))+\overline{N}(r,1;h^{m+n}\sum_{j=0}^{p}a_jh(z+c_j))+S(r,h)\\ &\leq N(r,0;h)+mT(r,h)+pT(r,h)+S(r,h)\\ &\leq (m+p+1)T(r,h)+S(r,h),\end{aligned}$$

which contradicts with $n>1$. Hence h is a constant. Since g is transcendental entire funtion, from (0.10) we have

$$h^{n+m}\sum_{j=0}^{p}a_jh(z+c_j)-1\equiv 0 \Longleftrightarrow h^{n}\sum_{j=0}^{p}a_jh(z+c_j)-1\equiv 0$$

and so $h^m=1$. Thus $f=tg$ for a constant t such that $t^m=1$.

Subcase 2. Let $P(\omega)=(\omega-1)^m$. Then from (0.8) we have

$$f^n(f-1)^m\sum_{j=0}^{p}a_jf(z+c_j)=g^n(g-1)^m\sum_{j=0}^{p}a_jg(z+c_j). \tag{0.12}$$

Let $h=\frac{f}{g}$. If $m=1$, then the result follows from Subcase 1. For $m\geq 2$: first we suppose that h is non-constant. Then from (0.12) we can say that f and g satisfy the algebraic equation $R(f,g)\equiv 0$, where

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m\sum_{j=0}^{p}a_j\omega_1(z+c_j)-\omega_2^n(\omega_2-1)^m\sum_{j=0}^{p}a_j\omega_2(z+c_j);$$

Next, we suppose that h is a constant, then from (0.12) we get

$$f^n\sum_{j=0}^{p}a_jf(z+c_j)\sum_{i=0}^{m}{}^mC_{m-i}f^{m-i}\equiv g^n\sum_{j=0}^{p}a_jg(z+c_j)\sum_{i=0}^{m}{}^mC_{m-i}g^{m-i}. \tag{0.13}$$

Now substituting $f=gh$ in (0.13) we get

$$\sum_{i=0}^{m}(-1)^{im}C_{m-i}g^{m-i}(h^{m+n+p-i}=1)\equiv 0.$$

which implies that $h=1$. Hence $f\equiv g$.

Case 2. $P(\omega)\equiv c_0$. Let $h=\frac{f}{g}$. Then from (0.8) we have

$$h^n(z)\equiv\frac{1}{\sum_{j=0}^{p}a_jh(z+c_j)}. \tag{0.14}$$

Thus from Lemmas 1 and 3 we have

$$nT(r,h)=T(r,\sum_{j=0}^{p}a_jh(z+c_j))+O(1)=pT(r,h)+S(r,h),$$

which is a contradiction since $n\geq 2$. Hence h must be a constant, which implies that $h^{n+p}=1$, thus $f=tg$ and $t^{n+p}=1$. This completes the proof. □

IV. Proof of Main Results

Proof of Theorem 11.

Proof. Let $F = \frac{f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)}{p}$ and $G = \frac{g^nP(g)\sum_{j=0}^{p} a_jg(z+c_j)}{p}$. Then F and G share $(1,2)$ except the zeris of $p(z)$. Now applying Lemma we see that one of the following three cases holds.

Case 1. Suppose

$$T(r,F) \leq N_2(r,0;F) + N_2(r,0;G) + S(r,F) + S(r,G).$$

Now by applying Lemmas 1 and we have

$$\begin{aligned}
T(r,F) &\leq N_2(r,0;F) + N_2(r,0;G) + S(r,F) + S(r,G)\\
&\leq N_2(r,0;f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j)) + N_2(r,0;g^nP(g)\sum_{j=0}^{p} a_jg(z+c_j)) + S(r,f) + S(r,g)\\
&\leq N_2(r,0;f^nP(f)) + N_2(r,0;g^nP(g)) + N_2(r,0;\sum_{j=0}^{p} a_jf(z+c_j)) + N_2(r,0;\sum_{j=0}^{p} a_jg(z+c_j)) + S(r,f) + S(r,g)\\
&\leq 2N(r,0;f) + N(r,0;P(f)) + N(r,0;\sum_{j=0}^{p} a_jf(z+c_j)) + 2N(r,0;g) + N(r,0;P(g))\\
&+ N(r,0;\sum_{j=0}^{p} a_jg(z+c_j)) + S(r,f) + S(r,g)\\
&\leq (2+m^*+p)T(r,f) + N(r,0;f) + (2+m^*+p)T(r,g) + N(r,0;g) + S(r,f) + S(r,g)\\
&\leq (2+m^*+p)T(r,f) + (2+m^*+p)T(r,g) + S(r,f) + S(r,g)\\
&\leq (4+2m^*+2p)T(r) + S(r).
\end{aligned}$$

From Lemmas 1 and Lemma 4 wwe have

$$(n+m^*+p)T(r,f) \leq (4+2m^*+2p)T(r) + S(r). \tag{0.15}$$

Similarly, we have

$$(n+m^*+p)T(r,g) \leq (4+2m^*+2p)T(r) + S(r). \tag{0.16}$$

Combining the inequalities (0.15) and (0.16), we get

$$(n+m^*+p)T(r) \leq (4+2m^*+2p)T(r) + S(r),$$

which contradicts with $n > m^* + p + 4$.

Case 2. $F \equiv G$. Then we have

$$f^nP(f)\sum_{j=0}^{p} a_jf(z+c_j) \equiv g^nP(g)\sum_{j=0}^{p} a_jg(z+c_j).$$

and so the result follows from Lemma 11. □

Case 3. $FG \equiv 1$. Then we have

$$f^n P(f)\sum_{j=0}^{p} a_j f(z+c_j) g^n P(g)\sum_{j=0}^{p} a_j g(z+c_j) \equiv p^2.$$

and so result from Lemma 9. This completes the proof.

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