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Original Article

Signed and Product Cordial Labeling of Pyramid Fibonacci Graph and Ripples Hamiltonian Graph

N. Jansirani[1], R. Rama[2]

[1,2]Department of Mathematics, Queen Mary's College, University of Madras, Chennai, Tamilnadu, India.

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***Abstract** - In this paper, patterns of subgraphs and some properties of Pyramid Fibonacci Graph(PFG) are analysed. Signed cordial labeling of PFG and Ripples Hamiltonian Graph(RHG) are derived. Furthermore, signed product cordial labeling of RHG is obtained.*

***Keywords** - PFG, RHG, Signed cordial, Signed product cordial.*

1. Introduction

The Graph labeling is one of the fascinating area of Graph Theory with wide range of applications. Graph labeling was first introduced in the 1960's by Rosa[20]. A labeling (or valuation) of a graph is a map that carries some set of graph elements to numbers, most often to the positive or nonnegative integers. The most common choices of domain are the set of all vertices and edges (such labelings are called totallabelings), the vertex-set alone (vertex-labelings), or the edge-set alone (edge-labelings). Labeled graph are becoming an increasingly useful family of mathematical models for a broad range of application. According to Beineks and Hegde[6] graph labelling serves as a frontier between number theory and structure of graph.

The concept of cordial graph was introduced by Cahit [2]. Shee and Ho [16] proved that path union of cycles, Petersen graphs, trees, wheels, unicyclic graphs is cordial. Vaidya et al. [17] proved that graph obtained by joining two copies of cycles by a path of arbitrary length is cordial. Harary introduced S-Cordiality with the first letter of Signed Cordiality. Devaraj et al.[4] proved that Petersen graph, complete graph, book graph, Jahangir graph and flower graph are signed cordial. The concept of signed product cordial labeling was introduced by Baskar Babujee [3]. P.Lawrence et al. [13] proved that arbitrary super subdivision of some graphs is signed product cordial.

2. Basic Definitions

In this section, some basic definitions are presented. If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. Most of the graph labeling problems have the following three common characteristics: a set of numbers for assignment of vertex labels, a rule that assigns a label to each edge and some condition(s) that these labels must satisfy.

Definition 2.1:

A mapping $f: V(G) \to \{0, 1\}$ is called binary vertex labeling of G under $f(v)$, called label of vertex v of G under f. The induced edge labelling $f^*: V(G) \to \{0, 1\}$ defined by $f^*(uv) = |f(u) - f(v)|$. Let $v_f(i)$ and $e_f(j)$ are respectively the number of vertices labeled with i and the number of edges labeled with j under f^*. A binary vertex labeling of graph G is cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is cordial if admits cordial labeling[3].

Definition 2.2:

A graph $G = (V, E)$ is called signed cordial if it is possible to label the edges with the number from the set $\mathbb{N} = \{+1, -1\}$ in such a way that at each vertex v, the algebraic product of the labels on the edges incident with V is either +1 or -1 and the inequalities $|v_f(+1) - v_f(-1)| \leq 1$ and $|e_{f^*}(+1) - e_{f^*}(-1)| \leq 1$ are also satisfied, where $v_f(i)$, i belongs to {+1, -1} and $e_f(j)$, j belongs to {+1, -1} are respectively the number of vertices labeled with i and the number of edges labeled with j. A graph is called signed cordial if it admits signed cordial labeling[3].

Definition 2.3:

A vertex labeling of graph G $f: V(G) \rightarrow \{-1, 1\}$ with induced edge labelling $f^*: E(G) \rightarrow \{-1, 1\}$ defined by $f^*(uv) = f(u)f(v)$ is called a signed product cordial labeling if $|v_f(-1) - v_f(+1)| \leq 1$ and $|e_{f^*}(-1) - e_{f^*}(+1)| \leq 1$ where $v_f(-1)$ is the number of vertices labeled with –1, $v_f(1)$ is the number of vertices labeled with 1, $e_{f^*}(-1)$ is the number of edges labeled with –1 and $e_{f^*}(+1)$ is the number of edges labeled with 1. A graph G is signed product cordial if it admits signed product cordial labeling[3].

Definition 2.4:

PFG and RHG are defined in [21, 22].

3. Patterns of Subgraphs in PFG

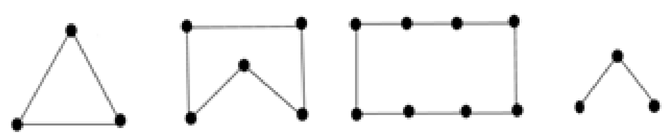
In this section, patterns of subgraphs of PFG and connection between subgraphs of PFG are defined. Further, some of the properties are Eulerian, Hamiltonian, k^{th} level of average vertex degree, block and matching of PFG are studied.

3.1. Subgraphs of PFG

Construction of PFG defined in [22] and is depicted in figure 2. Let $G_F^P(n)$ denotes the joint n^{th} level of PFG. The total number of vertices and edges at k^{th} level of $G_F^P(k)$ are $\sum_{i=1}^{k} f_{i+3}$ and $\sum_{i=1}^{k} f_{i+3} + 2(k-1)$ respectively. There are atmost three level patterns in this graph and the patterns are shown in figure 1. It is interested to note that, the subgraph in 1(d) (figure 1) follows the famous Fibonacci sequence from level 4 successively. Also the first 3 level patterns follows cycle graphs, where the number of vertices in each patterns follows the Fibonacci sequence f_4, f_5, f_6. The vertices of all levels, except level 1 and level 3 are placed in three horizontal parallel lines. H_j^i denotes the number of vertices at the i^{th} level on the j^{th} lines where $j = 1,2,3$. H_j^i satisfies the following equations

$$H_j^{i+1} = H_j^i + H_j^{i-1} \text{ where } i \geq 4, j = 1, 2, 3$$

H_2^i follows the Fibonacci Sequence $1, 1, 2, 3, 5, \ldots$ where $f_{n+1} = f_n + f_{n-1}$.



(a) (b) (c) (d)

Fig. 1 Subgraphs Patterns

Theorem 3.1.

$G_F^P(n)$ has the following properties.

1. $G_F^P(n)$ is a Hamiltonian for $n \geq 1$.
2. $G_F^P(n)$ is not Eulerian for $n \geq 2$.
3. Each level of PFG is block.

Proof:

1. Consider in figure 2. PFG is a connected graph, therefore there exists a path between any vertices of PFG. Since $v_1, v_2, v_3, v_5, v_4, v_7, v_6, v_8, v_{12}, v_{11}, v_{10}, v_9, v_{13}, v_{14}, v_{15}, v_{16}$ is a Hamiltonian path.
2. Given PFG is a connected graph.
   Assume that, PFG is Eulerian.
   By definition of Eulerian, every vertex of PFG has even degree. But some vertices of PFG has odd degree (see in figure. 2). Therefore, our assumption is wrong. Hence PFG is not Eulerian.
3. Given that, PFG be a connected graph. That is, there exists a path between any two vertices of PFG. We know that, the construction of PFG follows by the levels. The first level of PFG is a 3-cycle. If we remove any vertex in C_3 then PFG is connected. That is no cut vertex and hence PFG is block. Next consider the second level of PFG, if we remove any vertex then PFG is connected and block, that is, doesn't have cut vertex. The process following for each levels of PFG then each level of PFG is connected. Therefore, each level of PFG is block.

Theorem 3.2.
Let PFG is a connected planar graph . Then
(i). The average vertex degree of k^{th} level of PFG is

$$\frac{2(\sum_{i=1}^{k} f_{i+3} + 2(k-1))}{\sum_{i=1}^{k} f_{i+3}}$$

(ii). The average vertex degree at any level of PFG is lies between 2 and 3. That is

$$2 \leq \frac{2(\sum_{i=1}^{k} f_{i+3} + 2(k-1))}{\sum_{i=1}^{k} f_{i+3}} \leq 3$$

Proof:
(i). We know that, for any graph G satisfies the following condition

$$\sum_{v \in V(G)} d(V(G)) = 2E(G)$$

Therefore the average vertex degree of k^{th} level of PFG

$$= \frac{2E(PFG)}{V(PFG)}$$

$$= \frac{2(\sum_{i=1}^{k} f_{i+3} + 2(k-1))}{\sum_{i=1}^{k} f_{i+3}}$$

(ii). Further, in PFG has a minimum degree 2 and the maximum degree 3. Hence the average vertex degree in k^{th} level lies betwee 2 and 3.

Example 3.3.
The average vertex degree of 5^{th} level of PFG is

$$= \frac{2(\sum_{i=1}^{5} f_{i+3} + 2(5-1))}{\sum_{i=1}^{5} f_{i+3}}$$

$$= \frac{2\,(58)}{50} = 2.32.$$

Theorem 3.4.
Let PFG be a connected graph and $\mathcal{M}$ its matching. Then PFG is a perfect matching if $3i^{th}$ or $3(i-1)^{th}$level, where $i \geq 1$.
Proof:
Assume that, the total number of vertices in k^{th} level of PFG is odd and is a perfect matching. Therefore,

$$|V(PFG)| = 2n + 1$$

By definition of a perfect matching is which covers every vertex of the graph PFG. Each eadge in PFG have two end vertices. Since $2n + 1$ must divided by 2, but it is not true. Therefore our assumption is wrong. Hence the cardinality of PFG must have even.

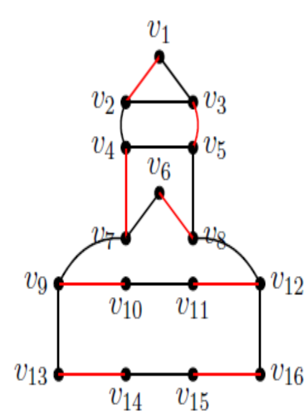


Fig.2 $G_F^P(3)$ **Perfect Matching**

In figure 2, the total number of vertices in 3^{rd} level of PFG is 16. Hence it has a perfect matching and the matching are

$$\{v_1v_2,\ v_3v_5,\ v_4v_7,\ v_6v_8,\ v_9v_{10},\ v_{11}v_{12},\ v_{13}v_{14},\ v_{15}v_{16}\}$$

4. Signed and Product Cordial Labeling PFG and RHG

In this section, we have proved that PFG and RHG admit signed cordial labeling. And also we obtained signed product cordial labeling of RHG.

Theorem 4.1:
PFG at level k ≥ 1 admits signed cordial labeling.
Proof:
Let *G* be a Pyramid Fibonacci Graph at level k ≥ 1 with *n* vertices and *m* edges, where

$$n = \sum_{i=1}^{k} f_{i+3}$$
$$m = \sum_{i=1}^{k} f_{i+3} + 2(k-1)$$

and f_i is the i^{th} term of the Fibonacci sequence.
Let $v_1, v_2, \dots, v_{n-1}, v_n$ and $e_1, e_2, \dots, e_{m-1}, e_m$ are respectively vertices and edges of PFG.
The labels are assigned for the vertices as follows

$$f: V(G) \to \{-1, 1\}$$
$$f(v_j) = \begin{cases} +1 ; & j \ is\ even \\ -1 ; & j \ is\ odd \end{cases} \quad \text{where } 1 \le j \le n.$$

Case (i): n is even
Hence $\frac{n}{2}$ vertices are labeled $+1$ and the remaining $\frac{n}{2}$ vertices are labeled -1.
Thus $v_f(+1) = v_f(-1)$. Therefore it satisfied the condition

$$|v_f(+1) - v_f(-1)| \le 1$$

Case (ii): n is odd
Hence $\frac{n-1}{2}$ vertices are labeled $+1$ and the remaining $\frac{n-1}{2} + 1$ vertices are labeled -1.
Thus $v_f(+1) = v_f(-1)$. Also it satisfied the condition

$$|v_f(+1) - v_f(-1)| \le 1$$

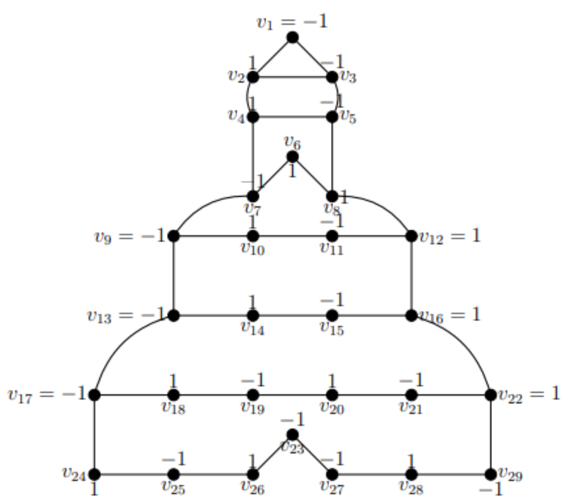


Fig. 3 Signed cordial labelling of $G_F^P(4)$

The labels are assigned for the edges as follows

$$f^*: E(G) \to \{+1, -1\}$$

$$f(e_j) = \begin{cases} +1 & ; j \ is\ odd \\ -1 & ; j\ is\ even \end{cases} \quad \text{where } 1 \le j \le m.$$

Case (i): m is even
The number of $\frac{m}{2}$ edges are labelled $+1$ and the remaining $\frac{m}{2}$ edges are labelled -1.
Thus $e_{f^*}(+1) = e_{f^*}(-1)$. Hence it is satisfied the condition

$$\left|e_{f^*}(+1) - e_{f^*}(-1)\right| \leq 1$$

Case (ii): m is odd

The number of $\frac{m-1}{2}$ edges are labelled $+1$ and the remaining $\frac{m-1}{2} + 1$ edges are labelled -1.

Hence it is satisfied the condition

$$\left|e_{f^*}(+1) - e_{f^*}(-1)\right| \leq 1$$

Theorem 4.2:

RHG with level $k \geq 1$ admits signed cordial labeling.

Proof:

Let G be a Ripples Hamiltonian Graph at level $k \geq 1$ with p vertices and q edges, where

$$p = \sum_{i=1}^{k} f_{i+3}$$
$$q = \sum_{i=1}^{k} f_{i+3} + 2(k-1)$$

and f_i is the i^{th} term of the Fibonacci sequence.

Let $v_1, v_2, \dots, v_{p-1}, v_p$ and $e_1, e_2, \dots, e_{q-1}, e_q$ are respectively vertices and edges of PFG.

The labels are assigned for the vertices as follows

$$f: V(G) \to \{-1, 1\}$$
$$f(v_l) = \begin{cases} +1\,; & l \text{ is even} \\ -1\,; & l \text{ is odd} \end{cases} \quad \text{where } 1 \leq l \leq p.$$

Case (i): p is even

Hence $\frac{p}{2}$ vertices are labeled $+1$ and the remaining $\frac{p}{2}$ vertices are labeled -1. Thus $v_f(+1) = v_f(-1)$. Therefore it satisfied the condition

$$\left|v_f(+1) - v_f(-1)\right| \leq 1$$

Case (ii): p is odd

Hence $\frac{p-1}{2}$ vertices are labeled $+1$ and the remaining $\frac{p-1}{2} + 1$ vertices are labeled -1. Hence it is satisfied the condition

$$\left|v_f(+1) - v_f(-1)\right| \leq 1$$

The edges of RHG are assigned for the labels as follows

$$f^*: E(G) \to \{+1, -1\}$$
$$f(e_l) = \begin{cases} +1\,; & l \text{ is odd} \\ -1\,; & l \text{ is even} \end{cases} \quad \text{where } 1 \leq l \leq q.$$

Case (i): q is even

The number of $\frac{q}{2}$ edges are labelled $+1$ and the remaining $\frac{q}{2}$ edges are labelled -1. Thus $e_{f^*}(+1) = e_{f^*}(-1)$. Hence it is satisfied the condition

$$\left|e_{f^*}(+1) - e_{f^*}(-1)\right| \leq 1$$

Case (ii): q is odd

The number of $\frac{q-1}{2}$ edges are labelled $+1$ and the remaining $\frac{q-1}{2} + 1$ edges are labeled -1.

Hence it is satisfied the condition

$$\left|e_{f^*}(+1) - e_{f^*}(-1)\right| \leq 1$$

Theorem 4.3:

RHG at level $k \geq 1$ admits signed product cordial labeling.

Proof:

Let G be a Ripples Hamiltonian Graph at level $k \geq 1$.

Let the vertices be v_{ij}, $0 \leq i \leq k-1$, $0 \leq j \leq f_{i+3} - 1$, where f_i is the i^{th} term of the Fibonacci Sequence.

The vertex labelling as follows

$$f: V(G) \to \{-1, 1\}$$

Case (i): i – even

$0 \leq i \leq k-1,\ 0 \leq j \leq f_{i+3} - 1$

$$v_{ij} = \begin{cases} +1 & ; j \text{ is even} \\ -1 & ; j \text{ is odd} \end{cases}$$

Case (ii): i-odd

$$v_{ij} = \begin{cases} -1 ; & j \text{ is even} \\ +1 ; & j \text{ is odd} \end{cases}$$

Label the edges as follows
Case(i): i-even

$$e_{ij,i(j+1)} = \begin{cases} -1 ; & j \text{ is even} \\ +1 ; & j \text{ is odd} \end{cases}$$

Case(ii): i-odd

$$e_{ij,i(j+1)} = \begin{cases} +1 ; & j \text{ is even} \\ -1 ; & j \text{ is odd} \end{cases}$$

Label the joined edges as follows

$$e_{ij,i0} = \begin{cases} -1 ; & i \text{ is even} \\ +1 ; & i \text{ is odd} \end{cases}$$

Label the level i and level $i+1$ connected edges as follows

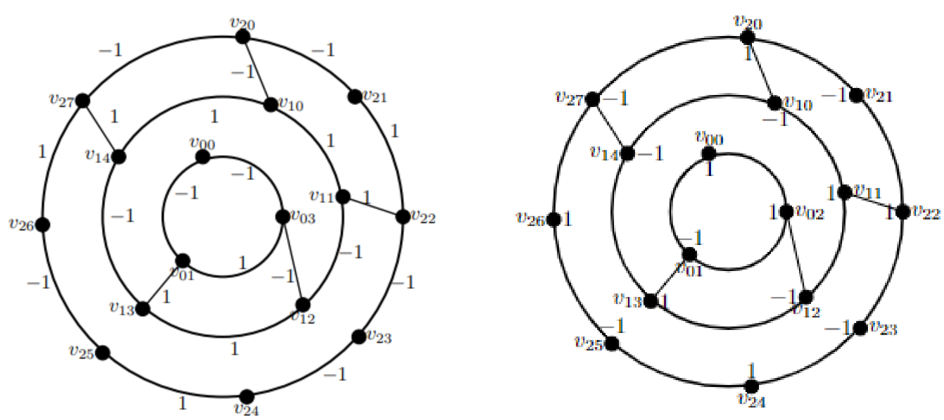
$$e_{ij,(i+1)(j\pm n)} = \begin{cases} -1 ; & j-n \\ +1 ; & j+n \end{cases} \text{ where } n \in \mathbb{W}$$

The graph *RHG* satisfies the condition

$$\left|v_f(+1) - v_f(-1)\right| \leq 1$$

$$\left|e_{f^*}(+1) - e_{f^*}(-1)\right| \leq 1$$

Hence *RHG* is admits Signed Product cordial.



(a) Edge labeling **(b) Vertex labeling**

Fig. 4 Signed Product Cordial labeling of RHG at level - 3

5. Conclusion

In this research article, subgraph patterns of PFG and Signed cordial labeling of PFG and RHG are studied. Further Signed product cordial labeling of RHG is derived.

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