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Original Article

The Application of Minimum Polynomial

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Abstract - *Matrix theory plays an important role in algebra research and linear control, and the minimum polynomial plays an important role in matrix. Therefore, it is necessary to study the application of minimal polynomials. In this paper, six applications of minimum polynomial include solving matrix function and its inverse function by using minimum polynomial, determining singularity of matrix polynomial, solving matrix equation, determining whether matrix can be diagonalized, solving basis and dimension of linear space, and solving differential equation with constant coefficient.*

Keywords - *The matrix function and its inverse, The singularity of matrix polynomials, Diagonalizable, The basis and dimension.*

1. Introduction

In the development of society and the development of mathematics, algebra is always very important. At the same time, the minimum polynomial plays an important role in algebra. Not only the theoretical research, the practical application of the minimum polynomial also solves many problems. For example, matrix polynomials can judge whether a matrix can be diagonalized, solve matrix functions and linear equations, etc., which are also very important in the study of automation control. Therefore, if you want to do new research in the field of algebra and engineering control, you must be very familiar with the minimum polynomial, but also grasp how the minimum polynomial can help us solve practical problems.

The concept of minimal polynomial is first given in the Hamilton-Cayley theorem. Therefore, [1-3] describes the origin and development of the Hamilton-Cayley theorem, as well as the rationality proof and other related properties of the theorem. Before proposing the minimum polynomials, we mainly study the characteristic polynomials of matrices. The minimum polynomial is a special case of the characteristic polynomial, so to understand the minimum polynomial we should start with the characteristic polynomial. Literature [4-10] has analyzed the impulse condition that the characteristic polynomial is equivalent to the minimum polynomial, and the method of solving the minimum polynomial. If you want to solve practical problems by minimal polynomials, you need to have a good understanding of their properties and particularity. Literature [11-20] introduces various properties of minimal multinomial and gives corresponding proofs. The application of minimum polynomial in control system and linear multivariable system is briefly introduced in literature [21-25]. Therefore, the main contents of this paper are as follows: Section 2: Using the minimum polynomial to solve the matrix function and its inverse function; Section 3: Judge whether the matrix polynomial has singularity; Section 4: Using minimal polynomial to solve matrix equation; Section 5: How to judge whether the matrix can be diagonalized by using the minimum polynomial; Section 6: Solving the basis and dimension of linear space according to the minimum polynomial; Section 7: Solving differential equations with constant coefficients by means of minimum polynomials; Finally, the eighth section summarizes the full text.

2. Solving the Matrix Function and its Inverse

For polynomials of high degree, it is very troublesome to solve the matrix function. If we want to do it directly, the minimum polynomial here makes it easy and fast.

Example 1. Setting $A=\begin{pmatrix}2&0&0\\1&1&1\\1&-1&3\end{pmatrix}$, $f(\lambda)=\lambda^6+2\lambda^5-\lambda^4+3\lambda^2-2\lambda+1$, try to solve $f(A)$.

Solution: First find the minimum polynomial: $f_A(\lambda)=|\lambda E-A|=(\lambda-2)^3$, the minimum polynomial of matrix A possible: $m_1=(\lambda-2)^3$, $m_2=(\lambda-2)^2$, $m_3=(\lambda-2)$.

Because $m_3(A)=(A-2E)\neq 0$, $m_2(A)=(A-2E)^2=0$, so $m_A(\lambda)=(\lambda-2)^2$.

Then divide $m_A(\lambda)$ by $f(\lambda)$ to get: $f(\lambda) = q(\lambda)m_A(\lambda) + r(\lambda)$, we can get $r(\lambda) = 330\lambda - 539$.

So $f(A) = r(A) = \begin{pmatrix} 121 & 0 & 0 \\ 330 & -209 & 330 \\ 330 & -330 & 451 \end{pmatrix}$.

3. Judge the Singularity of Matrix Polynomials

Theorem 3.1: Let matrix A be a matrix of order n, $f(\lambda)$ is a polynomial of degree λgreater than zero, and the smallest polynomial of matrix A be $m_A(\lambda)$, then we have:

(1) If $f(\lambda)$ can be divisible $m_A(\lambda)$, then $f(A)$ is degenerate;

(2) If the largest common factor of $f(\lambda)$and $m_A(\lambda)$is $d(\lambda)$, then the rank of $f(A)$and $d(A)$is equal;

(3) The necessary and sufficient condition for $f(A)$ to be non-degenerate is that$\left(m_A(\lambda), f(\lambda)\right) = 1$,that is $f(\lambda)$ and $m_A(\lambda)$ are prime to each other.

Example 2. $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix}$ we know that the matrix polynomials are $f(\lambda) = \lambda^6 + 2\lambda^5 - \lambda^4 + 3\lambda^2 - 2\lambda + 1$ and $g(\lambda) = \lambda^4 - 2\lambda^3 + 2\lambda^2 - 3\lambda - 2$, so is A degenerate?

Solution: Find the minimum polynomial in matrix A is $m_A(\lambda) = (\lambda - 2)^2$, and because $f(\lambda)$ and $m_A(\lambda)$ are prime to each other. And the greatest common factor of $m_A(\lambda)$ and $f(\lambda)$ is $d_1(\lambda) = \left(m_A(\lambda), f(\lambda)\right) = 1.$ So $f(A)$ is nonsingular. And the greatest common factor of $g(\lambda)$ and $f(\lambda)$ is $d_2(\lambda) = \left(m_A(\lambda), g(\lambda)\right) = (\lambda - 2)$, so $g(\lambda)$ is singular.

4. Solving a Matrix Equation

Using the minimum polynomial, the matrix$AX - XB = C$ can be solved more simply and easily.

Before solving the corresponding example, it is necessary to introduce the following definitions and theorems.

Definition 4.1 Let two linear Spaces X and Y have a mapping from X to Y, which we call T. Let $D(T)$ be the domain of the mapping T and $R(T)$ be the range of the mapping T. Take any $\alpha, \beta, x_1, x_2 \in D(T)$, if $\alpha x_1 + \beta x_2 \in D(T)$, $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$, then say is a linear operator from X to Y in the domain $D(T)$.

Theorem 4.1: Let $A, B, X \in F^{n\times n}$, if $\tilde{A}, \underset{\sim}{B}$be a linear operator from $F^{n\times n}$to$F^{n\times n}$, then $\tilde{A}X = AX, \underset{\sim}{B}X = BX$e

Theorem 4.2: According to $m_{\tilde{A}}(\lambda) = m_A(\lambda), m_{\underset{\sim}{B}}(\lambda) = m_B(\lambda)$, equation $AX - XB = C$ is equivalent to equation

$$\left(\tilde{A} - \underset{\sim}{B}\right)X = C.$$

Theorem 4.3: A necessary and sufficient condition for $\tilde{A} - \underset{\sim}{B}$ invertibility is $\left(m_A(\lambda), m_B(\lambda)\right) = 1$and $X = \left(\tilde{A} - \underset{\sim}{B}\right)^{-1} C = -\sum_{k=0}^{r-1}\sum_{s=0}^{k} \alpha_{r-s} A^{r-1-k} C B^{k-s} u(B)$,

Where $r$is the degree of the minimum polynomial $m_A(\lambda)$of matrix A, and $u(\lambda)$, $v(\lambda)$satisfies $m_A(\lambda)u(\lambda) + m_B(\lambda)v(\lambda) = 1$.

Example 3: $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -2 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, try to solve the matrix equation $AX - XB = C$.

The minimum polynomial of matrix A and matrix B are $m_A(\lambda) = \lambda^3 + 2\lambda, m_B(\lambda) = \lambda^3 + \lambda^2 + 2\lambda + 1$. Because $m_A(\lambda)(-\lambda^2 - \lambda - 1) + m_B(\lambda)(\lambda^2 + 1) = 1$

Take $u(\lambda) = -(\lambda^2 + \lambda + 1)$, so $u(B) = -(B^2 + B + E) = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, So let's substitute $\alpha_1 = 2, \alpha_2 = 0\ \alpha_3 = 1$, down here $X = \left(\tilde{A} - \underset{\sim}{B}\right)^{-1} C = -\sum_{k=0}^{r-1}\sum_{s=0}^{k} \alpha_{r-s} A^{r-1-k} C B^{k-s} u(B)$, we can get

$$X = -(-AC^2 + ACB + CB^2 + 2C)u(B) = \begin{pmatrix} 3 & 1 & 1 \\ -3 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

5. Determine whether the Matrix is Diagonalizable

The necessary and sufficient conditions of matrix diagonalization can be obtained by the relation property and theory of minimum polynomial.

Lemma 5.1: Let $A \in C^{n\times n}$, the matrix A be similar to the diagonal matrix $\Leftrightarrow m_A(\lambda)$has no repeated roots.

Example 4: Prove that idempotent matrices are similar to diagonal matrices
Proof: First we need to understand the definition of an idempotent matrix. Let $A \in C^{n\times n}$, let $A^2 = A$ be true of matrices called idempotent matrices. A zeroing polynomial of matrix A is $f(\lambda) = \lambda^2 - \lambda$, and the minimal polynomial$m_A(\lambda)$ of matrix A divisible $f(\lambda)$. Since $f(\lambda)$ has no repeated roots, $m_A(\lambda)$ has no repeated roots too. According to the lemma, matrix A is similar to diagonal matrices.

6. Solve for the Basis and Dimension of a Linear Space

For matrix polynomials of a given square matrix, we mainly study the form of matrix powers. If the matrix itself does not have very obvious characteristics, then it is difficult to find the formal characteristics of each power of the matrix. In this case, if we need to calculate the degree of the matrix polynomial, there are some difficulties, so it is difficult to determine the basis and dimension of the linear space generated by the matrix polynomial. In algebra, a basis of a linear space is generally represented by the maximum independent group of the linear space. The calculation process of this representation method is very complicated and tedious. However, the appearance of the minimum polynomial makes the process of solving the basis and dimension of the linear space simple and fast.

Theorem 6.1： Let the linear space formed by all polynomials of $A \in F^{n\times n}$ be W, and the minimum polynomial of matrix A be$m_A(\lambda)$, then:

(1) The dimension of W is equal to the degree k of $m_A(\lambda)$, namely $dim(W) = \partial\big(m_A(\lambda)\big) = k$;

(2) $E, A, A^2, \dots, A^{k-1}$ is a basis for W.

Example 5: The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{bmatrix}$, where $w = \frac{-1+\sqrt{3}i}{2}$,$w = \{f(\lambda) | f(\lambda) \in R(\lambda)\}$, the characteristic polynomial of matrix A is $f(\lambda)$, try to solve the dimension of W and its basis.

Solution: Find the characteristic polynomial of matrix A is $f_A(\lambda) = (\lambda - 1)(\lambda - w)(\lambda - w^2)$

And since $w^2 = \frac{-1-\sqrt{3}i}{2}$, the matrix A has three different eigenvalues, the minimum polynomial of the matrix A is $m_A(\lambda) = f_A(\lambda) = (\lambda - 1)(\lambda - w)(\lambda - w^2)$.

And since $\partial\big(m_A(\lambda)\big)$ is three, the dimension of W is $dim(W) = \partial\big(m_A(\lambda)\big) = 3$.

So, the basis for the linear space W generated by all the characteristic polynomials of the matrix A is E, A, A^2, and its dimension is 3.

7. Solve Ordinary Differential Equations with Constant Coefficients

Given a system of constant coefficient differential equations, the form is as follows:

$$\frac{dx(t)}{dt} = Ax(t), x(t) = \big(x_1(t), x_2(t), x_3(t), \dots, x_{n-1}(t), x_n(t)\big)^T, (i = 1,2,3,\dots,n-1,n)$$

According to the definition of matrix function, the matrix function $f(A)$is generally expressed by the matrix polynomial $g(A)$corresponding to the consistent polynomial $f(\lambda)$in the eigenvalues of matrix A. But there is a non-unique problem, such that the polynomial matrix used to define the matrix function is also non-unique. This problem is solved when we solve a system of differential equations with constant coefficients by using the smallest polynomial.

Assume that all the different eigenvalues of matrix A are $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{s-1}, \lambda_s$, $\partial\big(m_A(\lambda)\big)$is m, $g(\lambda) = q(\lambda)m_A(\lambda) + r(\lambda)$ is obtained by division with remainder, $f(A)$ can be obtained according to the Lagrange interpolation formula. Next, we divided it into two cases for study.

(1) When the minimum polynomial $m_A(\lambda)$ of matrix A has no repeated root.

$$f(A) = g(A) = r(A) = \sum_{i=1}^{s} f(\lambda_i)\, Z_i(A), \text{ where } Z_i(A) = \prod_s \frac{A-\lambda_j I}{\lambda_i - \lambda_j}, i, j = 1,2,3,\dots,s \quad (1).$$

(2) When the minimum polynomial $m_A(\lambda)$ of matrix A has repeated root.

$$m_A(\lambda) = (\lambda-\lambda_1)^{d_1}(\lambda-\lambda_2)^{d_2}(\lambda-\lambda_3)^{d_3}\dots(\lambda-\lambda_s)^{d_s}, \text{ where } d_1 + d_2 + d_3 + \dots + d_s = m \le n, \text{ then}$$

$$f(A) = \sum_{i=1}^{s} m_i(A)\left[a_{i1} I + a_{i2}(A-\lambda_i I)^{d_1} + a_{i3}(A-\lambda_i I)^{d_2} + \dots + a_{id_i}(A-\lambda_i I)^{d_{i-1}}\right],$$

$$m_i(A) = (A-\lambda_1 I)^{d_1}\dots(A-\lambda_s I)^{d_s},$$

$$a_{ij} = \frac{1}{(j-1)!}\frac{d^{j-1}}{d\lambda^{j-1}}\left[(\lambda-\lambda_i)^{d_i}\frac{f(\lambda)}{m_A(\lambda)}\right]|\lambda=\lambda_i, \quad i = 1,2,\dots,s; j = 1,2,\dots,d_s.$$

Example 6: Solve a system of linear differential equations $\frac{dx}{dt} = Ax, X(0) = (0,1,-1)^T$,

where $A = \begin{pmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$.

Solution: The minimum polynomial in matrix A is $m_A(\lambda) = (\lambda-4)(\lambda-2)$, and the basis solution matrix X is of the form $X = e^{At}c$, $c = (c_1, c_2, c_3), c_i (i = 1,2,3)$, not all zeros, as we know from the theory of differential equations. So, we just compute matrix functions $f(A) = e^{At}$ and vector c. So let $f(\lambda) = e^{\lambda t}, \lambda_1 = 4, \lambda_2 = 2$. And since the minimum polynomial has no repeated root, according to equation (1), we can get $f(A) = g(A) = r(A) = \sum_{i=1}^{s} f(\lambda_i)\, Z_i(A)$,

where $Z_1(A) = \frac{A-\lambda_2 I}{\lambda_1-\lambda_2}, Z_2(A) = \frac{A-\lambda_1 I}{\lambda_2-\lambda_1}$.

So, it's calculated $f(A) = e^{4t}\begin{pmatrix} 2 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - e^{2t}\begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$, when t=0, $f(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We know from the initial condition $c = (c_1, c_2, c_3) = (0,1,1)$, so the basis solution matrix that satisfies the initial conditions is $X(t) = e^{At}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ e^{At} \\ e^{At} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

8. Conclusion

This article introduces the application of six minimum polynomials, including solving matrix functions and their inverses, judging the singularity of polynomials, solving matrix equations, judging whether matrices can be diagonalized, solving the basis and dimension of linear spaces, and solving ordinary differential equations with constant coefficients. In order to better understand the application of minimal polynomials, this paper gives many related theorems, definitions and lemmas. The corresponding examples are also given for each application. Through the research and analysis of the application of minimal polynomials, we have a clearer understanding of the importance of minimal polynomials in the field of algebra. With the minimum polynomial, many complex problems can be simplified, thus saving manpower and material resources.

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