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Original Article

Construction of Positivity Preserving Full-Discrete Scheme for Stochastic Age-Structured Population Equations

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Abstract - *The numerical solutions of the stochastic age-structured population models may appear negative when the disturbance intensity is sufficiently large. This phenomenon goes against the biological meaning and people's cognition, but it is indeed the case. In order to circumvent this unreasonable situation, this paper aims to construct a full-discrete positive-preserving numerical scheme for stochastic age-structured population models. The technique we use is based on the balanced implicit method (BIM), a relatively mature tool that plays a crucial role in the issue of keeping positivity. Also, we proved the convergence of new numerical scheme we obtained, the strong convergence order is 1/2. At last, several numerical experiments are given to verify the accuracy of theoretical results.*

Keywords - *Stochastic age-structured population equations, Balanced implicit method, Numerical solution, Positivity preserving, Convergence.*

1. Introduction

It is important for humans to understand the laws regarding population dynamics, and age is actually one of the most natural and significant parameters for structuring the population. In the past few decades, several authors have achieved remarkable results in the study of age-dependent population systems, details can be seen in [1-3]. What's more, humans live in a real world with complex compositions so that the effects of stochastic perturbations on population cannot be ignored. Therefore, the stochastic age-structured population equations (SASPEs) has more practical and research meaning than the deterministic one. However, it is difficult to solve exact solutions of this system, people usually turn to find approximate solutions. In 2007, Zhang [4] studied the exponential stability of numerical solutions to a SASPEs with diffusion. Then, Wang et al. [5] researched the convergence of numerical solutions to SASPEs in 2008. We recommend readers to [11, 12] for more information about numerical simulation of SASPEs. Literature [15-18] describes the dynamics of population dynamics in age structure under discrete scenarios. Literature [19-22] introduces the dynamic analysis of infectious disease models with age structure. Literature [23-25] studies population persistence dynamics and optimal control of biological models with age structure.

When considering the numerical solutions, we should also pay attention to whether they conform to the practical meaning. In the biological sense, the population density of SASPEs is required nonnegative, but experiments show that the approximate solutions cannot satisfy this property well. In recent years, scholars have developed various numerical schemes to keep positivity, and the balanced implicit method (BIM) is one of them. In 1996, Schurz [6] proposed BIM in order to construct nonnegative solutions for stochastic ordinary differential equations (SODEs). Then, Milstein et al. [7] applied BIM to stiff SODEs and proved positiveness of solutions in 1998. Moreover Schurz [10] studied the convergence and stability of BIM for SODEs in 2005. Recently, some authors used BIM to real world models. Zhang et al. showed that BIM can preserve positivity of numerical solutions for stochastic SIQS epidemic models [8] and stochastic SIVS epidemic modes [9]. In 2017, Tan et al. [13] considered a positivity preserving numerical method for stochastic age-dependent population equations. However, all the above studies use semi-discrete numerical scheme, and no one has discussed the positivity preserving of fully discrete numerical format for SASPEs. From a computational point of view, the full-discrete scheme is easier to operate than the semi-discrete scheme. Therefore, the content of this paper is valuable to maintain positivity for full-discrete numerical scheme of SASPEs. In order to ensure the rigor of new scheme we constructed; we also prove the strong convergence order later.

The paper layout is listed below: In Section 2, we will give some necessary concepts and put forward several basic assumptions needed for this paper. Section 3 illustrates that the most commonly used numerical method: explicit Euler method cannot ensure the positivity of numerical solutions for stochastic age-structured population models. Therefore, we constructed a full-discrete BIM scheme to preserve positivity for numerical solutions of SASPEs. It is proved that the strong convergence and convergence order in Section 4. At last, several examples are given to verify the correctness of our conclusions.

2. Preliminaries

First, we will introduce some necessary concepts, notations and basic assumptions used in this paper. We consider the stochastic age-structured population equations as follows:

$$\begin{cases} (i)\, \dfrac{\partial P}{\partial t} = \left[-\dfrac{\partial P}{\partial a} - \mu(t,a)P\right]\mathrm{d}t + \sigma(t,a)P\,\mathrm{d}W_t, \qquad 0 \le t \le T, 0 \le a \le a^+, \\ (ii)\, P(t,0) = B(t) = \displaystyle\int_0^{a^+} \beta(t,a)P(t,a)\,\mathrm{da},\ 0 \le \mathrm{t} \le \mathrm{T}, \\ (iii)\, P(0,a) = P_0(a),\ 0 \le a \le a^+, \end{cases} \tag{2.1}$$

where $P(t,a)$ represents the age density function of age a at time $t, 0 < t < T,\ 0 < a < a^+ < \infty.$ $\mu(t,a), \beta(t,a)$ correspondingly denote the mortality and fertility of age a at time t. The number of births produced from all individuals per unit time at time t represented by $B(t).$ $\sigma(t,a)$ is a function of stochastic perturbation intensity. W_t is Wiener process defined on the filtered probability space $(\omega, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}(t))_{t\ge 0}$ is the natural filtration generated by W_t.

Due to the finite maximum age a^+, mortality rate μ is unbounded near the maximum age. Considering the fertility rate, from biological considerations we may assume that there exists an $\bar{a} < a^+$ such that $\beta(t,a) = 0$ for $a > \bar{a}$. An example in literature [1] illustrates that this assumption is objective. The figure of fertility $\beta(a)$ for Japanese woman at 2004 showed that the fertility decreases to 0 at age a, $a < a^+$. Therefore, the population renewal process only exists in the age interval $[0, \bar{a}]$ so that we may disregard older individuals if knowledge of the age distribution and the total number of individuals in the population past a certain age lager than a are not needed. Thus, in this paper, we will restrict ourselves to $[0, \bar{a}]$, where $\mu(t,a)$ can be assumed bounded.

Now, given some basic assumptions (1) The fertility rate $\beta(t,a) \in L^\infty((0,T),(0,\bar{a}))$, there exist constants $\underline{\beta}, \bar{\beta}, C$ for all $t_1, t_2 \in [0,T]$, $a_1, a_2 \in [0, \bar{a}]$, such that

$$0 < \underline{\beta} \le \beta(t,a) \le \bar{\beta} < +\infty;$$
$$|\beta(t_1,a_1) - \beta(t_2,a_2)|^2 \le C|(t_1-t_2)^2 + (a_1-a_2)^2|;$$

(2) The mortality $\beta(t,a) \in L^\infty((0,T),(0,\bar{a}))$, there exist constant $\underline{\mu}, \bar{\mu}, C$ for all $t_1, t_2 \in [0,T]$,

$a_1, a_2 \in [0, \bar{a}], C$, such that

$$0 < \underline{\mu} \le \mu(t,a) \le \bar{\mu} < +\infty;$$
$$|\mu(t_1,a_1) - \mu(t_2,a_2)|^2 \le C|(t_1-t_2)^2 + (a_1-a_2)^2|;$$

(3) The perturbation function $\sigma(t,a) \in L^\infty((0,T),(0,\bar{a}))$, there exist constant $\underline{\sigma}, \bar{\sigma}, C$ for all $t_1, t_2 \in [0,T]$, $a_1, a_2 \in [0, \bar{a}]$, such that

$$0 < \underline{\sigma} \le \sigma(t,a) \le \bar{\sigma} < +\infty;$$
$$|\sigma(t_1,a_1) - \sigma(t_2,a_2)|^2 \le C|(t_1-t_2)^2 + (a_1-a_2)^2|;$$

(4) The initial distribution $P_0 \in L^2(0,\bar{a}), P_0(a) \ge 0$ when $a \in [0,\bar{a}]$.

Lemma 2.1. Under the assumptions (1)-(4), the $Eq.\ (2.1)$ has a unique positive solution $P(t,a)$.

Moreover, for any given $0 < T < +\infty$, there exists C, such that

$$\mathbb{E}(\sup_{0 \le t \le T} \left| \int_0^{\bar{a}} P(t,a)\,da \right|^2) \le C\mathbb{E}||P_0||_1^2, \qquad (2.2)$$
$$\mathbb{E}|P(t_1,a_1) - P(t_2,a_2)|^2 \le C(|t_1-t_2| + |a_1-a_2|). \qquad (2.3)$$

3. Numerical Solution

For the convenience of description, we will briefly explain the necessary symbols at the beginning of this section. $\bar{a}$ represents the maximum age when the fertility rate $\beta(t,a) > 0$. Correspondingly, the positive integer $\overline{M}$ is the max number of age steps with age-time step $h = \frac{\bar{a}}{M}, T > 0$. Let T > 0 be the final time of approximation, $N = \frac{T}{h}$ is the max number of time steps. We denote the exact solution of (2.1) by $P(t,a)$, which only has nonnegative values. Now let $Q_j^n \approx P(t_n, a_j)$ as an approximate solution, where $t_n = nh, a_j = jh$. $\Delta W_n = W(t_{n+1}) - W(t_n)$, which is Gaussian distribution with mean 0 and standard deviation of the increment in time t, $\mu_j^n = \mu(t_n, a_j)$ and $\sigma_j^n = \sigma(t_n, a_j)$ are bounded, $0 < \underline{\mu} \le \mu_j^n \le \bar{\mu} < +\infty, 0 < \underline{\sigma} \le \sigma_j^n \le \bar{\sigma} < +\infty.$

3.1. Life time of numerical solutions

We generalize the definition about life time of numerical solutions from Schurz [14].

Definition 3.1. Assume that the $(P(t,a): 0 \le t \le T, 0 \le a \le \bar{a})$ satisfying (2.1) has only nonnegative values a.s. provided that $P_0 \ge 0$, i.e. it holds

$$\mathbb{P}(P(t,a) > 0) = 1, 0 \le t \le T, 0 \le a \le \bar{a}.$$

Then a numerical solution $Q_j^n \approx P(t_n, a_j)$ possesses an eternal life time if

$$\mathbb{P}\{Q_{j+1}^{n+1} > 0 \,|\, Q_j^n > 0\} = 1, 0 \le n \le N, 0 \le j \le M.$$

Otherwise, we say that a numerical solution has a finite life time.

3.2. Numerical solutions of explicit Euler schemes

Applying the explicit Euler method to system 2.1, we get the following numerical scheme

$$Q_{j+1}^{n+1} = Q_j^n - h\mu_j^n Q_j^n + \sigma_j^n Q_j^n \Delta W_n. \qquad (3.1)$$

Lemma 3.1. The explicit Euler numerical solution of (3.1) have a finite life time.

The explicit Euler approximation (3.1) may produce negative values. Next, we will discuss how to construct a numerical scheme to keep the numerical solution positive and have practical significance.

3.3. Construction of positive solutions for (2.1)

We consider the full-discrete BIM numerical scheme of system (2.1) as follows:

$$\begin{cases} (i)\, Q_{j+1}^{n+1} = Q_j^n - h\mu_j^n Q_j^n + \sigma_j^n Q_j^n \Delta W_n + C_j^n (Q_j^n - Q_{j+1}^{n+1}), \\ \;0 \le j \le \overline{M} - 1, 0 \le n \le N - 1, \\ (ii)\, Q_0^{n+1} = h \sum_{j=1}^{\overline{M}} \beta_j^n Q_j^n, \; 0 \le n \le N - 1, \\ (iii)\, Q_j^0 P_0(a_j), \; 0 \le j \le \overline{M}, \end{cases} \qquad (3.2)$$

where $\beta_j^n = \beta(t_n, a_j), C_j^n = C_j^0 h + C_j^1 |\Delta W_n|$. C_j^n, C_j^0, C_j^1 are called control functions, which are all the functions of Q_j^n. Next, we will prove the theorem about positivity preserving of full-discrete BIM scheme (3.2).

Theorem 3.1. The numerical solution Q_j^n of stochastic age-structured population equations (2.1) obtained by BIM ensures positivity if the following assumptions hold:

$$\begin{cases} (i)\, 1 + (C_j^0 - \mu_j^n) h > 0, \qquad \forall\; 0 \le j \le \overline{M}, 0 \le n \le N, \\ (ii)\, C_j^1 \ge \bar{\sigma}, \forall\; 0 \le j \le \overline{M}, 0 \le n \le N, \\ (iii) (1 + C_j^n)^{-1} > 0, \qquad \forall\; 0 \le j \le \overline{M}, 0 \le n \le N. \end{cases}$$

4. Convergence of the Numerical Scheme

In this section, we will discuss the strong convergence of BIM numerical scheme. We define the errors and related concepts for numerical approximation (3.2) in the beginning.

Definition 4.1. The errors of numerical approximation is denoted by $\varepsilon_j^n = P(t_n, a_j) - Q_j^n$, some standard norms related to errors are defined as follows:

$$||\varepsilon^n||_1^2 = h\sum_{j=0}^{\overline{M}}|\varepsilon_j^n|^2, \qquad ||\varepsilon||_{\infty,\infty} = \max_{\substack{0\le n\le N\\ 0\le j\le \overline{M}}}\{|\varepsilon_j^n|^2\}. \qquad (4.1)$$

Theorem 4.1. Based on the assumptions (1)-(4), the error satisfies the following properties:

$$\mathbb{E}||\varepsilon^n||_1^2 \le Ch, \qquad \mathbb{E}||\varepsilon||_{\infty,\infty} \le Ch.$$

where C is a positive constant independent of h.

5. Numerical Experiments

We consider the following equations

$$\begin{cases} \mathrm{d_t}P = \left[-\dfrac{\partial P}{\partial a} - \mu(t,a)P\right]\mathrm{dt} + \sigma(t,a)P\,\mathrm{d}W_t,\ 0\le t\le T, \\ \ 0\le a\le \bar{a} < a^+, \\ P(t,0) = B(t) = \displaystyle\int_0^{\bar{a}} \beta(t,a)P(t,a)\,\mathrm{da},\ 0\le \mathrm{t}\le \mathrm{T}, \\ P(0,a) = P_0(a),\ 0\le a\le \bar{a} < a^+, \end{cases} \qquad (5.1)$$

where W_t is a standard Brownian motion.

Example 1. In this example, let $T=2, a^+=2$, there exists an $\bar{a}=0.999$ such that $\beta(t,a)=0$ when age is $a\in(\bar{a},a^+]$. Therefore, we restrict ourselves on $a\in(0,\bar{a}]$. Let $\mu(t,a)=t+\frac{1}{(1-a)^2}, \beta(t,a)=\frac{1}{(1-a)^2}$, $a\in(0,\bar{a}]$, $P_0(a)=exp(-\frac{1}{1-a})$, $\sigma(t,a)$ is bounded, and $\sigma(t,a)<\bar{\sigma}=1$.

Now apply the explicit Euler scheme and full-discrete BIM scheme (3.2) to Example 1 respectively. We choose the control functions that satisfy the requirement of preserving positivity: $C_j^0 = t+\frac{1}{(1-a)^2}, C_j^1=\bar{\sigma}=1$. Using numerical simulation to draw the images of numerical solutions obtained by the above two methods in Figure 1. As shown in Figure 1, the explicit Euler numerical solutions appear negative values, however all of the BIM numerical solutions are positive. This verifies the accuracy of our conclusion.

Example 2. In this example, we discuss the convergence order of full-discrete BIM scheme. Let $\mu(t,a)=\beta(t,a) = \frac{1}{1+(2-a)^2}, P_0(a)=\frac{a}{1+a},\ 0\le a\le \bar{a},\ 0\le t\le T,$ where $\bar{a}=1, T=8$ (satisfaction (1)-(4)). The control functions $C_j^0=\mu_j^n=\frac{1}{1+(2-a_j)^2},\ C_j^1=\sigma=0.2$ (satisfaction Theorem 4.1). Because it is hard to get a true solution for the Example 2, The BIM numerical approximation of $B(T)$ is denoted by B_h^N with step size h. And B_h^N with $h=\frac{1}{2^{10}}$ is considered the true. In numerical experiments, we consider the absolute error of the number of newborns $B(T)$ at the final time $t=T$. The absolute error of $B(T)$ with step size h: $E_h = B(T)-B_h$. Here, $E_h = B_{\frac{1}{2^{10}}} - B_h$, B_h represents the numerical solution of step size h at time T. The convergence order of BIM scheme is by taking $M=1000$ sample paths on $[0,T]$, and five different step sizes are applied to each sample: $h=\frac{1}{2^p}$, $1\le p\le 5$. The formula for calculating convergence order is as follows:

$$r_h=\frac{log(\frac{1}{M}\sum_{i=1}^{M}(E_h/E_{\frac{h}{2}}))}{log2} \quad (5.2)$$

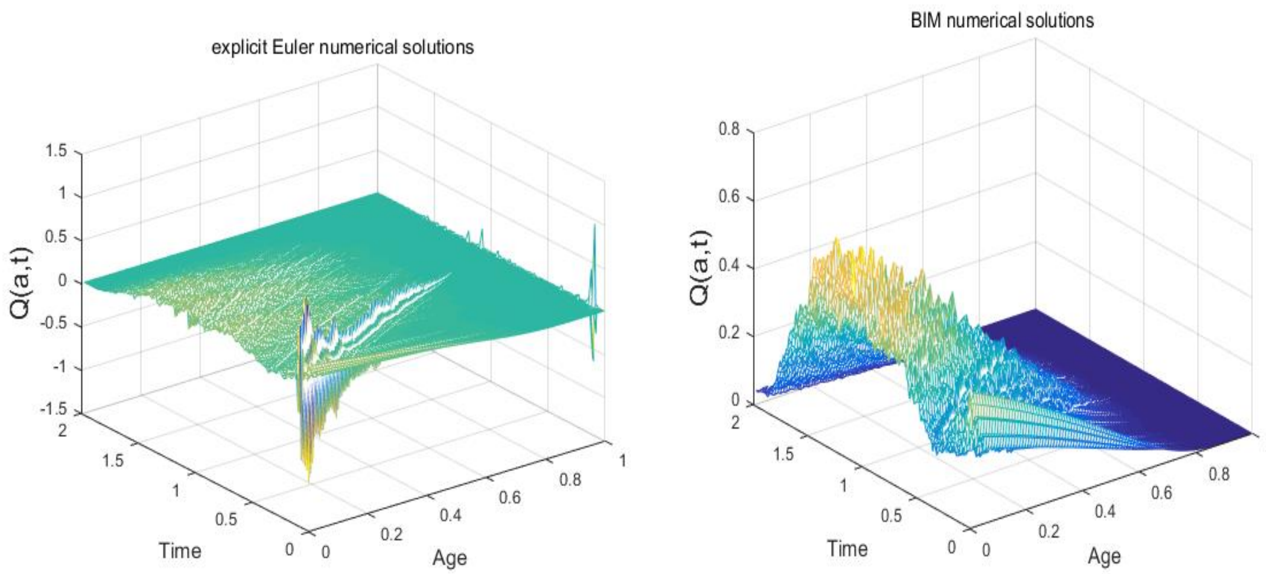

Fig. 5.1 Explicit Euler approximation and BIM for Example 1

Table 1. Convergence orders of BIM with σ = 0.05, σ = 0.1, σ = 0.15, σ = 0.2.

Convergence order	**σ = 0.05**	**σ = 0.1**	**σ = 0.15**	**σ = 0.2**
$r_{\Delta t}=\frac{1}{2^5}$	0.48345	0.48667	0.52641	0.57501
$r_{\Delta t}=\frac{1}{2^6}$	0.51316	0.60236	0.64074	0.68116
$r_{\Delta t}=\frac{1}{2^7}$	0.40311	0.46236	0.50681	0.55153
$r_{\Delta t}=\frac{1}{2^8}$	0.53966	0.58632	0.62547	0.66629

Then, we draw the errors of B_j^N with different step sizes: $h=\frac{1}{2^8},\frac{1}{2^7},\frac{1}{2^6},\frac{1}{2^5}$ in Figure 2, represented by solid points. The dashed lineis the reference line withe slope of 1/2. We see that the slopes of tw curves appear to match well. It satisfies the results of Theorem 4.1.

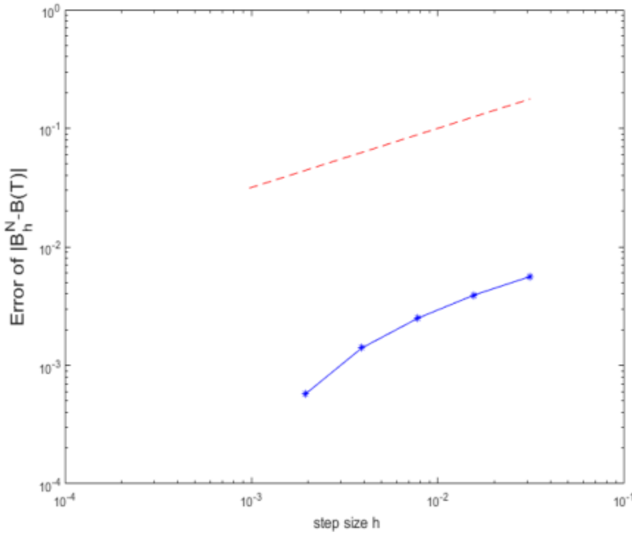

Fig. 5.2. The absolute error of B(T).

6. Conclusion

There are a lot of meaningful contents about preserving positivity of BIM numerical solutions, such as stability of BIM numerical solutions, positivity preserving of nonlinear system and so on. We can also consider the asymptotic and mean square stability regions of BIM numerical schemes in the future.

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