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Research Article

Cubic Vertex-Transitive bi-Cayley Graphs over a Nonabelian Group

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Abstract *- The vertex-transitive graph is a graph with high symmetry. A graph Γ is said to be a bi-Cayley graph over a group H if it admits H as a semiregular automorphism group with two orbits of equal size. And Γ is normal with respect to H if $R(H)$ is normal subgroup of $Aut(\Gamma)$. In this paper, we complete the classification of the cubic vertex-transitive normal bi-Cayley graphs over a group of order* pq^2, *where p and q be two primes with p>q. Furthermore, these cubic vertex-transitive bi-Cayley graphs are also a Cayley graph.*

Keywords *- Bi-Cayley graph, Normal, Cayley graph, Vertex-transitive, Isomorphism.*

1. Introduction

The vertex-transitive graph is a graph with high symmetry, and the symmetry of a graph is described by some transitivity properties of the graph. Cayley graph is a famous symmetric graph, which has important significance in many fields such as mathematical models, computer networks and communication technology. As a natural generalization of the Cayley graph, the bi-Cayley graph was proposed by Resmini and Jungnickel [1] when they studied the normality of Cayley graphs, and bi-Cayley graph is also important research tool for vertex-transitive graph, edge-transitive graph, semisymmetric graph and even symmetric graph. The symmetry of the bi-Cayley graph has been a hot topic, and the research focus being on classifying bi-Cayley graphs with specific symmetry properties over a given finite group *H*.

For a graph Γ, we denote by $V(\Gamma), E(\Gamma), \mathrm{Aut}(\Gamma)$ the vertex set, edge set and full automorphism group of Γ, respectively. The graph Γ is said to be *vertex-transitive* or *edge-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$ or $E(\Gamma)$, respectively. Initially, bi-Cayley graph over the cyclic group and abelian group were studied (see[2,3,4,5,6,7,8]). In recent years, the research focus being on classifying bi-Cayley graphs over finite nonabelian groups. For example, in [9], cubic symmetric bi-Cayley graphs on non abelian simple groups were classified and the full automorphism groups of these graphs were determined; trivalent vertex-transitive bi-Cayley graphs over dihedral groups were classified and Cayley property of trivalent vertex-transitive bi-dihedrants was presented in [10]. And for more results about it, we refer the reader to [11,12,13,14,15,16].

The normality of the bi-Cayley graph is an important property in the study of transitivity and full automorphism group of bi-Cayley graphs (see[17,18,19,20]). Much work has also been done on normal bi-Cayley graph. For example, it was shown that every finite group has a normal bi-Cayley graph in [21]. For a bi-Cayley graph Γ over a group *H*, the normalizer of group *H* in the full automorphism group of bi-Cayley graph Γ was determined in [22].

In this paper, we will apply the normality of bi-Cayley graph to study the bi-Cayley graph with respect to the vertex-transitive property and its full automorphism group. We present a classification of the vertex-transitive property of cubic normal bi-Cayley graphs Γ over a group of order pq^2, and it is shown that if Γ is vertex-transitive, it is also a Cayley graph.

Theorem 1.1 *Let* $\boldsymbol{H = \langle a, b | a^p = b^{q^2} = 1, b^{-1}ab = a^{r^{\frac{p-1}{q}}}, q|p-1\rangle}$, where p and q be two primes with q<p, and r is a primitive root of modulo p. Let $\boldsymbol{\Gamma}$ = BiCay (H, R, L, S) be a connected normal cubic bi-Cayley graph over group H, it is vertex-transitive if and only if the (R, L, S) is equivalent to the one of the triples in Table 1. Furthermore, all the graphs in Table 1 are both Cayley graph.

Table 1. Cubic vertex-transitive normal bi-Cayley graphs over group *H*.

No	(R,L,S)	conditions	Cayley
1	$(\emptyset, \emptyset, \{1, ab^{-s}, b^s\})$	$1 \le s \le q-1$	Yes
2	$(\emptyset, \emptyset, \{1, ab^{2s}, b^s\})$	$1 \le s \le q-1$	Yes
3	$(\emptyset, \emptyset, \{1, ab^k, b^{2k}\})$	$k = nq + t(t \neq 0), 0 \le n, t \le q-1$	Yes
4	$(\{ab^s, (ab^s)^{-1}\}, \{b^s, b^{-s}\}, \{1\})$	$1 \le s \le q-1$	Yes

2. Definition and Preliminaries

All graphs considered in this paper are finite, simple and undirected; some concepts and symbols not mentioned which will be used in the whole paper; see [23,24]. For a vertex *v* of graph Γ, its *neighborhood*, denoted by *N*(*v*). Given a finite group *G* and an inverse closed subset $S \subseteq G¥\{1\}$, the *Cayley graph X*=Cay(*G*,*S*) on *G* with respect to *S* is a graph with vertex set *G* and edge set $\{\{g, sg\}|g \in G, s \in S\}$.Define the *bi-Cayley* graph Γ =BiCay(*H*,*R*,*L*,*S*) with vertex set $V(\Gamma) = H_0 \cup H_1$ and edge set $E(\Gamma) = \{\{h_0, g_0\}|gh^{-1} \in R\} \cup \{\{h_1, g_1\}|gh^{-1} \in L\} \cup \{\{h_0, g_1\}|gh^{-1} \in S\}$, where $H_i = \{h_i|h \in H\}$,*i*=0,1; *R*, *L* and *S* is subsets of a group *H* such that $R = R^{-1}, L = L^{-1}$ and $R \cup L$ does not contain the identity element of *H*. The graph Γ is called *s- type bi-Cayley* if |*R*|=|*L*|=*s*, when *R*(*H*) is normal in Aut(Γ), the bi-Cayley graph Γ will be called a *normal bi-Cayley graph*. For the case when |*S*|=1, the graph Γ is also called *one-matching bi-Cayley graph*.

Proposition 2.1 [6] *Let* Γ = BiCay(*H, R, L, S*) *be a connected bi-Cayley graph over a group H. The following obvious facts are basic for graph* Γ.
(1) *H is generated by* $R \cup L \cup S$.
(2) *Up to graph isomorphism, S can be chosen to contain the identity element of H.*
(3) *For any automorphism* α of *H*, BiCay(*H, R, L, S*)≅ BiCay ($H, R^\alpha, L^\alpha, S^\alpha$).
(4) BiCay (*H, R, L S*)≅ BiCay(*H L, R*,S^{-1}).

In [22], let $\boldsymbol{\Gamma}$ = BiCay(*H, R, L, S*). It is easy to see that *R*(*H*) can be regarded as a group of automorphisms of $\boldsymbol{\Gamma}$ acting on its vertices by the rule

$$\boldsymbol{h_i}^{\boldsymbol{R(g)}} = (\boldsymbol{hg})_{\boldsymbol{i}}, \forall \boldsymbol{i} = \boldsymbol{0}, \boldsymbol{1}; \boldsymbol{h}, \boldsymbol{g} \in \boldsymbol{H}.$$

For an automorphism $\boldsymbol{\alpha}$ of *H* and *x, y, g*∈*H*, define two permutations on $\boldsymbol{V(\Gamma)} = \boldsymbol{H_0} \cup \boldsymbol{H_1}$ as following:

$$\boldsymbol{\delta_{\alpha,x,y}}: \boldsymbol{h_0} \mapsto (\boldsymbol{xh^\alpha})_{\boldsymbol{1}}, \boldsymbol{h_1} \mapsto (\boldsymbol{yh^\alpha})_{\boldsymbol{0}}, \quad \textbf{for each } \mathrm{h} \in \mathrm{H}. \qquad \textbf{(I)}$$

$$\boldsymbol{\sigma_{\alpha,g}}: \boldsymbol{h_0} \mapsto (\boldsymbol{h^\alpha})_{\boldsymbol{0}}, \boldsymbol{h_1} \mapsto (\boldsymbol{gh^\alpha})_{\boldsymbol{1}}, \quad \textbf{for each } \mathrm{h} \in \mathrm{H}. \qquad \textbf{(II)}$$

and then define

$$\boldsymbol{I} = \{\boldsymbol{\delta_{\alpha,x,y}}|\boldsymbol{R^\alpha} = \boldsymbol{x^{-1}Lx}, \boldsymbol{L^\alpha} = \boldsymbol{y^{-1}Ry} \textbf{ and } \boldsymbol{S^\alpha} = \boldsymbol{y^{-1}S^{-1}x}\} \subseteq \boldsymbol{Aut(\Gamma)}.$$
$$\boldsymbol{F} = \{\boldsymbol{\sigma_{\alpha,g}}|\boldsymbol{R^\alpha} = \boldsymbol{R}, \boldsymbol{L^\alpha} = \boldsymbol{g^{-1}Lg} \textbf{ and } \boldsymbol{S^\alpha} = \boldsymbol{g^{-1}S}\} \leq \boldsymbol{Aut(\Gamma)}.$$

Proposition 2.2 [6] *Let* $\boldsymbol{\Gamma}$ = BiCay(*H, R, L, S*). *be a connected bi-Cayley graph over the group H. If* I=∅*then* $\boldsymbol{N_{Aut(\Gamma)}} = (\boldsymbol{R(H)}) = \boldsymbol{R(H)} \rtimes \boldsymbol{F}$, *and If* $\boldsymbol{I} \neq \emptyset$,*then* $\boldsymbol{N_{Aut(\Gamma)}} = (\boldsymbol{R(H)}) = \boldsymbol{R(H)}\langle \boldsymbol{F}, \boldsymbol{\delta_{\alpha,x,y}}\rangle$ *for some* $\boldsymbol{\delta_{\alpha,x,y}} \in \boldsymbol{I}$. *For any* $\boldsymbol{\delta_{\alpha,x,y}} \in \boldsymbol{I}$, *x*, *y*∈*H*,$\boldsymbol{\alpha} \in \boldsymbol{Aut(\Gamma)}$, *we have the following*:
(1)$\langle \boldsymbol{R(H)}, \boldsymbol{\delta_{\alpha,x,y}}\rangle$*acts transitively on* $\boldsymbol{V(\Gamma)}$ *and* $\boldsymbol{\delta_{\alpha,x,y}}$*normalizes R*(*H*);
(2) *If* $\boldsymbol{o(\alpha)} = \boldsymbol{2}$*and x*=*y*=1, *then* $\boldsymbol{\Gamma}$*is isomorphic to the Cayley graph* Cay($\overline{\boldsymbol{H}}, \boldsymbol{R} \cup \boldsymbol{\alpha S}$), *where* $\overline{\boldsymbol{H}} = \boldsymbol{H} \rtimes \langle\boldsymbol{\alpha}\rangle$.

Lemma 2.3 *Let p and q be two primes with q<p. Let*

$$H = \langle a, b|a^p = b^{q^2} = 1, b^{-1}ab = a^{r^h}, q|p-1\rangle,$$

where r is a primitive root of modulo p and h=$\frac{p-1}{q}$. *If H*=$\langle x, y\rangle$, *then*$\{x, y\}$*is either*$\{a, b^s\}$,*or*$\{ab^k, b^u\}$,*or*$\{ab^l, b^w\}$,*where k*=*nq*+*t*, *l*=*mq*, *and s*, *u*, *w*, *t*, *m*=1, 2,⋯, *q*-1, *n*=*0*, 1,⋯, *q*-1.

Proof By Theorem in [25],

$$\mathrm{Aut}(H)=\{\alpha|a^\alpha = a^i, b^\alpha = a^l b^{nq+1}\},$$

where *i*=1, 2,⋯, *p*-1, *l*=1, 2,⋯, *p* and *n*=0, 1,⋯, *q*-1. Then *H* can be generated either by elements of order *p* and q^2, or elements of order $q^2$and q^2, or elements of order $q^2$and *pq*. If *H*=$\langle x, y\rangle$,then $\{x, y\}$is the following holds:

$$\{a^{i_1}, a^j b^{nq+s}\}, \{a^j b^{nq+s}, a^{j_1} b^{n_1 q+s_1}\}, \{a^{i_2} b^{n_2 q}, a^j b^{nq+s}\},$$

where $i_1, i_2 = 1,2,\cdots,p-1; j, j_1 = 0,1,\cdots,p-1; n, n_1 = 0,1,\cdots,q-1; s, s_1, n_2 = 1,2,\cdots,q-1$. There exist $\alpha_1, \alpha_2, \alpha_3 \in Aut(H)$ such that

$$\langle a, b^s\rangle^{\alpha_1} = \langle a^{i_1}, a^j b^{nq+s}\rangle, \quad \text{where } \alpha_1: a \mapsto a^{i_1}, b^s \mapsto a^j b^{nq+s},$$
$$\langle ab^{nq+s}, b^{s_1}\rangle^{\alpha_2} = \langle a^j b^{nq+s}, a^{j_1} b^{n_1 q+s_1}\rangle, \quad \text{where } \alpha_2: a \mapsto a^j, b^{s_1} \mapsto a^{j_1} b^{n_1 q+s_1},$$
$$\langle ab^{n_2 q}, b^s\rangle^{\alpha_3} = \langle a^{i_2} b^{n_2 q}, a^j b^{nq+s}\rangle, \quad \text{where } \alpha_3: a \mapsto a^{i_2}, b^s \mapsto a^j b^{nq+s}.$$

Hence, up to the automorphism of the group,$\{x, y\}$*is either* $\{a, b^s\}$,*or*$\{ab^k, b^u\}$,*or*$\{ab^l, b^w\}$, *where k*=*nq*+*t*, *l*=*mq*, *and s*, *u*, *w*, *t*, *m*=1, 2,⋯, *q*-1, *n*=0, 1,⋯, *q*-1.

According to structure of group *H*, there is no elements of order 2 in *H*. Thus, there is no 1-type bi-Cayley graphs over a group *H*. In the following; we only need to consider the 0-type and 2-type bi-Cayley graphs over a group *H*.

3. 0-Type vertex-transitive bi-Cayley graph

In this section, we shall give a characterization of connected cubic 0-type normal bi-Cayley graphs over a group H and show that cubic 0-type vertex-transitive normal bi-Cayley graphs over a group H. LetΓ = BiCay(H, R, L S) be a connected cubic 0-type normal bi-Cayley graph over group H. Firstly, determine R, L and S. Note that$\Gamma$is 0-type bi-Cayley graphs, it is clear that R=L=$\{\emptyset\}$. By connectivity of Γ, we can get H=$\langle S\rangle$. From Lemma 2.3, we can get S is one of the following holds:

(1) $S_1=\{1,a,b^s\}$;
(2) $S_2=\{1,ab^k,b^u\}, where\ k=nq+t$;
(3) $S_3=\{1,ab^l,b^w\}, where\ l=mq$,

where s, u, w, t, m=1, 2,⋯, q-1, n=0, 1, 2,⋯, q-1.

Lemma 3.1 *Let*$\Gamma_1=BiCay(H,\emptyset,\emptyset,S_1),\Gamma_2=BiCay(H,\emptyset,\emptyset,S_2)$ *be connected cubic* 0*-type bi-Cayley graphs over group* H*, if* s=u=k*, then*$\Gamma_1\cong\Gamma_2$.

Proof: Set $V(\Gamma_n)=\{(a^ib^j)_{0n},(a^ib^j)_{1n}|i=0,\cdots,p-1,j=0,\cdots,q^2-1\}$,where n=1, 2. Take a mapping$\rho$from$V(\Gamma_1)$ to$V(\Gamma_2)$ as follows:

$$\rho:(a^ib^j)_{01}\mapsto(a^{-i}b^j)_{12},(a^ib^j)_{11}\mapsto(a^{-ir^{hu}}b^{j-u})_{02}.$$

Firstly, we will show thatρ is a bijection. For any$(a^ib^j)_{12},(a^ib^j)_{02}\in V(\Gamma_2)$, there exist$(a^{-i}b^j)_{01},(a^{-ir^{-hu}}b^{j+u})_{11}\in V(\Gamma_1)$ such that$(a^{-i}b^j)_{01}^{\rho}=(a^ib^j)_{12}and(a^{-ir^{-hu}}b^{j+u})_{11}^{\rho}=(a^ib^j)_{02}$. Thus$\rho$is a surjection. For any$(a^ib^j)_{01},(a^{i'}b^{j'})_{01}\in V(\Gamma_1)$ and$(a^ib^j)_{11},(a^{i'}b^{j'})_{11}\in V(\Gamma_1)$, then

$$\rho((a^ib^j)_{01})=\rho((a^{i'}b^{j'})_{01})\Leftrightarrow((a^{-i}b^j)_{12})=(a^{-i'}b^{j'})_{12}\Leftrightarrow(a^ib^j)_{01}=(a^{i'}b^{j'})_{01}.$$
$$\rho((a^ib^j)_{11})=\rho((a^{i'}b^{j'})_{11})\Leftrightarrow((a^{-ir^{hu}}b^{j-u})_{02})=(a^{-i'r^{hu}}b^{j'-u})_{02}\Leftrightarrow(a^ib^j)_{11}=(a^{i'}b^{j'})_{11}.$$

Therefore, ρ is a bijection.

Next, we show thatρpreserves $E(\Gamma)$ if s=u=k. Note that

$$N((a^ib^j)_{01})^{\rho}=\{(a^ib^j)_{11},(a^{i+1}b^j)_{11},(a^{ir^{-hu}}b^{u+j})_{11}\}^{\rho}$$
$$=\{(a^{-ir^{hu}}b^{j-u})_{02},(a^{-(i+1)r^{hu}}b^{j-u})_{02},(a^{-i}b^j)_{02}\}=N((a^{-i}b^j)_{12}).$$
$$N((a^ib^j)_{11})^{\rho}=\{(a^ib^j)_{01},(a^{i-1}b^j)_{01},(a^{ir^{hu}}b^{j-u})_{01}\}^{\rho}$$
$$=\{(a^{-i}b^j)_{12},(a^{1-i}b^j)_{12},(a^{-ir^{hu}}b^{j-u})_{12}\}=N((a^{-ir^{hu}}b^{j-u})_{02}).$$

Therefore,ρ is an isomorphism from $V(\Gamma_1)$ to$V(\Gamma_2)$, then $\Gamma_1\cong\Gamma_2$.

Lemma 3.2 *Let* Γ = BiCay$(H,\emptyset,\emptyset,S_1)$ *be a connected cubic* 0*-type normal bi-Cayley graph over group H, then*Γ*is not vertex-transitive, and* $Aut(\Gamma)=R(H)\rtimes Z_2$.

Proof: By Proposition 2.2, first, we show that $I=\emptyset$.

Suppose that $I\neq\emptyset$. Since H is transitive on $\{(a^ib^j)_1|i=0,\cdots,p-1,j=0,\cdots,q^2-1\}$, without loss of generality, we can assume that $1_0^{\delta_{\alpha,x,y}}=1_1$, and it is easy to know that x=1. Moreover, $N(1_0)^{\delta_{\alpha,1,y}}=\{1_1,a_1,b_1^s\}^{\delta_{\alpha,1,y}}=\{1_0,(a^{-1})_0,(b^{-s})_0\}=N(1_1)$, the following three cases are discussed.

(i) If $1_1^{\delta_{\alpha,1,y}}=1_0$, it forces that y=1. By the definition of I, we have $S_1^{\alpha}=\{1,a,b^s\}^{\alpha}=\{1,a^{-1},b^{-s}\}=y^{-1}S_1^{-1}x$, and for any α ∈Aut(H), there is$1^{\alpha}=1$. And we know that there is no $\alpha\in Aut(H)$ such that $\{a,b^s\}^{\alpha}=\{a^{-1},b^{-s}\}$, a contradiction.

(ii) If $1_1^{\delta_{\alpha,1,y}}=(a^{-1})_0$, it forces that $y=a^{-1}$. Since$\alpha,x,y\in I$, it follows that $S_1^{\alpha}=\{1,a,b^s\}^{\alpha}=\{a,1,ab^{-s}\}=y^{-1}S_1^{-1}x$. For any α ∈Aut(H), we have$1^{\alpha}=1$. But there is no $\alpha\in Aut(H)$ such that $\{a,b^s\}^{\alpha}=\{a,ab^{-s}\}$, a contradiction.

(iii) If $1_1^{\delta_{\alpha,1,y}}=(b^{-s})_0$, it forces that$y=b^{-s}$. Then, we have$S_1^{\alpha}=\{1,a,b^s\}^{\alpha}=\{b^s,b^sa^{-1},1\}=y^{-1}S_1^{-1}x$. But there is no$\alpha\in Aut(H)$ such that$\{a,b^s\}^{\alpha}=\{b^sa^{-1},b^s\}$, a contradiction.

Based on the above,$I=\emptyset$. Next, we show that $Aut(\Gamma)=R(H)\rtimes\langle\sigma_{\alpha,a}\rangle$,where $\sigma_{\backslash\alpha,a}:h_0\mapsto(h^{\alpha})_0,h_1\mapsto(ah^{\alpha})_1$for each
h∈ H, where$\alpha:a\mapsto a^{-1},b^s\mapsto a^{-1}b^s$.

By equation (II) and the definition of F, for any $\sigma_{\alpha,g}\in F$, we have $S_1^{\alpha}=\{1,a,b^s\}^{\alpha}=g^{-1}S_1=\{g^{-1},g^{-1}a,g^{-1}b^s\}$. Note that$\alpha$ ∈Aut(H); then there is the identity element 1 in $\{g^{-1},g^{-1}a,g^{-1}b^s\}$.

(i) If g=1, then we can obtain $S_1^{\backslash\alpha}=\{1,a,b^s\}^{\alpha}=S_1$, where$\alpha$ is an identity mapping and α ∈Aut(H);

(ii) If g=a, then we can obtain ${S_1}^{\alpha}=\{1,a,b^s\}^{\backslash\alpha}=\{a^{-1},1,a^{-1}b^s\}=a^{-1}S_1$, where$\alpha:a\mapsto a^{-1},b^s\mapsto a^{-1}b^s$and α ∈Aut(H);

(iii) If g=b^s, then there exists no $\alpha\in$ Aut(H) such that $S_1^{\backslash\alpha}=\{1,a,b^s\}^{\backslash\alpha}=\{b^{-s}a,1,b^{-s}\}=b^{-s}S_1$.

SinceΓ is normal bi-Cayley graph, we can obtain $Aut(\Gamma) = R(H) \rtimes \langle\sigma_{\alpha,a}\rangle = R(H) \rtimes Z_2$. Moreover, by the definition of F, it is easy to see that Γ is not vertex-transitive.

Lemma 3.3 *Let*Γ = BiCay$(H, \emptyset, \emptyset, S_2)$ *be a connected cubic* 0*-type normal bi-Cayley graph over group H, where k≠u. If k=-u, k=2u or u=2k, then* Γ*is a vertex-transitive graph. Especially,* Γ*is a Cayley graph and* $\frac{Aut(\Gamma)}{R(H)} = Z_2$.

Proof: If k=-u, then $\Gamma = BiCay(H, \emptyset, \emptyset, \{1, ab^{-u}, b^u\})$.Set

$$\gamma: a \mapsto a^{-1}, b \mapsto a^{-r^{-\hbar u}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}}b$$

in Aut(H). Note that

$$a^{\gamma^2} = (a^{-1})^\gamma = a,\ \ b^{\gamma^2} = (a^{-r^{-\hbar u}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}}b)^\gamma = a^{r^{-\hbar u}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}}a^{-r^{-\hbar u}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}}b = b.$$

Clearly,$o(\gamma) = 2$. We take x=y=1, the n$o(\delta_{\gamma,1,1}) = 2$. Next, we show that $\delta_{\gamma,1,1} \in I$.

$$S_2^\gamma = \{1, ab^{-u}, b^u\}^\gamma = \{1, a^{-1-r^{-\hbar u}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}(\frac{r^{\hbar u}-1}{r^{-\hbar}-1})}b^{-u}, a^{-r^{-\hbar u}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})}b^u\}$$
$$= \{1, b^{-u}, a^{-r^{-\hbar u}}b^u\} = S_2^{-1}.$$

Hence $\delta_{\gamma,1,1} \in I$. By Proposition 2.2,$\langle R(H), \delta_{\gamma,1,1}\rangle$ acts transitively on$V(\Gamma)$ and $\Gamma$is isomorphic to a Cayley graph.

If k=2u, then $\Gamma = BiCay(H, \emptyset, \emptyset, \{1, ab^{2u}, b^u\})$. Take

$$\pi: a \mapsto a^{-r^{-\hbar u}}, b \mapsto a^{(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}}b$$

in Aut(H).

Next, we show that $\delta_{\pi,1,(ab^{2u})^{-1}} \in I$. Let$x = 1, y = (ab^{2u})^{-1}$, then

$$S_2^\pi = \{1, ab^{2u}, b^u\}^\pi = \{1, a^{-r^{-\hbar u}}a^{(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}(\frac{r^{-2\hbar u}-1}{r^{-\hbar}-1})}b^{2u}, a^{(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})}b^u\}$$
$$= \{1, ab^{2u}, ab^u\} = (ab^{2u})S_2^{-1}.$$

Therefore, $\delta_{\pi,1,(ab^{2u})^{-1}} \in I$. Consequently, Γ is vertex-transitive graph.

Next, we show that $\delta^2_{\pi,1,(ab^{2u})^{-1}} \in R(H)$. For any $h_0 = (a^ib^j)_0 \in H_0$, we have

$$(a^ib^j)_0^{\delta^2_{\pi,1,(ab^{2u})^{-1}}} = ((ab^{2u})^{-1}(a^ib^j)^{\pi^2})_0$$
$$= ((ab^{2u})^{-1}(a^{-ir^{-\hbar u}}a^{(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}(\frac{r^{-\hbar j}-1}{r^{-\hbar}-1})}b^j)^\pi)_0$$
$$= (a^{i-r^{2\hbar u-\hbar j}}b^{j-2u})_0.$$

Similarly, for any $h_1 = (a^ib^j)_1 \in H_1$, we have

$$(a^ib^j)_1^{\delta^2_{\pi,1,(ab^{2u})^{-1}}} = (((ab^{2u})^{-1})^\pi(a^ib^j)^{\pi^2})_1$$
$$= (((ab^{2u})^{-1})^\pi a^{ir^{-2\hbar u}-r^{-\hbar j}+1}b^j)_1$$
$$= (a^{(\frac{r^{-\hbar u}-1}{r^{-\hbar}-1})^{-1}(\frac{r^{2\hbar u}-1}{r^{-\hbar}-1})}b^{-2u}a^{r^{-\hbar u}}a^{ir^{-2\hbar u}-r^{-\hbar j}+1}b^j)_1$$
$$= (a^{i-r^{2\hbar u-\hbar j}}b^{j-2u})_1.$$

Hence, we can get that$\delta^2_{\pi,1,(ab^{2u})^{-1}} = R(a^{-r^{2\hbar u}}b^{-2u}) \in R(H)$. Therefore, we have $\langle R(H), \delta_{\pi,1,(ab^{2u})^{-1}}\rangle/R(H) \cong Z_2$and $|\langle R(H), \delta_{\pi,1,(ab^{2u})^{-1}}\rangle| = 2pq^2$. Hence, $|\langle R(H), \delta_{\pi,1,(ab^{2u})^{-1}}\rangle|$acts regularly on$V(\Gamma)$.Then $\Gamma$is isomorphic to a Cayley graph.

If u=2k, then$\Gamma = BiCay(H, \emptyset, \emptyset, \{1, ab^k, b^{2k}\})$.Let

$$\epsilon: a \mapsto a^{-r^{-\hbar k}}, \qquad b \mapsto b$$

be an automorphic mapping of H. Next, we show that $\delta_{\epsilon,1,b^{-2k}} \in I$. We take x=1,$y = b^{-2k}$, then

$$S_2^\epsilon = \{1, ab^k, b^{2k}\}^\epsilon = \{1, a^{-r^{-\hbar k}}b^k, b^{2k}\} = (b^{2k})S_2^{-1}.$$

Therefore,$\delta_{\epsilon,1,b^{-2k}} \in I$. Then $\Gamma$is vertex-transitive graph.

Next, we show that$\delta^2_{\epsilon,1,(b^{-2k})} \in R(H)$. For any $h_0 = (a^ib^j)_0 \in H_0$, we have

$$(a^ib^j)_0^{\delta^2_{\epsilon,1,b^{-2k}}} = (b^{-2k}(a^ib^j)^{\epsilon^2})_0$$
$$= (b^{-2k}(a^{-ir^{-\hbar k}}b^j)^\epsilon)_0$$
$$= (a^ib^{j-2k})_0.$$

Similarly, for any $h_1 = (a^ib^j)_1 \in H_1$, we have $(a^ib^j)_1^{\delta^2_{\epsilon,1,b^{-2k}}} = (a^ib^{j-2k})_1$.

Hence, we have that $\delta^2_{\epsilon,1,b^{-2k}} = R(b^{-2k}) \in R(H)$. Hence,$\langle R(H), \delta_{\epsilon,1,b^{-2k}}\rangle/R(H) \cong Z_2$and $|\langle R(H), \delta_{\epsilon,1,b^{-2k}}\rangle| = 2pq^2$. Then
$\langle R(H), \delta_{\epsilon,1,b^{-2k}}\rangle$acts regularly on $V(\Gamma)$, Γ is isomorphic to a Cayley graph.

By Proposition 2.2,$Aut(\Gamma) = N_{Aut(\Gamma)}(R(H)) = R(H)\langle F, \delta_{\gamma,1,1}\rangle$. Next, we show that $F = \{1\}$.Since k=-u, k=2u, u=2k, we have $S_2^{\alpha_1} = \{1, ab^{-u}, b^u\}^{\alpha_1} = g_1^{-1}\{1, ab^{-u}, b^u\} = g_1^{-1}S_2$, $S_2^{\alpha_2} = \{1, ab^{2u}, b^u\}^{\alpha_2} = g_2^{-1}\{1, ab^{2u}, b^u\} = g_2^{-1}S_2$, $S_2^{\alpha_3} = \{1, ab^k, b^{2k}\}^{\alpha_3} = g_3^{-1}\{1, ab^k, b^{2k}\} = g_3^{-1}S_2$, respectively. And we obtain that only $g_1, g_2, g_3 = 1, \alpha_1, \alpha_2, \alpha_3 \in Aut(H)$ is identity mapping is satisfied. That is $F = \{1\}$.Then$\frac{Aut(\Gamma)}{R(H)} = Z_2$.

Lemma 3.4 *Let $\Gamma = BiCay(H, \emptyset, \emptyset, \boldsymbol{S_2})$ be a connected cubic 0-type normal bi-Cayley graph over group H, where $k \neq -u, k \neq 2u$and $k \neq u/2$, then $\Gamma$is not vertex-transitive graph, and$Aut(\Gamma) = R(H)$.*

Proof If k=u, then$BiCay(H, \emptyset, \emptyset, S_2) \cong BiCay(H, \emptyset, \emptyset, S_1)$ by Lemma 3.1.

If $k \neq -u, 2u, u/2$. By Proposition 2.2, first, we show that $I = \emptyset$.

Suppose that $I \neq \emptyset$. Since H is transitive on $\{(a^ib^j)_1 | i = 0, \cdots, p-1, j = 0, \cdots, q^2-1\}$, without loss of generality, we can assume that$1_0^{\delta_{\alpha,x,y}} = 1_1$, and it is easy to know that x=1. Moreover, $N(1_0)^{\delta_{\alpha,1,y}} = \{1_1, (ab^k)_1, b_1^u\}^{\delta_{\alpha,1,y}} = \{1_0, (ab^k)_0^{-1}, b_0^{-u}\} = N(1_1)$, the following three cases are discussed.

(i) If $1_1^{\delta_{\alpha,1,y}} = 1_0$, it forces that y=1. By the definition of I, we have $S_2^\alpha = \{1, ab^k, b^u\}^\alpha = \{1, (ab^k)^{-1}, b^{-u}\} = y^{-1}S_2^{-1}x$, and for any $\alpha \in Aut(H)$, there is$1^\alpha = 1$. And we know that there is no$\alpha \in Aut(H)$ such that$\{ab^k, b^u\}^\alpha = \{(ab^k)^{-1}, b^{-u}\}$,a contradiction.

(ii) If $1_1^{\delta_{\alpha,1,y}} = (ab^k)_0^{-1}$, it forces that$y = (ab^k)^{-1}$. Since$\alpha, x, y \in I$, it follows that $S_2^\alpha = \{1, ab^k, b^u\}^\alpha = \{ab^k, 1, ab^{k-u}\} = y^{-1}S_2^{-1}x$. For any $\alpha \in Aut(H)$, we have$1^\alpha = 1$. But there is no $\alpha \in Aut(H)$ such that $\{ab^k, b^u\}^\alpha = \{ab^k, ab^{k-u}\}$,a contradiction.

(iii) If $1_1^{\delta_{\alpha,1,y}} = (b^{-u})_0$, it forces that$y = b^{-u}$. Then, we have $S_2^\alpha = \{1, ab^k, b^u\}^\alpha = \{b^u, b^{u-k}a^{-1}, 1\} = y^{-1}S_2^{-1}x$. But there is no $\alpha \in Aut(H)$ such that $\{ab^k, b^u\}^\alpha = \{b^{u-k}a^{-1}, b^u\}, a\ contradiction.$

Based on the above, $I = \emptyset$. Next, we show that $Aut(\Gamma) = R(H) \rtimes \langle \sigma_{\varepsilon,1}\rangle$,where$\sigma_{\varepsilon,1}: h_0 \mapsto (h^\varepsilon)_0, h_1 \mapsto (h^\varepsilon)_1$, and ε is an identity mapping.

By equation (II) and definition of F, for any $\sigma_{\alpha,g} \in F$, we have $S_2^\alpha = \{1, ab^k, b^u\}^\alpha = g^{-1}S_2 = \{g^{-1}, g^{-1}ab^k, g^{-1}b^u\}$. Note that $\alpha \in Aut(H)$, then there is the identity element 1 in $\{g^{-1}, g^{-1}ab^k, g^{-1}b^u\}$.

(i) If g=1, then we can obtain $S_2^\alpha = \{1, ab^k, b^u\}^{\backslash\alpha} = S_2$, where$\alpha$ is an identity mapping and $\alpha \in Aut(H)$;

(ii)If$g = ab^k$, there is no$\alpha \in Aut(H)$ such that $S_2^\alpha = \{1, ab^k, b^u\}^{\backslash\alpha} = \{(ab^k)^{-1}, 1, a^{-r^{hk}}b^{u-k}\} = (ab^k)^{-1}S_2$;

(iii)If$g = b^u$, there is no$\alpha \in Aut(H)$ such that $S_2^{\backslash\alpha} = \{1, ab^k, b^u\}^{\backslash\alpha} = \{b^{-u}, a^{r^{hu}}b^{k-u}, 1\} = b^{-u}S_2$.

SinceΓ is normal bi-Cayley graph, we can obtain $Aut(\Gamma) = R(H)$. Consequently, $\Gamma$is not vertex-transitive.

Lemma 3.5 *Let$\Gamma = BiCay(H, \emptyset, \emptyset, S_3)$ be a connected cubic 0-type normal bi-Cayley graph over group H, then$\Gamma$is not vertex-transitive, and $Aut(\Gamma) = R(H)$.*

Proof By Proposition 2.2, first, we show that$I = \emptyset$.

Suppose that $I \neq \emptyset$. Since H is transitive on $\{(a^ib^j)_1 | i = 0, \cdots, p-1, j = 0, \cdots, q^2-1\}$, without loss of generality, we can assume that $1_0^{\delta_{\alpha,x,y}} = 1_1$, and it is easy to know that x=1. Moreover, $N(1_0)^{\delta_{\alpha,1,y}} = \{1_1, ab_1^l, b_1^w\}^{\delta_{\alpha,1,y}} = \{1_0, (ab^l)_0^{-1}, b_0^{-w}\} = N(1_1)$, the following three cases are discussed.

(i) If $1_1^{\delta_{\alpha,1,y}} = 1_0, it\ forces\ that\ y = 1. By\ the\ definition\ of\ I,$ we have $S_3^\alpha = \{1, ab^l, b^w\}^\alpha = \{1, (ab^l)^{-1}, b^{-w}\} = y^{-1}S_3^{-1}x$, and for any $\alpha \in Aut(H), there\ is\ 1^\alpha = 1$. And we know that there is no$\alpha \in Aut(H)$ such that $\{ab^l, b^w\}^\alpha = \{(ab^l)^{-1}, b^{-w}\}$,a contradiction.

(ii) If$1_1^{\delta_{\alpha,1,y}} = (ab^l)_0^{-1}, it\ forces\ that\ y = (ab^l)^{-1}$. Since$\alpha, x, y \in I$, it follows that $S_3^\alpha = \{1, ab^l, b^w\}^\alpha = \{ab^l, 1, ab^{l-w}\} = y^{-1}S_3^{-1}x$.

For any$\alpha \in Aut(H)$, there is$1^\alpha = 1$. But there is no$\alpha \in Aut(H)$ such that $\{ab^l, b^{-w}\}^\alpha = \{ab^l, ab^{l-w}\}$, a contradiction.

(iii) If $1_1^{\delta_{\alpha,1,y}} = (b^{-w})_0$, it forces that$y = b^{-w}$. Then, we have $S_3^\alpha = \{1, ab^l, b^w\}^\alpha = \{b^w, b^{w-l}a^{-1}, 1\} = y^{-1}S_3^{-1}x$. But there is no $\alpha \in Aut(H)$ such that$\{ab^l, b^w\}^\alpha = \{b^w, b^{w-l}a^{-1}\}$,a contradiction.

Based on the above,$I = \emptyset$.Next, we show that $Aut(\Gamma) = R(H) \rtimes \langle \sigma_{\varepsilon,1}\rangle$,where $\sigma_{\varepsilon,1}: h_0 \mapsto (h^\varepsilon)_0, h_1 \mapsto (h^\varepsilon)_1$, and ε is an identity mapping.

By equation (II) and definition of F, for any$\sigma_{\alpha,g}$, we have $S_3^\alpha = \{1, ab^l, b^w\}^\alpha = g^{-1}S_3 = \{g^{-1}, g^{-1}ab^l, g^{-1}b^w\}$. Note that $\alpha \in Aut(H)$, then there is the identity element 1 in$\{g^{-1}, g^{-1}ab^l, g^{-1}b^w\}$.

(i) If g=1, then we can obtain $S_3^{\backslash\alpha} = \{1, ab^l, b^w\}^{\backslash\alpha} = S_3$, where α is an identity mapping;

(ii) If$g = ab^l$, there is no$\alpha \in Aut(H)$ such that $S_3^{\backslash\alpha} = \{1, ab^l, b^w\}^\alpha = \{(ab^l)^{-1}, 1, a^{-r^{hl}}b^{w-l}\} = (ab^l)^{-1}S_3$;
(iii) If$g = b^w$, then there is no$\alpha \in Aut(H)$ such that $S_3^\alpha = \{1, ab^l, b^w\}^\alpha = \{b^{-w}, a^{r^{hw}}b^{l-w}, 1\} = b^{-w}S_3$.
SinceΓ is normal bi-Cayley graph, we can obtain $Aut(\Gamma) = R(H)$. Consequently, Γ is not vertex-transitive.

4. 2-Type vertex-transitive bi-Cayley graph

In this section, we shall give a characterization of connected cubic 2-type normal bi-Cayley graphs over a group *H* and show that cubic 2-type vertex-transitive normal bi-Cayley graphs over a group *H*. Let $\boldsymbol{\Gamma = BiCay(H, R, L, S)}$ be a connected cubic 2-type normal bi-Cayley graph over group *H*. Firstly, determine *R*, *L* and *S*. Note that $\boldsymbol{\Gamma}$ is 2-type bi-Cayley graphs, then $\boldsymbol{S = \{1\}}$. By connectivity of $\boldsymbol{\Gamma}$ we can get $\boldsymbol{H = \langle R \cup L\rangle}$. From Lemma 2.3, we can get *R*, *L*, *S* is one of the following holds:

$$(1)\ R_1 = \{a, a^{-1}\}, L_1 = \{b^s, b^{-s}\}, S = \{1\};$$
$$(2)\ R_2 = \{ab^l, (ab^l)^{-1}\}, L_2 = \{b^w, b^{-w}\}, S = \{1\}, where\ l = mq;$$
$$(3)\ R_3 = \{ab^k, (ab^k)^{-1}\}, L_3 = \{b^u, b^{-u}\}, S = \{1\}, where\ k = nq + t;$$
$$where\ s, u, w, t, m = 1,2,\cdots, q-1, n = 0,1,2,\cdots, q-1.$$

Lemma 4.1 *Let* $\Gamma_1 = BiCay(H, \{ab^k, (ab^k)^{-1}\}, \{b^u, b^{-u}\}, \{1\})$*and* $\Gamma_2 = BiCay(H, \{(ab^{-k})^{-1}, ab^{-k}\}, \{b^u, b^{-u}\}, \{1\})$*be two connected cubic* 2*-type bi-Cayley graphs over a group H, then* $\Gamma_1 \cong \Gamma_2$.

Proof: Take $\alpha \in Aut(H)$ such that $\alpha: a \mapsto a^{-r^{-hk}}, b \mapsto b$. It is clear that $\{1\}^{\backslash\alpha} = \{1\}$. Furthermore, we have

$$\{ab^k, (ab^k)^{-1}\}^\alpha = \{a^{-r^{-hk}}b^k, a^{(-r^{hk})(-r^{-hk})}b^{-k}\} = \{ab^{-k}, (ab^{-k})^{-1}\},$$
$$\{b^u, b^{-u}\}^\alpha = \{b^u, b^{-u}\}.$$

By Proposition 2.1(3), it follows that $\Gamma_1 \cong \Gamma_2$.

Lemma 4.2 *Let*$\Gamma_i = BiCay(H, R_i, L_i, S)$, *where i*=1,2, *be a connected cubic* 2*-type normal bi-Cayley graph over group H, then* Γ*is not vertex-transitive, and* $Aut(\Gamma) = R(H) \rtimes Z_2$.

Proof: By Proposition 2.2, first, we show that $I = \emptyset$.

Suppose that $\delta_{\alpha,x,y} \in I$. By the definition of I, we have $S^\alpha = \{1\}^\alpha = y^{-1}S^{-1}x = \{1\}$, this forces that *y*=*x*.
Since $R_i^\alpha = x^{-1}L_ix = L^x = L^{\sigma(x)}$, it follows that $R_i^{\alpha\sigma(x^{-1})} = L_i$, where$\sigma(x)$ is inner automorphism induced by *x*. Note that the order of the element in $R_i$is different from the order of the element in L_i. Then there is no $\delta_{\alpha,x,y}$ in I such that
$R_i^\alpha = x^{-1}L_ix = L^x =$
$L^{\sigma(x)}$, a contradiction.

Based on the above, $I = \emptyset$. Next, we will determine F. For any $\sigma_{\backslash\alpha,g} \in F$, we have $S^\alpha = \{1\} = g^{-1}S$, this forces that *g*=1.

Note that $R_1^\alpha = \{a, a^{-1}\}^\alpha = \{a, a^{-1}\} = R_1, L_1^\alpha = \{b^s, b^{-s}\}^{\backslash\alpha} = \{b^s, b^{-s}\} = L_1$. This forces that α is as follows:$\alpha: a \mapsto a^{-1}, b^s \mapsto b^s$.

Note that $R_2^\alpha = \{ab^l, (ab^l)^{-1}\}^\alpha = \{ab^l, (ab^l)^{-1}\} = R_2, L_2^\alpha = \{b^w, b^{-w}\}^\alpha = \{b^w, b^{-w}\} = L_2$, it is easy to see that$\alpha$is identity mapping.

Since Γ_i is normal bi-Cayley graph, we can obtain $Aut(\Gamma_1) = R(H) \rtimes \langle\sigma_{\alpha,1}\rangle = R(H) \rtimes Z_2$and $Aut(\Gamma_2) = R(H)$. Moreover, Γ_i is not vertex-transitive, where$i = 1,2$.

Lemma 4.3 *Let* $\Gamma = BiCay(H, R_3, L_3, S)$ *be a connected cubic* 2*-type normal bi-Cayley graph over group H*. *If k*=*u, then* Γ *is vertex-transitive*. *Especially,* Γ *is a Cayley graph, and* $Aut(\Gamma) = \langle R(H), \delta_{\phi,1,1}\rangle$.

Proof: If *k*=*u*, then $\Gamma = BiCay(H, \{ab^u, (ab^u)^{-1}\}, \{b^u, b^{-u}\}, \{1\})$. Take

$$\phi: a \mapsto a^{-1}, b \mapsto a^{(\frac{r^{-hu}-1}{r^{-h}-1})^{-1}}b$$

in Aut(*H*). Note that

$$a^{\phi^2} = (a^{-1})^\gamma = a, b^{\phi^2} = (a^{(\frac{r^{-hu}-1}{r^{-h}-1})^{-1}}b)^\phi = a^{-(\frac{r^{-hu}-1}{r^{-h}-1})^{-1}}a^{(\frac{r^{-hu}-1}{r^{-h}-1})^{-1}}b = b.$$

Clearly,$o(\phi) = 2$. We take *x*=*y*=1, then $o(\delta_{\phi,1,1}) = 2$. Next, we show that $\delta_{\phi,1,1} \in I$.

$$R_2^\phi = \{ab^u, (ab^u)^{-1}\}^\phi = \{b^u, b^{-u}\} = L_2,$$
$$L_2^\phi = \{b^u, b^{-u}\}^\phi = \{ab^u, (ab^u)^{-1}\} = R_2,$$
$$S^\phi = \{1\} = S^{-1}.$$

Hence, $\delta_{\phi,1,1} \in I$. By Proposition 2.2,$\langle R(H), \delta_{\phi,1,1}\rangle$ acts transitively on $V(\Gamma)$, and Γ is isomorphic to a Cayley graph.
By Proposition 2.2, $Aut(\Gamma) = N_{Aut(\Gamma)}R(H)) = R(H)\langle F, \delta_{\phi,1,1}\rangle$. Next, we show that F={1}. Since $S = \{1\} = g^{-1}S$, this

force that g=1. Note that $R_2^\alpha = \{ab^u, (ab^u)^{-1}\}^{\backslash\alpha} = \{ab^u, (ab^u)^{-1}\} = R_2, L_2^{\backslash\alpha} = \{b^u, b^{-u}\}^{\backslash\alpha} = \{b^u, b^{-u}\} = L_2$. It is easy to see that there is no nonidentity element in Aut(H) satisfying the above equation. That is F={1}. Then $Aut(\Gamma) = \langle R(H), \delta_{\phi,1,1}\rangle$.

Lemma 4.4 *Let $\Gamma = BiCay(H, R_3, L_3, S)$ be a connected cubic 2-type normal bi-Cayley graph over group H. If $k \neq u$, then Γ is not vertex-transitive, and $Aut(\Gamma) = R(H)$.*

Proof: Suppose that $\delta_{\alpha,x,y} \in I$. Since H is transitive on $\{(a^i b^j)_1 | i = 0, \cdots, p-1, j = 0, \cdots, q^2 - 1\}$, without loss of generality, we assume that $1_0^{\delta_{\alpha,x,y}} = 1_1$, and it is easy to know that x=1. By the definition of I, we have $S^\alpha = \{1\}^\alpha = \{1\} = y^{-1}S^{-1}x$, this forces that y=1. Since $k \neq u$, there is no $\alpha \in Aut(H)$ such that $R_2^\alpha = \{ab^k, (ab^k)^{-1}\}^\alpha = \{b^u, b^{-u}\} = L_2, L_2^\alpha = \{b^u, b^{-u}\}^\alpha = \{ab^k, (ab^k)^{-1}\} = R_2$, a contradiction.
Suppose that $\delta_{\alpha,g} \in F$. By equation (II), if $S^\alpha = \{1\} = g^{-1}S$, this forces that g=1.
Note that $R_2^\alpha = \{ab^k, (ab^k)^{-1}\}^\alpha = \{ab^k, (ab^k)^{-1}\} = R_2, L_2^\alpha = \{b^u, b^{-u}\}^\alpha = \{b^u, b^{-u}\} = L_2$, it is easy to see that only identity mapping in Aut(H) satisfies the above equation. If Γ is normal bi-Cayley graph, we can obtain $Aut(\Gamma) = R(H)$. Consequently, Γ is not vertex-transitive.

In the end, by Lemma 3.1, 3.2, 3.3, 3.4, 3.5, 4.1, 4.2, 4.3 and 4.4, we complete the proof of Theorem 1.1.

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