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Original Article

Study Equivalence Between the Solution of Integral Equations and Ordinary Differential Equations

Muntaha Khudair Abbass

Technical College of Management/Baghdad , Middle Technical University

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Abstract - *The present article aims to study the types of linear integral equations, like" Fredholm integral equation" (FIE) and "Volterra integral equation" (VIE). Also Equivalence among Integral Equations and ordinary Differential Equations was studied. It was shown that there is relation between Reduction an Integral Equations to ordinary Differential Equations and vice versa. Some types of kernel of integral equations like" iterated kernel ", and " Resolved kernel " are mentioned by given some examples of these kernels. An analytical and numerical methods for solving the Fredholm integral equation was the method of successive approximations. Some examples about solving fredholm integral equation, Volterra integral equation are mentioned.*

Keywords **-** *Differential Equations, Kernel, Volterra integral equation, Fredholm integral equation, Successive approximation.*

1. Introduction

An integral equation is an equation in which the unidentified function appears under an integral symbol. According to Bocher [1914], The integral equation can be classifying into two classes. First, it is called Volterra integral equation (VIE) where the Volterra's significant work in this area was complete in 1884-1896 and the second, called "Fredholm integral equation" (FIE) where the Fredholm's important contribution was made in 1900-1903. Fredholm developed the theory of this integral equation as a limit to the linear system of equations [1]. Integral equations diverse evolved directly linked to the number of many branches of mathematics in the differential account, integral account, differential equations. Problems in which integral equations are encountered include radiative transfer, and the oscillation of a string, membrane, or axle. Oscillation problems may also be solved as differential equations [3,4]. There is equivalence relation between Integral equations and ordinary differential equations. There is a close connection between differential and integral equations, and some problems may be formulated either way. For example, Green's function, FredHolm theory, and Maxwell's equations[6]. Integral equations play an important role in many branches of sciences such as mathematics, biology, chemistry, physics, mechanics and engineering [5,7]. There is a close relationship among differential and integration neutralization, and several issues may be features either way[8,14] . Therefore, there exist approximate and numerical method for solving integral equations.

In 2018, Muntaha A. and Massia Ali [15] studied the subject of Reduces Solution of FreedHolm Integral Equation to a System of Linear Algebraic Equation . The objective of this study is to find the solution of fredholm complete neutralization for kernel and it reduces the solution of fredholm integration neutralization to a system of linear algebraic equation.

2. Definition of Integral equation

A general form of an integral equation in f(x) can be offered as:

$$h(x)f(x) = g(x) + \lambda \int_{a}^{b(x)} k(x,y) f(y)dy$$

Where: h ,g are given functions , K(x , y) function of two variables called the kernel of the integral equation , λ is a scalar parameter, the given function K(x , y) which depends up on the current variables x as well as the variables y is known as the kernel or nucleus of the integral equation[3, 6].

3. Classification of the integral equations

Integral equations are classified according to different kinds:

The Linear integral equations divided into two types :

1) both fixed: Fredholm equation $h(x)f(x) = g(x) + \lambda \int_a^{b(x)} k(x,y)f(y)dy$, $a \le x \le b$

2) one variable: Voltera equation $h(x)f(x) = g(x) + \lambda \int_a^{b(x)} k(x,y)f(y)dy$, when b(x)=x $a \le x < \infty$

3) only inside integral: if h(x)=0 Then the voltera and fredholm Integral equations is called of the first kind.

A) Fredholm equation $g(x) = \int_a^{b(x)} k(x,y)f(y)dy$.

B) Volterra equation $g(x) = \lambda \int_a^{x} k(x,y)f(y)dy$

4) both inside and outside integral: if h(x) =1 Then the voltera and fredholm Integral equations is called of the second kind [16].

A) Fredholm equation , $f(x) = g(x) + \lambda \int_a^{b} k(x,y)f(y)dy$

B) "Volterra equation $f(x) = g(x) + \lambda \int_a^{x} k(x,y)f(y)dy$

5) identically zero: homogeneous: first kind , if g(x)=0

A) Fredholm equation $\int_a^{b} k(x,y)f(y)dy = 0$

B)Volterra equation $\lambda \int_a^{x} k(x,y)f(y)dy = 0$

6) not identically zero: inhomogeneous

Both Fredholm and Volterra equations are linear integral equations, due to the linear behavior of y(*x*) under the integral [15] , [17]..

Examples

1) $\int_0^{x} (xy + x^2) f(y)dy = 0$ linear homo. Volterra integral eq. of the first kind $\lambda = 1, k(x,y) = (xy + x^2)$

2) $2\int_0^{1} (xy + x\ \ ) f(y)dy = 0$ linear homo. Fredholm integral equation .

4. Types of kernels : There are many types of kernels

4.1. The Iterated Kernel

A) $y(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt$ (1)

B) Consider the following non-homo-fredholm integral equation of the second kind

The Iterated Kernels of equation. 1 defined as :$k_1(x,t)=k(x,t)$

$$k_n(x,t) = \int_a^b k(x,z) k_{n-1}(z,t) dz$$

…n=2 ,3,...

$$k_n(x,t) = \int_a^b k_{n-1}(x,z) k(z,t) dz$$

B) Consider the following non-homo-Volterra integral equation of the second kind:

$$y(x) = f(x) + \lambda \int_a^x k(x,t) y(t) dt \quad (2)$$

The Iterated Kernels of equ. (2) defined as :$k_1(x,t)=k(x,t)$

$$k_n(x,t) = \int_t^x k(x,z) k_{n-1}(z,t) dz$$

…n=2 ,3,...

$$k_n(x,t) = \int_t^x k_{n-1}(x,z) k(z,t) dz$$

4.2. Resolvent Kernel

A) Consider the following non-homo-fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt \quad (3)$$

The Iterated Kernels is defined as : $R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$

Where: this series convergent absolutely and uniformly in the case of continuous k(x,t) ,$k_n(x,t)$ are iterated kernels as[1], [18]..

Then : the solution of equation. (3) is defined as: $y(x) = f(x) + \lambda \int_a^b R(x,t,\lambda) f(t) dt$ "- B) Consider the following non-homo-volterra integral equation of the second kind

$$y(x) = f(x) + \lambda \int_a^x k(x,t) y(t) dt \quad (4)$$

The Iterated Kernels is defined as : $R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$

:Then : the solution of equ. (4) is defined as :

$$y(x) = f(x) + \lambda \int_a^x R(x,t,\lambda) f(t) dt$$

4.3. Degenerate Kernel

A kernel k(x,t) is called degenerate or separable kernel if it can be expressed as the sum of a finite number of terms , each of which is the product of a function of x only and a function of t only[5, 9] .

$$k(x,t) = \sum_{i=1}^{n} g_i(x) h_i(x)$$

The functions $g_i(x)$can be regarded as linearly independent. The set of functions is assumed to be linearly independent if

$c_1 g_1(x) + c_2 g_2(x) + \ldots + c_n g_n(x) = 0$ where $c_1, c_2, \ldots, c_n$ are arbitrary constants then $C_1 = c_2 = \ldots = c_n = 0$

4.4. Examples:

Example 1: The iterated kernel for " $y(x) = f(x) + \lambda \int_a^b k(x,t) y(t) dt$ "

Where k(x,t)=sin(x-2t) $\quad 0 \le x \le 2\pi, \quad 0 \le t \le 2\pi$

Solution: k1(x,t)=k(x,t) sin(x-2t) , $k_n(x,t) = \int_a^{2\pi} k(x,z) k_{n-1}(z,t) dz$

$$\therefore k_2(x,t) = \int_a^{2\pi} \sin(x-2z)\sin(z-2t) dz = 2\pi - 0 = 0$$

$$\therefore k_3(x,t) = \int_a^{2\pi} \sin(x-2z).0.dz = 0$$

$$\vdots$$

$$k_n(x,t) = 0$$

"The Iterated Kernels is $R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t)$ =0

Example2: This example about the resolvent kernel for a volterra integral equation where :k(x,t)=1.

Answer: k1(x,t)=k(x,t)=1 , $k_n(x,t) = \int_t^x k(x,z) k_{n-1}(z,t) dz$

$$k_2(x,t) = \int_t^x 1.dz = \frac{x-t}{1!} = x - t \quad , \quad k_3(x,t) = \int_t^x 1.(x-z)dz = \frac{(x-t)^2}{2!}$$

$$k_n(x,t) = \frac{(x-t)^{n-1}}{(n-1)!} \quad , \quad R(x,t,\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x,t) = \sum_{n=1}^{\infty} \frac{(\lambda(x-t))^{n-1}}{(n-1)!}$$

5. Equivalence Between Integral Equations and ordinary Differential Equations

Certain Integral Equations can be deduced from and reduced to Differential Equations, In order to this reduction is necessary to make use of two theorems to convert Integral Equations to Differential Equations [10,14]**:**

Libeneze Formula:

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,y)dy = \int_{A(x)}^{B(x)} \frac{\partial F(x,y)}{\partial x} dy + F(x,B(x))\frac{dB}{dx} - F(x,A(x))\frac{dA}{dx}$$

Where F and $\frac{\partial F}{\partial x}$ are continuous functions of x and y .

Fundamental Theorem of Integral Calculus:

If f is continuous on [a,b], then the functions :

F(x) = $\int_a^x f(t)dt$ has a derivative at every point in [a,b] and $\quad \frac{dF}{dx} = \frac{dF}{dx}\int_a^x f(t)dt = f(x)$

5.1. Reduction an Integral Equations to ordinary Differential Equations:

Consider the following integral equation

$$I_n(x)=\int_a^x (x-y)^{n-1}.f(y)dy \qquad (5)$$

Where n is positive integer $n \geq 1$, a is constant , $I_n(x)=I_n{}'(x)=\ldots=I_n^{(n-1)}=0$.

If n = 1 $\rightarrow$ $I_1(x)=\int_a^x f(y)dy$, $\frac{dI_1(x)}{dx}=\frac{d}{dx}\int_a^x f(y)dy=f(x)$

If n > 1 $\frac{dI_n(x)}{dx}=\frac{d}{dx}\int_a^x (x-y)^{(n-1)} f(y)dy$

$$=\int_a^x (n-1)(x-y)^{(n-2)} f(y)dy+0+0$$

$$\frac{dI_n(x)}{dx}=\frac{d}{dx}\int_a^x (x-y)^{(n-1)} f(y)dy$$

$$\frac{dI_n(x)}{dx^2}=(n-1)(n-2)I_{n-2}$$

.

$$\frac{dI_n(x)}{dx^k}=(n-1)(n-2)\ldots.(n-k)I_{n-k} \qquad , \ n>k$$

$$\frac{d^{n-1}I_n(x)}{dx^{n-1}}=(n-1)!I_n(x)$$

$$\frac{d^{n-1}I_n(x)}{dx^{n-1}}=(n-1)!f(x) \qquad (6) \qquad , \quad I_n(a)=I'_n(a)=\ldots=I_n^{(n-1)}(a)=0$$

The Equation(6) Equivalent to Equation (5).

5.2. Reduction an ordinary Differential Equations to an Integral Equations

If we have the I.v.p. $\frac{d^nI_n(x)}{dx^n}=(n-1)!f(x)$ (7)

$I_n(a)=I'_n(a)=\ldots=I_n^{(n-1)}(a)=0$

To reduce this equation to integral eq. $I_n(x)=(n-1)!\int_a^x\int_a^x\ldots\int_a^x f(y)dy.dy\ldots dy$

$$I_n(x)=\int_a^x (x-y)^{(n-1)} f(y)dy \qquad , \qquad I_1(x)=\int_a^x f(y)dy$$

$$\therefore I_n(x)=\frac{1}{(n-1)!}\int_a^x (x-y)^{n-1} f(y)dy \qquad (8)$$

The Equation(8) Equivalent to Equation (7).

5.3. The Connection Between Differential and Integral Equations (First-Order)

Example 1: Consider the differential equation (initial value problem)

$$y'(x) = f(x, y), y(x_0) = y_0 isI.v.p$$

$$\int_{x0}^{x} y'(t)dt = \int_{x0}^{x} f(t, y(t))dt \quad , \quad y(x) - y(x0) = \int_{x0}^{x} f(t, y(t))dt$$

$$y(x) = y_0 + \int_{x0}^{x} f(t, y(t))dt$$

5.4. The Connection Between Differential and Integral Equations (second -Order)

Example 2: Reduce the I.V.P y" – 2y = 0 , y(0) , y'(0) = 0 to integral Equation

Solution : $y'' = 2y$

$$\int_{0}^{x} y''(t)dt = \int_{0}^{x} 2y(t)dt \quad , \quad y'(x) - y'(0) = \int_{0}^{x} 2y(t)dt$$

$$y'(x) = 0 + \int_{0}^{x} 2y(t)dt \quad , \quad \int_{0}^{x} y'(x) = \int_{0}^{x}\int_{0}^{x} 2y(t)dt$$

y(t) - y(0) = $\int_{0}^{x} (x-t)2y(t)dt$

$\therefore y(x) = 1 + \int_{0}^{x} (x-t)2y(t)dt$ this is integral equation equivalent differential equation

6. Equivalence Between the Solution of Integral Equations and ordinary Differential Equations

6.1. Theorem

A function ϴ(x) a solution of I.V.P. yon an integral I , y'= f(x,y) , y(x0) =y0
If and only if a solution of integral equation [9,10] .

$$y = y_0 + \int_{x0}^{x} f(t, y(t))dt \quad \text{on I}$$

Proof : $\rightarrow \because$ ϴ is a solution of I.V.P. y'(x)=f(x,y) , y(x0)= y0 on I
ϴ'(x) = f(x,ϴ(x)) , ϴ(x0) =y0 , $\forall x \in I$ (9)
ϴ is a continuous on I , f is a continuous on R . by integration (9)

$$\int_{x0}^{x} \vartheta'(t)dt = \int_{x0}^{x} f(t, y(t))dt \quad \text{on I}$$

$$\phi(x) = y_0 + \int_{x0}^{x} f(t, y(t))dt$$

$\therefore$ ϴ is a solution of integral eq. $y = y_0 + \int_{x0}^{x} f(t, y(t))dt$

$\leftarrow$ $\because$ ϴ is a solution of integral eq. $y = y_0 + \int_{x0}^{x} f(t, y(t))dt$ on I

$$\therefore \phi(x) = y_0 + \int_{x0}^{x} f(t, y(t))dt \qquad (10)$$

By differential (**) w.r.t. x ϴ'(x) f(x ,ϴ(x)) $\forall x \in I$

$\because$ ϴ(x0) = y0 , ϴ is a solution of I.V.P. y' f(x ,y) , y(x0) =y0 .

Example 1 : prove that y = e^x is a solution of integral eq. :

$$\therefore y(x) = 1 + \int_0^x y(t)dt \quad , \quad e^x = 1 + \int_0^x e^x dt = 1 + e^x + e^0 = e^x$$

$\therefore$ y(x) = e^x is a solution of integral eq. : $y(x) = 1 + \int_0^x e^x dt$

7. The Method of successive approximations: (Picard's Method) for successive approximations

The successive approximation method, which was successfully applied to Volterra integral equations, can be applied even more easily to the basic Fredholm integral equations:

If we have the I.V.P. : y' f(x,y) , y(x0) = y0 ….(11) , Reduce eq. (11) to integral eq. we obtain :

$$y(x) = y_0 + \int_{x0}^{x} f(t, y(t))dt \qquad (12)$$

Then eq.(12) which equivalence to eq. (11).

To solve (**) eq. by using Picard successive approximations[6, 14]. Or using this eq.

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t)\varphi(t)\,dt$$

We set $\varphi_0(x) = f(x)$ .Note that the zeroth approximation can be any selected real-valued function $\varphi_0(x), a \le x \le b$ .Accordingly, the first approximation $\varphi_1(x)$ of the solution of $\varphi(x)$ is defined by

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x,t)\varphi_0(t)\,dt$$

The second approximation $\varphi_2(x)$ of the solution $\varphi(x)$ can be obtained by replacing $\varphi_0(x)$ in equation (2.19) by the previously obtained $\varphi_1(x)$; [1, 12] hence we find

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t)\,\varphi_1(t)\,dt$$

This process can be continued in the same manner to obtain the nth approximation. In other words, the various approximation can be put in a recursive scheme given by $\varphi_0(x)$ = any selective real valued function

$$\varphi_n(x) = f(x) + \lambda \int_a^b K(x,t)\, \varphi_{n-1}^{(t)}\, dt, \qquad n \geq 1$$

Even though we can select any real-valued function for the zeroth approximation $\varphi_0(x)$, the most commonly selected functions for $\varphi_0(x)$ are $\varphi_0(x)$ = 0,1, *or x*. We have noticed that with the selection of $\varphi_0(x)$ =0, the first approximation $\varphi_1(x) = f(x)$. The final solution $\varphi(x)$ is obtained by

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$$

So that the resulting $\varphi(x)$ is independent of the choice of $\varphi_0(x)$. This is known as Picard's method [12,13].

The Neumann series is obtained if we set $\varphi_0(x) = f(x)$ such that

$$\varphi_1(x) = f(x) + \lambda \int_a^b K(x,t)\varphi_0(t)\, dt$$

$$= f(x) + \lambda \int_a^b K(x,t) f(t) dt$$

$$= f(x) + \lambda\psi_1(x)$$

Where

$$\psi_1(x) = \int_a^b K(x,t) f(t) dt$$

The second approximation $\varphi_2(x)$ can be obtained as

$$\varphi_2(x) = f(x) + \lambda \int_a^b K(x,t)\varphi_1(t)\, dt$$

$$= f(x)\lambda \int_a^b K(x,t)\{f(t) + \lambda\, \psi_1(t)\}dt$$

$$= f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x)$$

Where

$$\psi_2(x) = \int_a^b K(x,t)\psi_1(t)\, dt$$

Proceeding in this manner, the final solution $\varphi(x)$ can be obtained

$$u(x) = f(x) + \lambda\psi_1(x) + \lambda^2\psi_2(x) + \cdots + \lambda^n\psi_n(x) + \cdots$$

$$= f(x) \sum_{n=1}^{\infty} \lambda^n \psi_n(x)$$

Where

$$\psi_n(x) = \int_a^b$$

Example 1:

Solve the Fredholm Integral equation :

$$\varphi(x) = 1 + \int_0^1 x\varphi(t)\; dt$$

By using the successive approximation method [11, 12].

Solution : let us consider the zeroth approximation is $<p_0(x)$ = 1, and then the first approximation can be computed as:

$$\varphi_1(x) = 1 + \int_0^1 x\varphi_0(t)\; dt$$

$$= 1 + \int_0^1 x\, dt$$

$$= 1 + x$$

Proceeding in this manner, we find the second approximation is

$$\varphi_2(x) = 1 + \int_0^1 x\varphi_1(t)\ dt$$

$$= 1 + \int_0^1 x(1+t)\, dt$$

$$= 1 + x\left(1 + \frac{1}{2}\right)$$

Similarly, the third approximation is

$$\varphi_3(x) = 1 + x\int_0^1 \left(1 + \frac{3t}{2}\right) dt$$

$$= 1 + x\left(1 + \frac{1}{2} + \frac{1}{4}\right)$$

Thus, we get

$$\varphi_n(x) = 1 + x\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots\ldots + \frac{1}{2^{n-1}}\right\}$$

And hence

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$$

$$= 1 + \lim_{n\to\infty} x\sum_{d=0}^{n} \frac{1}{2^d}$$

$$= 1 + x\left(1 - \frac{1}{2}\right)^{-1}$$

$$= 1 + 2x$$

Example 2 : solve y' = xy , y(0) =1 By using the successive approximation method [11,14].

Solution : convert different ion equ. . to integral equ. That is : $y(x) = 1 + \int_0^x y(t)dt$

the first approximation can be computed as

$$\phi_0(x) = 1 \quad , \quad \phi_1(x) = 1 + \int_0^x t dt = 1 + \frac{x^2}{2} \quad ,$$

The second approximation is the function :

$$\phi_2(x) = 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right)dt = 1 + \frac{x^2}{2} + \frac{x^4}{24}$$

The third approximation is the function

$$\phi_3(x) = 1 + \int_0^x t\left(1 + \frac{t^2}{2} + \frac{t^4}{24}\right)dt = 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right) + \frac{1}{3!}\left(\frac{x^2}{2}\right)^3$$

If k →∞ then $\phi_k(x) \to \varphi(x)$, $\phi_k(x) = 1 + \frac{x^2}{2} + \ldots + \frac{1}{k!}\left(\frac{x^2}{2}\right)^k = e^{x^2/2}$

To check the solution : $e^{\frac{x^2}{2}} = 1 + \int_0^x t e^{\frac{t^2}{2}} dt$

8. Conclusion

1. When the integral equation has difficulty to find a solution for O,D.E. It is better to transform the last to integral equation which is equivalent to it
2. The solution of the integral equation depend on the kernel and the Resolvent kernel R(x, t , λ) was satisfied the integral

equation.: $R(x,t,\lambda) = k(x,t) + \lambda \int_{t}^{x} k(x,z) R(z,t,\lambda) dz$

3. In many cases a non –degenerate kernel k(x,y) may be approximate by a degenerate kernel as a partial sum of the Taylor series expansion or other of k(x,y)and truncate the series
4. It has found that the initial value problem reduce to velterra integral equation and boundary value problem to freddholm integral equation.

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