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Original Article

Stability Analysis of Time-Delayed Systems Based on a Negative-Determination Quadratic Function

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Abstract **-** *This paper addresses the problem of stability analysis of time-delay systems. The quadratic function combined with reciprocally convex lemma has been constructed to establish the stability criteria for the linear time-delay system. Finally, a numerical example is given to demonstrate the effectiveness of the proposed criterion.*

Keywords - *Delay systems, Time-varying delay, Stability analysis, Lyapunov-Krasovskii functional, Quadratic function.*

1. Introduction

Over the past years, stability analysis of time-delay systems have acted an important part in the real systems, due to the fact that the time delay is a potential harm to system stability. Thus, how to reduce the conservatism of time-delay systems have attracted more and more scholars' concentration [1-4]. Time-delay is always regarded as a common phenomenon, which can result in performance degradation and instability [5, 6, 25]. In the literatures, many methods have been proposed to reduce the conservatism, such as: partition delay method [5, 7, 8], new LKF choices [9-15, 28], free-weighting matrices [16-18], improved quadratic integral inequality [19-22], Reciprocally convex combination lemma[23], Bessel-Legendre-based inequality [24], improved Jensen inequality[26], improved reciprocally convex inequality [27,30], negative definite lemmas of quadratic functions [29].

Among the above research, stability analysis of time-delay systems based on Lyapunov-Krasovskii functional (LKF) and LMI is the most popular. Under this framework, the main contributions of this paper show as follows. By considering the crucial information about delay states and integral terms, a novel LKF is proposed. The negative definite lemma of quadratic function combined with reciprocally convex lemma are used to establish the stability criteria for the linear time-delay system. A less conservative stability criteria is proposed and expressed as negativity conditions for quadratic function, which uses a quadratic negative definite lemma with a adjustable parameters to handle negativity conditions for quadratic function. Finally, the effectiveness of the proposed stability criteria are proved in the numerical example.

Notation: The superscript "T" means the transpose of a matrix; R^n denotes the n-dimensional Euclidean space; $\mathrm{P} > 0 (\geq 0)$ means that P is a real symmetric and positive definite (semi-positive definite) matrix; symmetric term in a symmetric matrix is denoted by $*$ and $\mathrm{sym} = \{\mathrm{Y}\} = \mathrm{Y} + \mathrm{Y}^{\mathrm{T}}$.

2. Problem Formulation and Preliminaries

The following time-delay system is considered in this paper:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)) \\ x(t) = \phi(t), t \in [0,h] \end{cases} \tag{1}$$

where $x(t) \in R^n$ is the state variable; $A, A_d \in R^{n\times n}$ are constant matrices; $\phi(t)$ is an initial condition; the time-varying delay $d(t)$ satisfies the following constraints:

$$h_1 \le d(t) \le h_2, d_1 \le \dot{d}(t) \le d_2, \tag{2}$$

which is a continuous function with constant scalars h_1, h_2, d_1 and d_2.

Before developing the stability criteria for system (1), the following lemmas are given as follows:

Lemma 2.1.[29] For a quadratic function $f(y) = a_2y^2 + a_1y + a_0$ with $a_0 \in R$, $f(y) < 0$ is ensured for all $y \in [h_1, h_2]$ if the following inequalities hold:

$$f(h_i) < 0, i = 1,2 \tag{3}$$

$$f(h_1) - h_{12}^2 a_2 < 0 \tag{4}$$

Lemma 2.2.[29] For a quadratic function $f(y) = a_2y^2 + a_1y + a_0$ with $a_0 \in R$, $f(y) < 0$ is ensured for all $y \in [h_1, h_2]$, if the following inequalities hold:

$$f(h_i) < 0, i = 1,2 \tag{5}$$

$$f(h_1) - \lambda^2 h_{12}^2 a_2 < 0 \tag{6}$$

$$f(h_2) - (1-\lambda)^2 h_{12}^2 a_2 < 0 \tag{7}$$

Remark 1: The above two negative definite lemmas of quadratic functions can be used as the basis for determining the stability conditions of system (1). Lemma 2.1 can be derived from Lemma 2.2 by choosing a suitable value for the unknown quantity λ. In other words, the results derived by using Lemma 2.2 are less conservative than those by Lemma 2.1.

Lemma 2.3. (Second Order Bessel-Legendre Inequality). For any $R \in S^n$ and different x in $[a,b] \to R^n$, the following inequality holds:

$$\int_a^b \dot{x}^T(u) R \dot{x}(u) \ge \frac{1}{b-a} \Omega^T diag(R, 3R, 5R) \Omega \tag{8}$$

where $\Omega = \mathrm{col}\{\Omega_1, \Omega_2, \Omega_3\}$, with

$$\Omega_1 = x(b) - x(a) \tag{9}$$

$$\Omega_2 = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(u)du \tag{10}$$

$$\Omega_3 = \Omega_1 - \frac{6}{b-a}\int_a^b x(u)du + \frac{12}{(b-a)^2}\int_a^b (b-u)x(u)du \tag{11}$$

Lemma 2.4. (Reciprocally Convex lemma). Let $R_1, R_2 \in S_+^n$; $\sigma_1, \sigma_2 \in R^m$ and a scalar $\alpha \in (0,1)$. If there exist matrices $X_1, X_2 \in S^m$ and $Y_1, Y_2 \in R^{m\times m}$ such that

$$\begin{bmatrix} R_1 - X_1 & Y_1 \\ * & R_2 \end{bmatrix} \geq 0, \begin{bmatrix} R_1 & Y_2 \\ * & R_2 - X_2 \end{bmatrix} \geq 0 \tag{12}$$

then the following inequality holds

$$\frac{1}{\alpha}\sigma_1^T R_1 \sigma_1 + \frac{1}{1-\alpha}\sigma_2^T R_2 \sigma_2 \geq \sigma_1^T [R_1 + (1-\alpha)X_1]\sigma_1 + \sigma_2^T [R_2 + \alpha X_2]\sigma_2 + 2\sigma_1^T [\alpha Y_1 + (1-\alpha)Y_2]\sigma_2 \tag{13}$$

3. Main results

In this part, by selecting an appropriate LKF, the stability analysis of the system (1) is researched in Theorem 3.1. Then, a corollary is presented based on the Theorem 3.1.

Theorem 3.1. For given scalars d_j, h_j $(j = 1,2)$, and a parameter λ selected within $[0,1]$, $\forall d(t) \in [h_1, h_2]$, system (1) is asymptotically stable if there exist matrices $P_i \in s^{3n}, Q_i \in S^n, R_i \in S^n, i = 1,2$, such that the following conditions hold:

$$\begin{bmatrix} \tilde{R}_2 - X_1 & Y_1 \\ * & \tilde{R}_2 \end{bmatrix} \geq 0, \begin{bmatrix} \tilde{R}_2 & Y_2 \\ * & \tilde{R}_2 - X_2 \end{bmatrix} \geq 0 \tag{14}$$

$$\begin{aligned} &\Delta_1 = h_1^2 \Psi_2(d_j) + h_1 \Psi_1(d_j) + \Psi_0(d_j) < 0, \\ &\Delta_2 = h_2^2 \Psi_2(d_j) + h_2 \Psi_1(d_j) + \Psi_0(d_j) < 0, \\ &\Delta_3 = \Delta_1 - \lambda^2 h_{12}^2 \Psi_2(d_j) < 0, \\ &\Delta_4 = \Delta_2 - (1-\lambda)^2 h_{12}^2 \Psi_2(d_j) < 0. \end{aligned} \tag{15}$$

where $\tilde{R}_i = diag\{R_i, 3R_i, 5R_i\}$, $i = 1,2$ and

$$\begin{aligned} &\Psi_0 = sym\{A_{11}^T P_1 A_2\} + sym\{A_{31}^T P_2 A_4\} + e_1^T P_1 e_1 + e_2^T (Q_1 - Q_2) e_2 - e_5^T Q_2 e_5 \\ &+ h_{12}^2 e_c^T \mathrm{R}_2 e_c - \Gamma_0^T \tilde{R}_1 \Gamma_0 - \Gamma_1^T \tilde{R}_2 \Gamma_1 + \frac{h_1}{h_{12}} \Gamma_1^T X_1 \Gamma_1 - \frac{h_2}{h_{12}} sym\{\Gamma_1^T Y_1 \Gamma_2\} + \frac{h_1}{h_{12}} sym\{\Gamma_1^T Y_2 \Gamma_2\} \\ &\Gamma_2^T \tilde{R}_2 \Gamma_2 - \frac{h_2}{h_{12}} sym\{\Gamma_1^T X_2 \Gamma_2\} \end{aligned} \tag{16}$$

$$\begin{aligned} &\Psi_1 = sym\{A_{12}^T P_2 A_2\} + sym\{A_{32}^T P_2 A_4\} - \frac{1}{h_{12}} \Gamma_1^T X_1 \Gamma_1 + \frac{1}{h_{12}} sym\{\Gamma_1^T Y_1 \Gamma_2\} \\ &- \frac{1}{h_{12}} sym\{\Gamma_1^T Y_2 \Gamma_2\} + \frac{1}{h_{12}} \Gamma_2^T X_2 \Gamma_2 \end{aligned} \tag{17}$$

$$\Psi_2 = sym\{A_{33}^T P_2 A_4\} + e_c^T R_1 e_c \tag{18}$$

with,

$$\Gamma_0 = col\{e_1 - e_2, e_1 + e_2 - 2e_8, e_1 - e_2 + 6e_8 - 12e_{11}\},$$
$$\Gamma_1 = col\{e_3 - e_5, e_3 + e_5 - 2e_7, e_3 - e_5 + 6e_7 - 12e_{10}\},$$
$$\Gamma_2 = col\{e_2 - e_3, e_2 + e_3 - 2e_6, e_2 - e_3 + 6e_6 - 12e_9\},$$
$$e_c = Ae_1 + A_d e_3,$$
$$A_{11} = col\{e_1, -h_1 e_6, h_1^2 e_{11}\},$$
$$A_{12} = col\{e_0, e_6, e_0\},$$
$$A_2 = col\{e_c, e_2 - (1 - \dot{d}(t))e_3, h_1(e_1 - e_8)\},$$
$$A_{31} = col\{e_1, h_2 e_7, h_1^2 e_9 + h_2^2 e_{10} + h_2 e_{12}\},$$
$$A_{32} = col\{e_0, -e_7, -2h_1 e_9 + -2h_2 e_{10} - e_{12}\},$$
$$A_{33} = col\{e_0, e_0, e_9 + e_{10}\},$$
$$A_4 = col\{e_c, (1 - \dot{d}(t))e_3 - e_5, h_{12} e_2 - e_{12} - e_{13}\}$$

and $e_i = [0_{n\times(i-1)n}\ \ I_n\ \ 0_{n\times 13n}]$, $i = 1, 2, \ldots, 13$; $e_0 = 0_{n\times 13n}$; $h_{12} = h_2 - h_1$.

Proof. For systems (1), we consider the following Lyapunov-Krasovskii functional:

$$V(t) = \sum_{i=1}^{3} V_i(t) \tag{19}$$

where

$$V_1(t) = \zeta_1^T(t) P_1 \zeta_1(t) + \zeta_2^T(t) P_2 \zeta_2(t)$$
$$V_2(t) = \int_{t-h_1}^{t} x^T(s) Q_1 x(s) ds + \int_{t-h_2}^{t-h_1} x^T(s) Q_2 x(s) ds$$
$$V_3(t) = h_1 \int_{-h_1}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta + h_{12} \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta$$

with

$$\zeta_1(t) = col\{x(t), \int_{t-d(t)}^{t-h_1} x(s) ds, h_1^2 \int_{t-h_1}^{t} \int_{\theta}^{t} \frac{x(s)}{h_1^2} ds d\theta\}$$
$$\zeta_2(t) = col\{x(t), \int_{t-h_2}^{t-d(t)} x(s) ds, h_{12}^2 \int_{t-h_2}^{t-h_1} \int_{\theta}^{t-h_1} \frac{x(s)}{h_{12}^2} ds d\theta\}$$

Since $P_i > 0, Q_i > 0, R_i > 0, i = 1, 2, V(t) \geq \varepsilon \|x(t)\|^2$ can be derived for a constant $\varepsilon > 0$.

Calculating the derivative of the $V(t)$ along the solution of (1), yield:

$$\begin{aligned}\dot{V}_1(t) &= 2\zeta_1^T(t) P_1 \dot{\zeta}_1(t) + \zeta_2^T(t) P_2 \dot{\zeta}_2(t) \\ &= \xi^T(t)\big\{sym\{A_{11}^T P_1 A_2\} + d(t) sym\{A_{12}^T P_2 A_2\} + sym\{A_{31}^T P_2 A_3\} + d(t) sym\{A_{32}^T P_2 A_4\} + \\ &\quad d^2(t) sym\{A_{33}^T P_2 A_4\}\big\}\xi(t)\end{aligned} \tag{20}$$

$$\dot{V}_2(t) = x^T(t) Q_1 x(t) - x^T(t-h_1)(Q_1 - Q_2) x(t - t_1) - x^T(t-h_2) Q_2 (t - h_2)$$

$$= \xi^T(t)\{e_1^T Q_1 e_1 + e_2^T(Q_1 - Q_2)e_2 - e_5^T Q_2 e_5\}\xi(t) \tag{21}$$

$$\begin{aligned}\dot{V}_3(t) &= \dot{x}^T(t)(h_1^2 R_1 + h_{12}^2 R_2)\dot{x}(t) - J_1 - J_2 \\ &\le \xi^T(t)\{e_c^T(d^2(t)R_1 + h_{12}^2 R_2)e_c\}\xi(t) - J_1 - J_2\end{aligned} \tag{22}$$

Where

$$\begin{aligned}\xi(t) = \ & col\{x(t), x(t-h_1), x(t-d(t)), \dot{x}(t-d(t)), x(t-h_2), u(d(t),h_1,t), u(h_2,d(t),t), \\ & u(h_1,0,t), v(d(t),h_1,t), v(h_2,d(t),t), v(h_1,0,t), (d(t)-h_1)u(d(t),h_1,t), \\ & \big(h_2-d(t)\big)u(h_2,d(t),t)\},\end{aligned}$$

$$u(a,b,t) = \int_{t-a}^{t-b} \frac{x(s)}{a-b} ds, \quad v(a,b,t) = \int_{t-a}^{t-b}\int_{\theta}^{t-b} \frac{x(s)}{(a-b)^2} ds d\theta$$

$J_1 = \int_{t-h_1}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds$, $J_2 = \int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_1\dot{x}(s)ds$.

Let $\alpha = \dfrac{h_2 - d(t)}{h_{12}}$, J_1 and J_2 can be rewritten as following by lemma 2.3,

$$\begin{aligned}J_1 &= h_1\int_{t-h_1}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds \\ &\ge [x(t)-x(t-h_1), x(t)+x(t-h_1)-2u(h_1,0,t), x(t)-x(t-h_1)+6u(h_1,0,t)-12v(h_1,0,t)] \\ &\quad \tilde{R}_1[x(t)-x(t-h_1), x(t)+x(t-h_1)-2u(h_1,0,t), x(t)-x(t-h_1)+6u(h_1,0,t)-12v(h_1,0,t)]^T\end{aligned} \tag{23}$$

$$\begin{aligned}J_2 = h_{12}\int_{t-h_2}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds &= h_{12}\left(\int_{t-h_2}^{t-d(t)} \dot{x}^T(s)R_2\dot{x}(s)ds + \int_{t-d(t)}^{t-h_1} \dot{x}^T(s)R_2\dot{x}(s)ds\right) \\ &\le \xi^T(t)\frac{h_{12}}{h_2-d(t)}\Gamma_1^T\tilde{R}_2\Gamma_1\xi(t) + \xi^T(t)\frac{h_{12}}{d(t)-h_1}\Gamma_2^T\tilde{R}_2\Gamma_2\xi(t) \\ &= \frac{1}{\alpha}\xi^T(t)\Gamma_1^T\tilde{R}_2\Gamma_1\xi(t) + \frac{1}{1-\alpha}\xi^T(t)\Gamma_2^T\tilde{R}_2\Gamma_2\xi(t)\end{aligned} \tag{24}$$

Hence, we can apply lemma 2.4 to obtain

$$\begin{aligned}&\frac{1}{\alpha}\xi^T(t)\Gamma_1^T\tilde{R}_2\Gamma_1\xi(t) + \frac{1}{1-\alpha}\xi^T(t)\Gamma_2^T\tilde{R}_2\Gamma_2\xi(t) \ge \xi^T(t)\{\Gamma_1^T[\tilde{R}_1 + (1-\alpha)X_1]\Gamma_1] + 2\Gamma_1^T[\alpha Y_1 + (1-\alpha)Y_2]\Gamma_2 \\ &+ \Gamma_2^T[\tilde{R}_2 + \alpha X_2]\Gamma_2]\xi(t)\end{aligned}$$

Then

$$J_2 \ge \xi^T(t)\{\Gamma_1^T[\tilde{R}_1 + (1-\alpha)X_1]\Gamma_1] + 2\Gamma_1^T[\alpha Y_1 + (1-\alpha)Y_2]\Gamma_2 + \Gamma_2^T[\tilde{R}_2 + \alpha X_2]\Gamma_2]\xi(t) \tag{25}$$

Thus, based on the previous inequalities (20)-(25) we have that

$$\dot{V}(t) \le \xi^T(t)[d^2(t)\Psi_2 + d(t)\Psi_1 + \Psi_0]\xi(t) \tag{26}$$

with $\Psi_2, \Psi_1.\Psi_0$ are given in (16), (17), and (18), respectively. Therefore, under the constraints (14) and (15), $\dot{V}(t) < 0$ is satisfied.

By selecting the parameter $\lambda = 1$ in Theorem 3.1, we can obtain the following corollary. Therefore, the stability analysis of corollary 3.1 is a special case of Theorem 3.1.

Corollary 3.1. For given scalars d_j, h_j $(j = 1,2)$, $\forall d(t) \in [h_1, h_2]$, system (1) is asymptotically stable if there exist matrices $P_i \in s^{3n}, Q_i \in S^n, R_i \in S^n, i = 1,2$, such that the following conditions hold:

$$\begin{bmatrix} \tilde{R}_2 - X_1 & Y_1 \\ * & \tilde{R}_2 \end{bmatrix} \geq 0, \begin{bmatrix} \tilde{R}_2 & Y_2 \\ * & \tilde{R}_2 - X_2 \end{bmatrix} \geq 0$$

$$\hat{\Delta}_1 = h_1^2 \Psi_2(d_j) + h_1 \Psi_1(d_j) + \Psi_0(d_j) < 0,$$
$$\hat{\Delta}_2 = h_2^2 \Psi_2(d_j) + h_2 \Psi_1(d_j) + \Psi_0(d_j) < 0,$$
$$\hat{\Delta}_3 = \hat{\Delta}_1 - h_{12}^2 \Psi_2(d_j) < 0,$$

where Ψ_0, Ψ_1, Ψ_2 are given in Theorem 3.1.

4. Numerical Example

In this section, a numerical example is given to illustrate the effectiveness of Theorem 3.1 and Corollary 3.1.

Example 1. Consider system (1) with the following parameters:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

For given bounds of the delay derivative $\dot{d}(t) \in [d_1, d_2]$, by setting $d_1 = -d_2$, we search for the maximum admissible delay upper bound h_2 with $h_1 = 0$ according to the value of variable λ.

Table 1. Maximum admissible upper bound h_2 of the delay $d(t)$ for given $d_1 = -d_2$

d_2	d_2 =0.1	d_2 =0.4	d_2 =0.9
[25, IQC analysis]	6.494	0.886	0.439
[26, Theorem 1]	6.668	1.542	1.263
[12, Proposition 1]	7.176	2.496	1.922
[27, Proposition 2]	7.230	2.509	1.940
[22, Theorem 2]	7.308	2.664	2.072
[28, Theorem 1]	7.400	2.717	2.089
Corollary 2.1	6.111	9.768	6.331
Theorem 2.1($\lambda = 0.4$)	8.989	1.677	6.031
Theorem 2.1($\lambda = 0.6$)	9.979	1.599	4.631
Theorem 2.1($\lambda = 0.8$)	8.848	11.139	3.661

Remark 2: For system (1), $\lambda \in [0, 1]$ was selected in Theorem 2.1. For different values of λ, we can get different results of the maximum admissible upper bound h_2. The above table indicates that the proposed criteria in Theorem 2.1 can lead to the less conservative results than those in literatures [12,22,25-28].

5. Conclusion

The stability analysis of time-delay systems is considered in this paper. Based on an appropriate LKF, the negative definite lemma of quadratic function combined with reciprocally convex lemma are used to establish the stability criteria for the linear time-delay system. An adjustable parameter has been adopted to reduce the conservatism, and its advantages have been shown in the numerical example.

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