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Original Article

Domain Decomposition Method for a Nonlocal Coupled System in the Stationary Case

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Abstract : *In this paper, we propose and study a domain decomposition method for a system of coupled equations with a nonlocal term in the stationary case. This algorithm combines an alternating direction scheme and a domain decomposition. The underlying idea is that with a good domain decomposition, we can better describe for each point the interaction neighbourhood. Moreover, the second step of this method is a good preconditioner of the first step. Indeed, the matrices of the first step are poorly dimensioned because of the nonlocal term. Finally, although combining the alternating direction scheme and the domain decomposition, our algorithm has a good degree of parallelism.*

Keywords : *Domain decomposition method; a coupled system equation with a nonlocal term; alternating direction scheme; preconditionner; a good degree of parallelism.*

1 Introduction

The importance of the nonlocal term in the modelization of many phenomenas physics is best known in the scientific world. Indeed it permits to take better the interactions between the different elements that contribute to the realization of the phenomena (propagation of the epidemics, competition between two populations that occupying the same territory, ...). But the introduction of the nonlocal term in the model gives many difficulties as soon as theoretically than numerically. Indeed for example on the theorie plan, it is very difficult to show the existence and the uniqueness of the solution [2], [3], [5], [7], [21]. We can give as an example of works dealing with these questions [4], [19], [36], [37]. The presence of the nonlocal term introduces for each point of the domain an interaction neighbourhood, i.e. a ball composed of the points which are in interaction with it. Numerically, the interaction neighbourhood is not easy to describe because it is not the same for the different points of the domain. The domain decomposition methods can be described as an artificial division of a given domain where a partial differential equation has to be solved (see [14], [15], [16], [17], [24], [25], [39]). According to the subdomain division, domain decomposition methods are classified into two categories: overlapping and nonoverlapping methods. The well-known overlapping method is the Schwarz method [26], [31], [32], [33]. The nonoverlaping methods are more recent than overlapping methods. As works on nonoverlaping methods we have [8],[9], [10], [11], [12],[13],[18], [23]. The well-known nonoverlaping methods is Lions method [27], [31], [34], [35]. Using domain decomposition methods, we propose an approach that circumvents this difficulty. By reducing the study domain the domain decomposition method allows a better approximation of the nonlocal term than the approximation made in the global domain. Indeed, with the decomposition of the global domain into subdomains, the approximation that consists in taking the interaction neighbourhood of each point as being the sub-domain of study is a good consideration. Moreover the second step of this algorithm by solving two Neumann problems gives a good preconditioner for the first step of this method [28], [29], [38]. Indeed, with the presence of the nonlocal term, the matrices of the first step are badly conditioned. Finally our algorithm, although combining the alternating direction scheme and the domain decomposition, presents a good degree of parallelism.

2 Existence and Uniqueness of the Local Solution

Let consider the nonlocal coupled system defined by:

$$
\begin{aligned}
-\, div(\kappa(l_r(u))\nabla u) + f(u-v) &= \alpha(u-v) \quad \text{in } \Omega, & (2.1)\\
-div(\kappa(l_r(v))\nabla v) - f(u-v) &= \alpha(v-u) \quad \text{in } \Omega, & (2.2)\\
u = v &= 0 \quad \text{on } \Gamma_0. & (2.3)
\end{aligned}
$$

where $u(x)$ and $v(x)$ are real valued functions, $\Omega \subset \mathbb{R}^n$ is a bounded domain. The boundary $\Gamma_0 = \partial\Omega$ is supposed to be class C^2. The real functions f and κ, defined on $\mathbb{R}$, are Lipschitz continuous such that

$$|f(t) - f(s)| \leq \gamma_1|t-s| \,\forall s, t \in \mathbb{R} \tag{2.4}$$

$$|\kappa(t) - \kappa(s)| \leq \gamma_2|t-s| \,\forall s, t \in \mathbb{R} \tag{2.5}$$

Moreover, the function κ verifies that

$$0 < m \leq a(\varepsilon) \leq M \,\forall \varepsilon \in \mathbb{R} \tag{2.6}$$

with m and M two positive constants. And $l_r : L^2(\Omega) \to \mathbb{R}$ is the continuous linear form. The problem ((2.1)-(2.3)) can be consider as an asymptotic case of the problem studied by C. A. Raposo et al. [37] in the evolution case. This mathematical model have been used to describe many phenomenas in physical, chemical, biological and ecological systems. The originality of this approach is in the term $\kappa(l_r(u))$. Indeed some publications on the subjet assume that the matrix of diffusion is a diagonal matrix so that coupling between the equations are present only through the term f (see [36]). However, many problems could be treated as equation in which the diffusion matrix is not diagonal (see [19], [22]). The new approach is to see the term $\kappa(l_r(u))$ as a nonlocal quantity with

$$l_r(u) = \int_{\Omega_r} u(x,t)\, dx$$

where $\Omega_r \subset \Omega$ (the diffusion depends on the global population in Ω_r but not on the global population in Ω). In [4], Andami and Rougirel studie the following nonlocal problem

Find $u \in H_0^1(\Omega)$ such that

$$-\,\kappa(l_r(u))\Delta u = f \quad \text{in } H^{-1}(\Omega) \tag{2.7}$$

where

$$l_r(u)(x) = \int_{\Omega \cup B(x,r)} g(x,y)u(y)dy.$$

For the model problem (2.1)-(2.3), we have the following existence and uniqueness theorem.

Theorem 2.1. *Assume (2.4), (2.5) and (2.6) hold true and $r \in (0,\, L)$. Then (2.1)-(2.3) admits a solution (u_r, v_r) in $H_0^1(\Omega) \times H_0^1(\Omega)$. If in addition solution (u_r, v_r) of (2.1)-(2.3) satisfy*

$$||u_r||_{H_0^1(\Omega)} \leq \frac{m - c(\Omega)(1+\gamma_1+2\alpha)}{\gamma_2|\Omega|^{1/2}c(\Omega)^{1/2}} \tag{2.8}$$

$$||v_r||_{H_0^1(\Omega)} \leq \frac{m - c(\Omega)(1+\gamma_1+2\alpha)}{\gamma_2|\Omega|^{1/2}c(\Omega)^{1/2}} \tag{2.9}$$

where $c(\Omega)$ denotes the Poincaré Sobolev constante and $|\Omega|$ the measure of Ω. Then the solution is unique.

Proof. Existence:
The proof can be obtained by a paper Galerkin scheme. We refer the reader to [20].

Uniqueness:
Let (u_1, v_1) and (u_2, v_2) be solutions of (2.1)-(2.3). Setting

$$\begin{aligned} \bar{u} &= u_1 - u_2 \quad w_1 = u_1 - v_1 \\ \bar{v} &= v_1 - v_2 \quad w_2 = u_2 - v_2 \end{aligned}$$

we have

$$-\,div(\kappa(l_r(u_1))\nabla u_1) + f(w_1) = \alpha w_1 \quad \text{in } \Omega \tag{2.10}$$

$$-div(\kappa(l_r(v_1))\nabla v_1) - f(w_1) = -\alpha w_1 \quad \text{in } \Omega \tag{2.11}$$

$$u_1 = v_1 = 0 \text{ on } \Gamma \tag{2.12}$$

and

$$-\,div(\kappa(l_r(u_2))\nabla u_2)) + f(w_2) = \alpha w_2 \quad \text{in } \Omega \tag{2.13}$$

$$-div(\kappa(l_r(v_2))\nabla v_2)) - f(w_2) = -\alpha w_2 \quad \text{in } \Omega \tag{2.14}$$

$$u_2 = v_2 = 0 \text{ on } \Gamma \tag{2.15}$$

By testing (2.10)-(2.12) and (2.13)-(2.15), we get

$$\begin{aligned} \int_\Omega (\kappa(l_r(u_1))\nabla u_1 - \kappa(l_r(u_2))\nabla u_2)\nabla\varphi_1\,dx &= -\int_\Omega (f(w_1) - f(w_2))\varphi_1\,dx \\ &+ \alpha\int_\Omega (w_1 - w_2)\varphi_1\,dx \quad \forall\varphi_1 \in H_0^1(\Omega) \end{aligned} \tag{2.16}$$

$$\begin{aligned} \int_\Omega (\kappa(l_r(v_1))\nabla v_1 - \kappa(l_r(v_2))\nabla v_2)\nabla\varphi_2\,dx &= \int_\Omega (f(w_1) - f(w_2))\varphi_2\,dx \\ &+ \alpha\int_\Omega (w_2 - w_1)\varphi_2\,dx \quad \forall\varphi_2 \in H_0^1(\Omega) \end{aligned} \tag{2.17}$$

Since

$$\kappa(l_r(u))\nabla u - \kappa(l_r(v))\nabla v = (\kappa(l_r(u)) - \kappa(l_r(v)))\nabla u + \kappa(l_r(v))\nabla(u - v), \tag{2.18}$$

if follows that

$$\begin{aligned} \int_\Omega (\kappa(l_r(u_1)) - \kappa(l_r(u_2)))\nabla u_1\nabla\varphi_1 dx + \int_\Omega \kappa(l_r(u_2))\nabla(\bar{u})\nabla\varphi_1 dx &= -\textstyle\int_\Omega (f(w_1) - f(w_2))\varphi_1 dx \\ &+\alpha\int_\Omega (w_1 - w_2)\varphi_1 dx, \qquad \forall\varphi_1 \in H_0^1(\Omega), \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} \int_\Omega (\kappa(l_r(v_1)) - \kappa(l_r(v_2)))\nabla v_1\nabla\varphi_2 dx + \int_\Omega \kappa(l_r(v_2))\nabla(\bar{u})\nabla\varphi_2 dx &= \textstyle\int_\Omega (f(w_1) - f(w_2))\varphi_2 dx \\ &+\alpha\int_\Omega (w_2 - w_1)\varphi_2 dx, \qquad \forall\varphi_2 \in H_0^1(\Omega). \end{aligned} \tag{2.20}$$

Setting $\varphi_1 = \bar{u}$ and $\varphi_2 = \bar{v}$, we get

$$\begin{aligned} m||\bar{u}||^2_{H_0^1(\Omega)} \leq \int_\Omega |\kappa(l_r(u_1)) - \kappa(l_r(u_2))||\nabla u_1||\nabla\bar{u}|dx &+ \int_\Omega |f(w_1) - f(w_2)||\bar{u}|dx + \alpha\int_\Omega |\bar{u}|^2 dx \\ &+ \alpha\int_\Omega |\bar{v}||\bar{u}|dx \end{aligned} \tag{2.21}$$

From (2.4), (2.5) and (2.21)

$$m||\bar{u}||^2_{H^1_0(\Omega)} \leq \gamma_2 \int_\Omega |l_r(u_1) - l_r(u_2)||\nabla u_1||\nabla \bar{u}|dx + \gamma_1 \int_\Omega |\bar{u}|^2 dx + \gamma_1 \int_\Omega |\bar{v}||\bar{u}|dx + \alpha \int_\Omega |\bar{u}|^2 dx + \alpha \int_\Omega |\bar{v}||\bar{u}|dx \tag{2.22}$$

We can easily get by using Cauchy-Schwarz inequality

$$|l_r(u)| \leq |\Omega|^{1/2} c(\Omega)^{1/2} ||u||_{H^1_0(\Omega)} \tag{2.23}$$

Using (2.21), (2.22) and (2.23), we obtain

$$m||\bar{u}||^2_{H^1_0(\Omega)} \leq \gamma_2 |\Omega|^{1/2} c(\Omega)^{1/2} ||\bar{u}||^2_{H^1_0(\Omega)} ||u_1||_{H^1_0(\Omega)} + \gamma_1 ||\bar{u}||^2_{L^2(\Omega)} + \gamma_1 ||\bar{v}||_{L^2(\Omega)} + \alpha ||\bar{u}||^2_{L^2(\Omega)} + \alpha ||\bar{v}||_{L^2(\Omega)} ||\bar{u}|\bar{u}|_{L^2(\Omega)} \tag{2.24}$$

And also by Young inequality

$$\begin{aligned} m||\bar{u}||^2_{H^1_0(\Omega)} &\leq \gamma_2 |\Omega|^{1/2} c(\Omega)^{1/2} ||\bar{u}||^2_{H^1_0(\Omega)} ||u_1||_{H^1_0(\Omega)} + c(\Omega) ||\bar{u}||^2_{H^1_0(\Omega)} + \frac{\gamma_1 c(\Omega)}{2} ||\bar{v}||^2_{H^1_0(\Omega)} \\ &+ \frac{\gamma_1 c(\Omega)}{2} ||\bar{u}||^2_{H^1_0(\Omega)} + \alpha c(\Omega) ||\bar{u}||^2_{H^1_0(\Omega)} + \frac{\alpha c(\Omega)}{2} ||\bar{v}||^2_{H^1_0(\Omega)} + \frac{\alpha c(\Omega)}{2} ||\bar{u}||^2_{H^1_0(\Omega)} \end{aligned} \tag{2.25}$$

Using the same arguments, we also obtain

$$\begin{aligned} m||\bar{v}||^2_{H^1_0(\Omega)} &\leq \gamma_2 |\Omega|^{1/2} c(\Omega)^{1/2} ||\bar{v}||^2_{H^1_0(\Omega)} ||v_1||_{H^1_0(\Omega)} + c(\Omega) ||\bar{v}||^2_{H^1_0(\Omega)} + \frac{\gamma_1 c(\Omega)}{2} ||\bar{u}||^2_{H^1_0(\Omega)} \\ &+ \frac{\gamma_1 c(\Omega)}{2} ||\bar{v}||^2_{H^1_0(\Omega)} + \alpha c(\Omega) ||\bar{v}||^2_{H^1_0(\Omega)} + \frac{\alpha c(\Omega)}{2} ||\bar{u}||^2_{H^1_0(\Omega)} + \frac{\alpha c(\Omega)}{2} ||\bar{v}||^2_{H^1_0(\Omega)} \end{aligned} \tag{2.26}$$

Additing (2.25) and (2.26) , we get

$$\begin{aligned} m(||\bar{u}||^2_{H^1_0(\Omega)} + ||\bar{v}||^2_{H^1_0(\Omega)}) &\leq (\gamma_2 |\Omega|^{1/2} c(\Omega)^{1/2} ||u_1||_{H^1_0(\Omega)} + c(\Omega) + \gamma_1 c(\Omega) + 2\alpha c(\Omega)) ||\bar{u}||^2_{H^1_0(\Omega)} \\ &+ (\gamma_2 |\Omega|^{1/2} c(\Omega)^{1/2} ||v_1||_{H^1_0(\Omega)} + c(\Omega) + \gamma_1 c(\Omega) + 2\alpha c(\Omega)) ||\bar{v}||^2_{H^1_0(\Omega)} \end{aligned} \tag{2.27}$$

Using $(Eq\,1)$ and $(Eq\,2)$ in (2.27), we deduce the uniqueness. □

3 Domain decomposition

Let $\boldsymbol{u} = (u, v)$ and

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u})) = \begin{pmatrix} \kappa(l_r(u)) & 0 \\ 0 & \kappa(l_r(v)) \end{pmatrix}$$

$$\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f(u-v) - \alpha(u-v) \\ -f(u-v) + \alpha(u-v) \end{pmatrix}$$

With the above notations, (2.1)-(2.3) can be written as

$$-\,div(\boldsymbol{\kappa}(l_r(\boldsymbol{u}))\nabla \boldsymbol{u}) + \boldsymbol{f}(\boldsymbol{u}) = 0 \quad \text{in } \Omega \tag{3.1}$$

$$\boldsymbol{u} = 0 \quad \text{in } \Gamma_0. \tag{3.2}$$

Let $\{\Omega_1,\, \Omega_2\}$ be a partition of Ω. We set $\Gamma_i = \Omega_i \cap \Gamma_0$. We set $\boldsymbol{u}_i = (u_i, v_i)$ where $u_i = u_{|\Omega_i}$ and $v_i = v_{|\Omega_i}$. In each suddomain Ω_i, we propose to solve

$$\begin{aligned} -\,div(\boldsymbol{\kappa}(l_r(\boldsymbol{u}_1))\nabla \boldsymbol{u}_1) + \boldsymbol{f}(\boldsymbol{u}_1) &= 0 \quad \text{in } \Omega_1 && (3.3)\\ \boldsymbol{u}_1 &= 0 \quad \text{on } \Gamma_1 && (3.4)\\ \boldsymbol{u}_1 &= \boldsymbol{u}_2 \quad \text{on } \Gamma_{12} && (3.5) \end{aligned}$$

and

$$\begin{aligned} -\,div(\boldsymbol{\kappa}(l_r(\boldsymbol{u}_2))\nabla \boldsymbol{u}_2) + \boldsymbol{f}(\boldsymbol{u}_2) &= 0 \quad \text{in } \Omega_2 && (3.6)\\ \boldsymbol{u}_2 &= 0 \quad \text{in } \Gamma_2 && (3.7)\\ \frac{\partial \boldsymbol{u}_2}{\partial n_2} &= -\frac{\partial \boldsymbol{u}_1}{\partial n_1} \quad \text{on } \Gamma_{12} && (3.8) \end{aligned}$$

where n_i, $(i = 1, 2)$ is the unit outward normal to Ω_i.

To show the equivalence between the multi-domain formulation (3.3)-(3.5) and (3.6)-(3.8), and the global formulation (2.1)-(2.3), we needs to write weak formulations. We then introduce the following spaces, for $\boldsymbol{u}_i \in H_0^1(\Omega_i)^2$

$$\begin{aligned} \mathbb{V}^1 &= \left\{\boldsymbol{w} \in (H_0^1(\Omega_1))^2, \boldsymbol{w} = \boldsymbol{u}_2 \text{ on } \Gamma_{12}\right\}\\ \mathbb{V}^2 &= \left\{\boldsymbol{w} \in (H_0^1(\Omega_2))^2, \frac{\partial \boldsymbol{w}}{\partial n_2} = -\frac{\partial \boldsymbol{u}_1}{\partial n_1} \text{ on } \Gamma_{12}\right\} \end{aligned}$$

The variational formulation of the global problem (3.1)-(3.2) is

Find $\boldsymbol{u} \in H_0^1(\Omega)^2$ such that

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u}))\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{l}(\boldsymbol{w}) = 0 \quad \forall \boldsymbol{w} \in (H_0^1(\Omega))^2, \tag{3.9}$$

where $\boldsymbol{u} = (u, v)$ and

$$\begin{aligned} \boldsymbol{a}(\boldsymbol{u}, \boldsymbol{w}) &= \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{w}\, dx\\ \boldsymbol{l}(\boldsymbol{w}) &= \int_\Omega \boldsymbol{f}(\boldsymbol{u})\boldsymbol{w}\, dx \end{aligned}$$

Setting

$$\begin{aligned} \boldsymbol{a}_i(\boldsymbol{u}_i, \boldsymbol{w}) &= \int_{\Omega_i} \nabla \boldsymbol{u}_i \boldsymbol{w}\, dx\\ \boldsymbol{l}_i(\boldsymbol{w}) &= \int_{\Omega_i} \boldsymbol{f}(\boldsymbol{u}_i)\boldsymbol{w}\, dx \end{aligned}$$

the weak formulations for multi-domain formulation is then

Find $\boldsymbol{u}_i \in \mathbb{V}_i$ such that :

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u}_i))\boldsymbol{a}_i(\boldsymbol{u}_i, \boldsymbol{w}) + \boldsymbol{l}_i(\boldsymbol{w}) = 0, \quad \forall \boldsymbol{w} \in \mathbb{V}_i, \quad i = 1, 2. \tag{3.10}$$

Theorem 3.1. *Problem (3.9) is equivalent to Problem (3.10).*

Proof. If $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ is the solution of (3.10), we have

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{w}) = \int_\Omega \nabla \boldsymbol{u} \nabla \boldsymbol{w}\, dx = \int_{\Omega_1 \cup \Omega_2} \nabla \boldsymbol{u} \nabla \boldsymbol{w}\, dx = \int_{\Omega_1} \nabla \boldsymbol{u}_1 \nabla \boldsymbol{w}_1\, dx + \int_{\Omega_2} \nabla \boldsymbol{u}_2 \nabla \boldsymbol{w}_2\, dx,$$

for all $\boldsymbol{w} \in (H_0^1(\Omega))^2$. Then, we get

$$\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{a}_1(\boldsymbol{u}_1, \boldsymbol{w}_1) + \boldsymbol{a}_2(\boldsymbol{u}_2, \boldsymbol{w}_2).$$

We can show by using the same procedure that $\boldsymbol{l}(\boldsymbol{w}) = \boldsymbol{l}_1(\boldsymbol{w}_1) + \boldsymbol{l}_2(\boldsymbol{w}_2)$. Then

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u}))\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{l}(\boldsymbol{w}) = 0$$

is equivalent to

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u}_1))\boldsymbol{a}_1(\boldsymbol{u}_1, \boldsymbol{w}_1) + \boldsymbol{l}_1(\boldsymbol{w}_1) + \boldsymbol{\kappa}(l_r(\boldsymbol{u}_2))\boldsymbol{a}_2(\boldsymbol{u}_2, \boldsymbol{w}_2) + \boldsymbol{l}_2(\boldsymbol{w}_2) = 0 \tag{3.11}$$

Let $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2) \in (H_0^1(\Omega))^2$. If we set $\boldsymbol{w}_2 = 0$ in (3.11) we get

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u}_1))\boldsymbol{a}_1(\boldsymbol{u}_1, \boldsymbol{w}_1) + \boldsymbol{l}_1(\boldsymbol{w}_1) = 0.$$

In other words, $\boldsymbol{u}_1$ is solution of (3.10) for $i = 1$. If we set $\boldsymbol{w}_1 = 0$, we get

$$\boldsymbol{\kappa}(l_r(\boldsymbol{u}_2))\boldsymbol{a}_2(\boldsymbol{u}_2, \boldsymbol{w}_2) + \boldsymbol{l}_2(\boldsymbol{w}_2) = 0$$

with the contitinuity condition on the normal derivatives on Γ_{12}. We then deduce that $\boldsymbol{u}_2$ is solution of (3.10) for $i = 2$.

If $\boldsymbol{u}_i$ $(i = 1, 2)$ is solution of (3.10), then it is easy to see that $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ solution of (3.9). □

4 Domain Decomposition Algorithm

Algorithm

For the numerical resolution of the problems ((2.1)- (2.3)), we propose the following algorithm. Starting with Given $(\boldsymbol{u}_i^0) \in (H_0^1(\Omega^\alpha))^2$ and $\lambda_0 \in H^{1/2}(\Gamma_i)$, we build a sequence of approximate solutions $\boldsymbol{u}_{i\ k\geq 0}^k \in (H_0^1(\Omega^\alpha))^2$, for $i = 1, 2$ by solving the following three-step algorithm.

Step 1.

$$\begin{cases} \text{find } \boldsymbol{u}_1^k \text{ such that:} \\ \boldsymbol{\kappa}(l_r(\boldsymbol{u}_1^{k-1}))\boldsymbol{a}_1(\boldsymbol{u}_1^k, \boldsymbol{v}) + \boldsymbol{l}_1(\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in (H_0^1(\Omega^1))^2, \\ \boldsymbol{u}_1^k = \lambda_{k-1} \quad \text{on } \Gamma_{12} \end{cases} \tag{4.1}$$

$$\begin{cases} \text{find } \boldsymbol{u}_2^k \text{ such that:} \\ \boldsymbol{\kappa}(l_r(\boldsymbol{u}_2^{k-1}))\boldsymbol{a}_2(\boldsymbol{u}_2^k, \boldsymbol{v}) + \boldsymbol{l}_2(\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in (H_0^1(\Omega^2))^2, \\ \boldsymbol{u}_2^k = \lambda_{k-1} \quad \text{on } \Gamma_{12} \end{cases} \tag{4.2}$$

Step 2.

$$\begin{cases} \text{find } \boldsymbol{w}_1^k \in (H^1(\Omega^1))^2 \text{ such that} \\ \boldsymbol{a}_1(\boldsymbol{w}_1^k, \boldsymbol{v}) = \dfrac{1}{2}(\boldsymbol{a}_1(\boldsymbol{u}_1^k, \boldsymbol{v}) - \boldsymbol{l}_1(\boldsymbol{v}) + \boldsymbol{a}_2(\boldsymbol{u}_2^k, R^2(\boldsymbol{v})) - \boldsymbol{l}_2(R^2(\boldsymbol{v}))) \\ \forall \boldsymbol{v} \in \mathbb{V}_0^1, \end{cases} \tag{4.3}$$

$$\begin{cases} \text{find } \boldsymbol{w}_2^k \in (H^1(\Omega^2))^2 \text{ such that} \\ \boldsymbol{a}_2(\boldsymbol{w}_2^k, \boldsymbol{v}) = \dfrac{1}{2}(-\boldsymbol{a}_1(\boldsymbol{u}_1^k, R^1(\boldsymbol{v})) + \boldsymbol{l}_1(R^1(\boldsymbol{v})) - \boldsymbol{a}_2(\boldsymbol{u}_2^k, \boldsymbol{v}) + \boldsymbol{l}_2(\boldsymbol{v})) \\ \forall \boldsymbol{v} \in \mathbb{V}_0^2, \end{cases} \tag{4.4}$$

where $R^\alpha : (H^{1/2}(\Gamma_i))^2 \to (H^1(\Omega_i))^2$, $i = 1, 2$ such that $\forall \varphi \in (H^{1/2}(\Gamma_i))^2$, $R^\alpha(\varphi) = \tilde{\boldsymbol{w}}_i$ with $\tilde{\boldsymbol{w}}_i$ solution of the problem

$$\begin{cases} \text{find } \boldsymbol{w}_i \in (H^1(\Omega^2))^2 \text{ such that} \\ \boldsymbol{a}_i(\tilde{\boldsymbol{w}}_i, \boldsymbol{v}) = 0, \ \forall \boldsymbol{v} \in \mathbb{V}_0^\alpha \\ \tilde{\boldsymbol{w}}_i = \varphi \text{ on } \Gamma_{12} \end{cases} \tag{4.5}$$

and $\mathbb{V}_0^\alpha = \{v \in (H^1(\Omega^\alpha))^2; \quad v = 0 \text{ on } \Gamma_i\}$.

Step 3. Update the interface

$$\lambda_k = \lambda_{k-1} - \theta(\boldsymbol{w}_1^k - \boldsymbol{w}_2^k) \quad \text{on } \Gamma_{12} \tag{4.6}$$

with $\theta \in [0, 1]$ the relaxation parameter.

Convergence

To prove the convergence of this algorithm, we need to show that the sequences of functions $\{\boldsymbol{u}_i^k\}_{k\geq 0}$ and $\{\boldsymbol{w}_i^k\}_{k\geq 0}$ converge if the sequence $\{\lambda_k\}_{k\geq 0}$ converges. We first rewrite the problem (4.1) as

$$\begin{cases} \text{find } \boldsymbol{u}_1^{\boldsymbol{l}_1} \text{ such that} \\ \boldsymbol{\kappa}(l_r(\boldsymbol{u}_1^{k-1}))\boldsymbol{a}_1(\boldsymbol{u}_1^{\boldsymbol{l}_1}, \boldsymbol{v}) + \boldsymbol{l}_1(\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in (H_0^1(\Omega^1))^2, \end{cases} \tag{4.7}$$

and

$$\begin{cases} \text{find } \boldsymbol{u}_1^\lambda \text{ such that} \\ \boldsymbol{\kappa}(l_r(\boldsymbol{u}_1^{k-1}))\boldsymbol{a}_1(\boldsymbol{u}_1^\lambda, \boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in (H_0^1(\Omega^1))^2, \\ \boldsymbol{u}_1^\lambda = \lambda_{k-1} \qquad\qquad \text{on } \Gamma_i \end{cases} \tag{4.8}$$

Indeed we have to separate the influence of the volume forces and the harmonic's behavior of the domain Ω^1 (See [30]). Then $\boldsymbol{u}_1^k = \boldsymbol{u}_1^{\boldsymbol{l}_1} + \boldsymbol{u}_1^\lambda$. Using the definition of the mapping R^i, we get $\boldsymbol{u}_1^\lambda = R^1(\lambda_{k-1})$. It follows that

$$\boldsymbol{u}_1^k = \boldsymbol{u}_1^{\boldsymbol{l}_1} + R^1(\lambda_{k-1}) \tag{4.9}$$

By doing the same for the (4.2), we obtain

$$\boldsymbol{u}_2^k = \boldsymbol{u}_2^{\boldsymbol{l}_2} + R^2(\lambda_{k-1}) \tag{4.10}$$

For (4.3)-(4.4), we have to use the definition of Steklov-Poincaré's operator and we obtain

$$\begin{cases} \boldsymbol{w}_1^k = \dfrac{1}{2} S_1^{-1}(S_1\boldsymbol{u}_1^k + S_2\boldsymbol{u}_2^k) \quad \text{on } \Gamma_i \\ \boldsymbol{w}_2^k = -\dfrac{1}{2} S_2^{-1}(S_1\boldsymbol{u}_1^k + S_2\boldsymbol{u}_2^k) \quad \text{on } \Gamma_i \end{cases} \tag{4.11}$$

where S_i, $i = 1, 2$ is the Steklov-Poincaré's operator define by $S_i : (H^{1/2}(\Gamma_i))^2 \to (H^{-1/2}(\Gamma_i))^2$, such that $\forall \mu \in (H^{1/2}(\Gamma_i))^2$, $S_i(\mu) = \dfrac{\partial \boldsymbol{w}_i}{\partial n_i}$ with $\boldsymbol{w}_i$ solution of the problem

$$\begin{cases} \text{find } \boldsymbol{w}_i \in (H^1(\Omega_i))^2 \text{ such that} \\ \boldsymbol{a}_i(\boldsymbol{w}_i, \boldsymbol{v}) = 0, \ \forall bv \in (H_0^1(\Omega_i))^2 \\ \boldsymbol{w}_i = \mu \text{ on } \Gamma_i \end{cases} \tag{4.12}$$

and S_i^{-1} the inverse operator of Steklov-Poincaré's operator.

Using the mapping R_i, $i = 1, 2$, (4.9)-(4.11) lead to

$$\begin{cases} \boldsymbol{w}_1^k = \dfrac{1}{2} R^1 S_1^{-1}(S_1\boldsymbol{u}_1^{\boldsymbol{l}_1} + S_2\boldsymbol{u}_2^{\boldsymbol{l}_2} + S_1\lambda_{k-1} + S_2\lambda_{k-1}) \\ \boldsymbol{w}_2^k = -\dfrac{1}{2} R^2 S_2^{-1}(S_1\boldsymbol{u}_1^{\boldsymbol{l}_1} + S_2\boldsymbol{u}_2^{\boldsymbol{l}_2} + S_1\lambda_{k-1} + S_2\lambda_{k-1}) \end{cases} \tag{4.13}$$

Concerining S_i and R^i, we recall the following theorems
Theorem 4.1. *The operators S_i, S_i^{-1} are linear, bounded, bijective, self-adjoint and coercive.*

Proof. See, e.g., [1]. □

Lemma 4.2. *The mapping R^i, $i = 1, 2$ is linear, bounded and bijective.*

Proof. See, e.g., [6], [28]. □

Lemma 4.3. *Assuming the existence of $\theta_{max} \in]0, 1[$ such that for all $\theta \leq \theta_{max}$, the sequence $\{\lambda_k\}_{k\geq 0}$ converges in $(H^{1/2}(\Gamma_i))^2$. Then $\{\boldsymbol{u}_i^k\}_{k\geq 0}$, $i = 1, 2$ converges in $(H^1(\Omega_i))^2$.*

Proof. We need to show that $\{\boldsymbol{u}_i^k\}_{k\geq 0}$ is a Cauchy sequence in the Banach space $(H^1(\Omega_i))^2$.

$$\begin{aligned}
\| \boldsymbol{u}_i^k - \boldsymbol{u}_i^\ell \|_1 &= \| \boldsymbol{u}_i^{\boldsymbol{l}_i} + R^i(\lambda_{k-1}) - (\boldsymbol{u}_i^{\boldsymbol{l}_i} + R^i(\lambda_{\ell-1})) \|_1 \\
&= \| R^i(\lambda_{k-1} - \lambda_{\ell-1}) \|_1 \\
&\leq \| R^i \| \| \lambda_{k-1} - \lambda_{\ell-1} \|_{1/2} \quad \text{(using the lemma 4.2)}
\end{aligned}$$

□

Theorem 4.4. *Using the same assumptions as in theorem 4.3, the sequence $(\boldsymbol{w}_i^k)_{k\geq 0}$ converges in $(H^1(\Omega^\alpha))^2$.*

Proof. We need to show that $(\boldsymbol{w}_i^k)_{k\geq 0}$ is a Cauchy sequence in the Banach $(H^1(\Omega_i))^2$.

$$\begin{aligned}
\| \boldsymbol{w}_i^k - \boldsymbol{w}_i^\ell \|_1 &= \frac{1}{2} \| R^i S_i^{-1}(S_1(\lambda_{k-1} - \lambda_{\ell-1}) + S_2(\lambda_{k-1} - \lambda_{\ell-1})) \|_1 \\
&\leq \frac{1}{2} \| R^i S_i^{-1} \| . (\| S_1 \| + \| S_2 \|) \| \lambda_{k-1} - \lambda_{\ell-1} \|_{1/2} \\
&\leq C \| \lambda_{k-1} - \lambda_{\ell-1} \|_{1/2}
\end{aligned}$$

with

$$C = \frac{1}{2} \| R^i S_i^{-1} \| (\| S_1 \| + \| S_2 \|)$$

and by using the lemmas 4.1 and 4.2. □

Now, it remains to prove the existence of θ_{max} such that the sequence $\{\lambda_k\}_{k\geq 0}$ converges for $\theta \leq \theta_{max}$. Let be $T_\theta : (H^{1/2}(\Gamma_i))^2 \to (H^{1/2}(\Gamma_i))^2$ the mapping defined by

$$T_\theta(\lambda_k) = \lambda_{k-1} - \theta(\boldsymbol{w}_1^k - \boldsymbol{w}_2^k). \tag{4.14}$$

with $\theta \in]0, 1[$. To apply the Banach fixe point to T_θ, let introduce the operator M define by $M = S_1^{-1} + S_2^{-1}$ and M^{-1} the inverse operator of M. We are endow the space $(H^{1/2}(\Gamma_i))^2$ with the scalar product

$$< \varphi, \psi >_{M^{-1}} = < M^{-1}\varphi, \psi > = < \varphi, M^{-1}\psi >, \ \forall \varphi, \psi \in H^{1/2}(\Gamma_i))^2$$

and the norm

$$\| \varphi \|_{M^{-1}} = (< M^{-1}\varphi, \varphi >)^{1/2}.$$

We are the following technical lemmas.

Lemma 4.5. *The operator M is bijective, self-adjoint and coercitive. Moreover there exists a constant $c > 0$ such taht*

$$\| M\mu \|_{1/2} \leq c||\mu||_{-1/2}, \quad \forall \mu \in (H^{-1/2}(\Gamma_i))^2$$

Lemma 4.6. *The norms $\| \cdot \|_{M^{-1}}$ and $\| \cdot \|_{1/2}$ are equivalent.*

Proof. The operator $S_i^{-1}, i = 1, 2,$ is bijective, sel-autoadjoint and coercive (see Theorem 4.1). Moreover

$$\begin{gathered} S_i^{-1} : (H^{-\frac{1}{2}}(\Gamma_c))^2 \longrightarrow (H^{\frac{1}{2}}(\Gamma_c))^2 \\ \mu \longmapsto S_i^{-1}\mu = \frac{\partial \boldsymbol{w}_i}{\partial n_i}, \end{gathered} \tag{4.15}$$

with $\boldsymbol{w}_i$ is the unique solution of the Neumann's problem

$$\boldsymbol{a}_i(\boldsymbol{w}_i, \boldsymbol{v}) =< \mu, \frac{\partial \boldsymbol{w}_i}{\partial n_i} >_{-\frac{1}{2},\frac{1}{2}} \forall \boldsymbol{v} \in (H^1(\Omega^\alpha))^2, \tag{4.16}$$

where $< \cdot, \cdot >_{-\frac{1}{2},\frac{1}{2}}$ is the duality product between the spaces $H^{\frac{1}{2}}(\Gamma_c)$ and his dual $H^{-\frac{1}{2}}(\Gamma_c)$.

We have

$$||S_i^{-1}\mu||_{\frac{1}{2},\Gamma_c} = ||\frac{\partial \boldsymbol{w}_i}{\partial n_i}||_{\frac{1}{2},\Gamma_c}$$

Using trace theorem, we get

$$||\frac{\partial \boldsymbol{w}_i}{\partial n_i}||_{\frac{1}{2},\Gamma_c} \leq C||\boldsymbol{w}^i||_i.$$

From (4.16), we obtain

$$||\boldsymbol{w}^i||_i^2 \leq C_2||\mu||_{-\frac{1}{2},\Gamma_c}||\frac{\partial \boldsymbol{w}_i}{\partial n_i}||_{\frac{1}{2},\Gamma_c} \leq C_3||\mu||_{-\frac{1}{2},\Gamma_c}||\boldsymbol{w}^i||_i.$$

So

$$||S_i^{-1}\mu||_{\frac{1}{2},\Gamma_c} \leq C_1||\mu||_{-\frac{1}{2},\Gamma_c}.$$

That is proove the Lemma 4.5.
The operator M^{-1} is clairely bijective, self-autoadjoint, coercive as the operator M. The Lemma 4.6 is the consequence of the continuty and coervicity of M^{-1}. □

We have the following existence theorem.

Theorem 4.7. *There exists $\theta_{max} \in]0, 1[$ such that $\forall \theta \leq \theta_{max}$, the mapping T_θ is a contraction.*

Proof. Let λ and μ such that we have (dropping the iterations index k)

$$\begin{aligned} T_\theta(\lambda) &= \lambda - \theta(\boldsymbol{w}_1 - \boldsymbol{w}_2) \\ T_\theta(\mu) &= \mu - \theta(\tilde{\boldsymbol{w}}_1 - \tilde{\boldsymbol{w}}_2) \end{aligned}$$

Using (4.11), we obtain that

$$\begin{cases} T_\theta(\lambda) = \lambda - \dfrac{\theta}{2}(S_1^{-1} + S_2^{-1})(S_1U_{L^1}^1 + S_2U_{L^2}^2 + (S_1 + S_2)\lambda) \\ T_\theta(\beta) = \beta - \dfrac{\theta}{2}(S_1^{-1} + S_2^{-1})(S_1U_{L^1}^1 + S_2U_{L^2}^2 + (S_1 + S_2)\beta) \end{cases} \tag{4.17}$$

From (4.17) and the lemmas 4.5 and 4.6 there follows

$$||T_\theta(\lambda) - T_\theta(\beta)||_{M^{-1}}^2 = ||\lambda - \beta - \frac{\theta}{2}(S_1^{-1} + S_2^{-1})(S_1 + S_2)(\lambda - \beta)||_{M^{-1}}^2$$

$$\leq ||\lambda - \beta||_{M^{-1}}^2 - \theta < (S_1 + S_2)(\lambda - \beta), \lambda - \beta > + \frac{\theta^2}{4}||(S_1^{-1} + S_2^{-1})(S_1 + S_2)(\lambda - \beta)||_{M^{-1}}^2$$

$$\leq (1 - C_{S_1+S_2}\theta + |||S_1 + S_2|||.|||S_1^{-1} + S_2^{-1}|||\frac{\theta^2}{4})||\lambda - \beta||_{M^{-1}}^2$$

where $C_{S_1+S_2}$ is the constante of coercitivity of the operator $S_1 + S_2$. Then T_θ is a contraction if

$$\theta \leq \frac{4C_{S_1+S_2}}{|||S_1 + S_2|||.|||S_1^{-1} + S_2^{-1}|||} = \theta_{max}$$

□

5 Numerical experiments

In this section, we shall compare performancies of domain decomposition method and the global resolution. All computations are performed in Matlab 7.

Let us consider the problem ((2.1)-(2.3)) in the two dimensional space Ω defined by $\Omega = (0,1)\times(0,1)$. We choose to work with uniform mesh for the simplicity of simulated models. We take $f(s) - \alpha(s) = \beta s(q-s)\,\forall s \in \Omega$ with $\beta = 1$ and $q = 10$. For the beginning of the algorithm, we need to have u_0 and v_0. So, we supposed that $\forall (x,y) \in \Omega$

$$\begin{aligned} u_0(x,y) = &\quad 0.01 \\ v_0(x,y) = &\quad 0.01\cos(x-x_0)\cos(y-y_0) \end{aligned}$$

where $x_0 = 0.25$ and $y_0 = 0.25$.

For the nonlocal term $a(l_r(u))$, we defined l_r by

$$l_r(u(X)) = \frac{1}{meas(\Omega)}\int_0^1 u(X)dX$$

with $X = (x,y)$.

To compare the behaviour of Domain Decomposition Method and the global resolution, we have to do two tests: in the first numerical test, we take a constant i.e $a(\varphi) = 1, \forall\varphi$ and in the second test $a(\varphi) = 0.1\varphi$. The table gives the number of iterations before the convergence of the algorithm.

Case 1: $a(\varphi) = 1$

The behaviour of the Domain Decomposition Method is seen in Table 1. The results demonstrates the numerical scalability and robustness for various values of the relaxation parameter θ. The figures 1 shown that with the constant diffusion the behaviour of the population that the density is given by $u(x,y)$ are symmetric and similar in each subdomain. The axis $y = 0.5$ is clearly the axis of symmetrize. It is the same thing for the population that the density is given by $v(x,y)$. For the global method, the Table 2 shown that the global resolution is less performante than the Domain Decomposition Method. Indeed the number of iteration for the convergence of the global resolution varies with the step of discretization. Global resolution is not numerical scalability.

N_v/N_e	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.7$	$\theta = 0.9$
9/8	11	11	11	11	11	11
16/18	11	11	11	11	11	11
25/32	11	11	11	11	11	11
81/128	11	11	11	11	11	11
625/1152	11	11	11	11	11	11
2500/4802	11	11	11	11	11	11
5625/10952	11	11	11	11	11	11

Table 1: Results of the Domain Decomposition Method in the case of $a(\varphi) = 1$

where N_v, N_e are the number of vertices and the number of elements on each subdomain respectively.

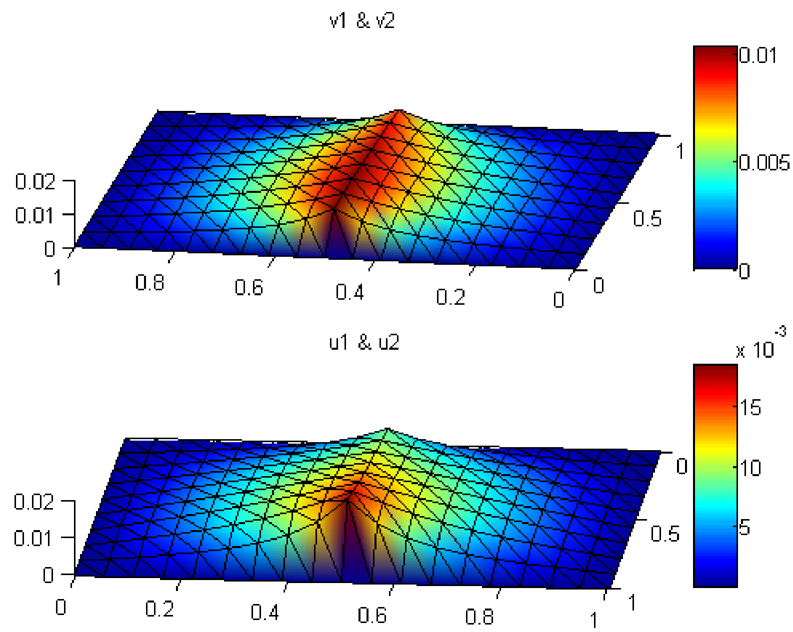

Figure 1: Evolution of the population of u(x,y) and v(x,y) in the two subdomains for the case with $a(\varphi) = 1$

The figures 1 shown that with the constant diffusion the behaviour of the population that the density is given by $u(x, y)$ are symmetric and similar in each subdomain. The axis $y = 0.5$ is clearly the axis of symmetrize. It is the same thing for the population that the density is given by $v(x, y)$.

N_v/N_e	$iterations$
9/8	7
16/18	12
25/32	19
81/128	66
625/1152	1436
2500/4802	1049
5625/10952	1028

Table 2: Global algorithm in the case of $a(\varphi) = 1$

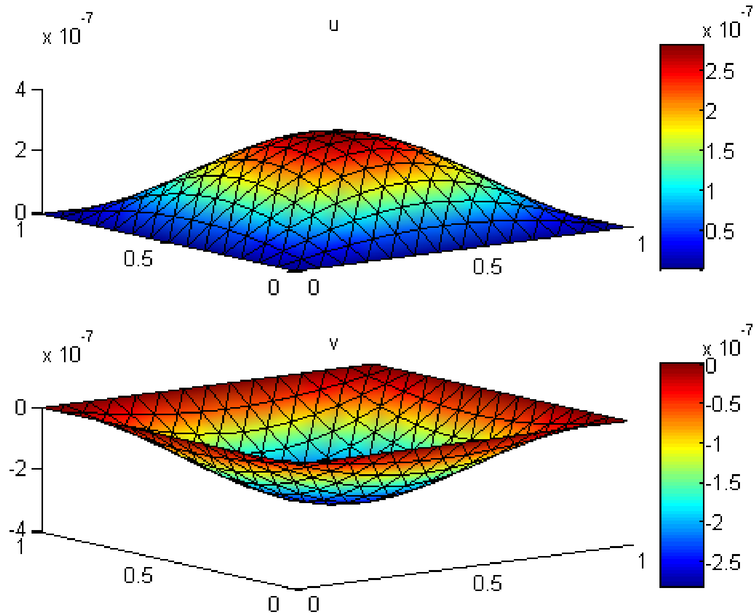

Figure 2: Evolution of u(x,y) and v(x,y) for case with constant diffusion

The figure 2 obtain with the global method in the case of the constant diffusion confirms that the behaviour of the population that the density is given by $u(x, y)$ and $v(x, y)$ are symmetric and similar in each subdomain.

Case 2: $a(\varphi) = 0.1\varphi$
The numerical results shown that the behaviour of the population $u(x, y)$ and $v(x, y)$ are symmetric and similar in each subdomain. If we compare to the numerical results with constant diffusion, we can see that the axis $y = 0.5$ is not the axis of symmetrize in the case of the nonlocal diffusion. The density of the population $u(x, y)$ and $v(x, y)$ are more or less important in the center of each subdomain. In the nonlocal diffusion case, the results confirms the numerical scalability and robustness of the Domain Decomposition Method. Indeed with the presence of the nonlocal term, the matrix are badly scaled but the Domain Decomposition Method needs the same number of iterations for the convergence than the case of constant diffusion. The global resolution needs a preconditionner in the case of nonlocal diffusion.

N_v/N_e	$\theta = 0.1$	$\theta = 0.2$	$\theta = 0.3$	$\theta = 0.4$	$\theta = 0.7$	$\theta = 0.9$
9/8	11	11	11	11	11	11
16/18	11	11	11	11	11	11
25/32	11	11	11	11	11	11
81/128	11	11	11	11	11	11
625/1152	11	11	11	11	11	11
2500/4802	11	11	11	11	11	11
5625/10952	11	11	11	11	11	11

Table 3: Results for Domain Decomposition Method in the case of $a(\varphi) = 0.1\varphi$

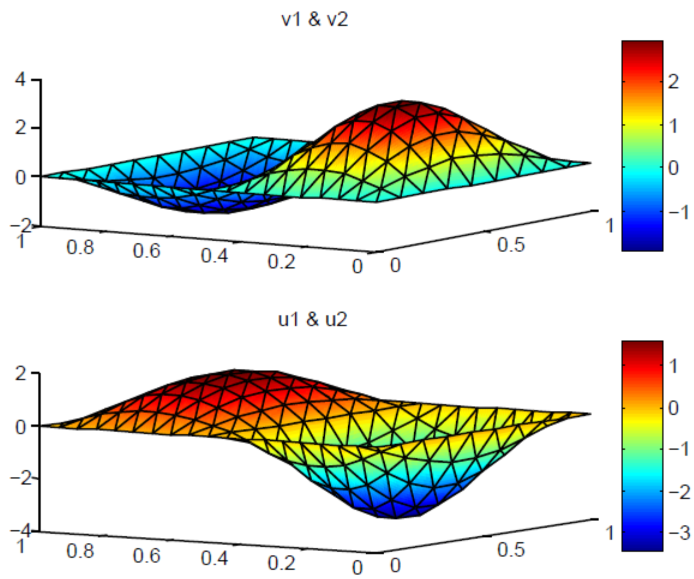

Figure 3: Evolution of u(x,y)and v(x,y) in the first subdomains for case with $a(\varphi) = 0.1\varphi$

The numerical results shown that the behaviour of the population $u(x, y)$ and $v(x, y)$ are symmetric and similar in each subdomain. If we compare to the numerical results with constant diffusion, we can see that the axis $y = 0.5$ is not the axis of symmetrize in the case of the nonlocal diffusion. The density of the population $u(x, y)$ and $v(x, y)$ are more or less important in the center of each subdomain. In the nonlocal diffusion case, the results confirms the numerical scalability and robustness of the Domain Decomposition Method. Indeed with the presence of the nonlocal term, the matrix are badly scaled but the Domain Decomposition Method needs the same number of iterations for the convergence than the case of constant diffusion. The global resolution needs a preconditionner in the case of nonlocal diffusion.

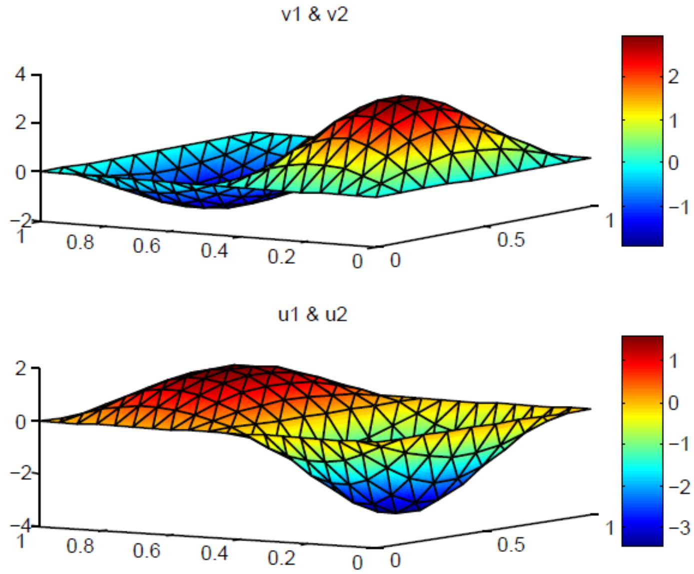

Figure 4: Evolution of u(x,y)and v(x,y) in the first subdomains for case with nonlocal diffusion

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