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Remarks on the Peripheral Spectrum of Operators in Hilbert Spaces

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Abstract - *In this paper, we investigate the peripheral spectrum of some operators, which is a boundary-type subset of the spectrum of a bounded linear operator in a separable complex Hilbert space. We characterize this set for operators in some equivalence relations and give conditions under which equivalent classes of operators have the same peripheral spectrum.*

Keywords - *Resolvent, Spectrum, Spectral radius, Peripheral spectrum, Peripheral point spectrum*

1. Introduction

Let H denote a Hilbert space and $B(H)$ denote the Banach algebra of bounded linear operators. If $T \in B(H)$, then T^* denotes the adjoint of T, while $Ker(T), Ran(T), \overline{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H, respectively. We denote by $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T$ is not invertible$\}$, $\|T\| = \{\|Tx\| : x \in H, \|x\| \leq 1\}$ and $W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}$, the spectrum, norm and numerical range of T, respectively. The elements contained in the spectrum of T are called the spectral values of T, and may or may not be eigenvalues of T. The resolvent set $\rho(T)$ is defined by $\rho(T) = \mathbb{C} \setminus \sigma(T)$. By $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ we denote the spectral radius of T. This is the radius of the smallest circle in $\mathbb{C}$, centred at the origin and contains $\sigma(T)$.

By $|T|$ we denote the absolute value of T which is the square root of the positive operator $P = \sqrt{T^*T}$.

Two operators $A \in B(H)$ and $B \in B(K)$ are said to be similar (denoted $A \sim B$) if there exists an invertible operator $N \in B(H,K)$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are unitarily equivalent (denoted by $A \cong B$)if there exists a unitary operator $U \in B_+(H,K)$ (Banach algebra of all invertible operators in $B(H)$) such that $UA = BU$ (i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$. Two operators $A \in B(H)$ and $B \in B(K)$ are said to be metrically equivalent (denoted by $A \sim_m B$) if $\|Ax\| = \|Bx\|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in H$). Equivalently, A and B are metrically equivalent if $A^*A = B^*B$ (i.e. $\langle Ax, Ax\rangle = \langle Bx, Bx\rangle$ for all $x \in H$ (see [21]).

Two operators $T \in B(H)$ and $S \in B(K)$ are said to be intertwined by an operator $X \in B(H,K)$ if $SA = AT$. An operator $X \in B(H,K)$ is a quasi-affinity if it is injective with dense range. An operator $B \in B(H)$ is a quasiaffine transform of an operator $A \in B(K)$ if there exists a quasiaffinity $X \in B(H,K)$ such that $AX = XB$. Two operators

$A \in B(H)$ and $B \in B(K)$ are quasisimilar if there exist quasiaffinities $X \in B(H,K)$ and $Y \in B(K,H)$ such that $XA = BX$ and $AY = YB$.

Clearly, unitary equivalence, similarity, quasisimilarity and metric equivalence are equivalence relations on $B(H)$.

An operator $T \in B(H)$ is said to be positive if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. An operator $T \in B(H)$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$ and subnormal if there exists a normal operator N acting on a Hilbert space K containing H such that H is N-invariant (i.e., $NH \subseteq H$) and that $T = N|_H$, the restriction of N to H. An operator $T \in B(H)$ is said to be normaloid if $r(T) = \|T\|$.

Let $T \in B(H)$. Then the peripheral spectrum of T is defined as $\sigma_{per}(T) = \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$ (see [26], [20], [7], [19], [5]). This consists of all spectral values with maximal modulus. That is, it is the set of points in the spectrum which have modulus equal to the spectral radius of T, equivalently, the non-empty part of the spectrum of T that lies on the circle centred at the origin with radius $r(T)$. Clearly, $\sigma_{per}(T) = \sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = r(T)\}$. The elements of $\sigma_{per}(T)$ are called the peripheral spectral values of T. The set $\sigma_{per-point}(T) = \sigma_{per}(T) \cap \sigma_p(T)$, where $\sigma_p(T)$ denotes the point spectrum of T is called the peripheral point spectrum of T, and its elements are called peripheral eigenvalues of T (see [12], [4], [25], p. 34).

Note that the peripheral spectrum of $T \in B(H)$ is compact and non-empty (in complex Hilbert spaces) (from the definition of spectral radius $r(T)$ (see [20]) and that $\sigma_{per}(T^*) = (\sigma_{per}(T))^*$, where by definition $(\sigma(S))^* = \{\overline{w} : w \in \sigma(S)\}$ for any $S \in B(H)$. We note that the peripheral point spectrum $\sigma_{per-point}(T)$ could be an empty set. $T \in B(H)$ is said to be primitive if $\sigma_{per}(T) = \{r(T)\}$ and imprimitive if the peripheral spectrum consists of more than one point (see [1]). Clearly every quasinilpotent operator $T \in B(H)$ is primitive, since $\sigma_{per}(T) = \{r(T)\} = \{0\} = \sigma(T)$. Also, every scalar operator $T = \alpha I$ is primitive since $\sigma_{per}(T) = \sigma(T) = \{\alpha\}$.

We define $r_{per}(T) = \sup\{|\lambda| : \lambda \in \sigma_{per}(T)\}$ the peripheral spectral radius of $T \in B(H)$.

2. Basic properties of the Peripheral Spectrum

In this section we explore some basic results about the peripheral spectrum of operators.

Theorem 2.1 Let H be a complex Hilbert space and let $T \in B(H)$. Then $\sigma_{per}(T)$ is a well-defined non-empty subset of the spectrum of T.

To prove Theorem 2.1, we need the following results.

Proposition 2.2 For any $T \in B(H)$, we have $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.

Note that a consequence of Proposition 2.2 is that $r(T) \leq \|T\|$ for any $T \in B(H)$ see [23], [18], [17]).

Proposition 2.3 Let $T \in B(H)$. Then there exists $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$ (see [11]).

Proof. This follows from the fact that the spectrum $\sigma(T)$ of T is non-empty.

Proof [Theorem 2.1] This follows immediately from Proposition 2.2 and Proposition 2.3.

Remark. We note that for any $T \in B(H)$, the peripheral spectrum $\sigma_{per}(T)$ is a subset of the topological boundary $\partial(\sigma(T))$ of the spectrum of T.

Example 2.4 Consider the linear operator $T \in B(\ell^2(\mathbb{N}))$ given by the infinite diagonal matrix

$$T = diag(1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...).$$

Then $\sigma_{per}(T) = \{1\} \subseteq \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} = \sigma(T)$.

Proposition 2.5 If $T \in B(H)$, then $\sigma_{per}(T) \neq \varnothing$.

Proof. Since $\sigma(T) \neq \varnothing$, by Proposition 2.3, there exists $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$ and so $\lambda \in \sigma_{per}(T)$ ([17], Corollary 6.23(b)). This shows that $\sigma_{per}(T) \neq \varnothing$, which proves the claim.

Note that $\sigma(T)$ is a closed subset of the disk $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$. That is $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$.

Proposition 2.6 Let $T \in B(H)$ and let $\lambda \in \mathbb{C}$. If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$.

Proof. Indeed, suppose such a λ exists with $|\lambda| > \|T\|$ and that $\lambda \in \sigma(T)$. Then the formula $T - \lambda I = (-\lambda)(I - \frac{T}{\lambda})$ together with the fact that $\left\|\frac{T}{\lambda}\right\| < 1$ implies that the operator $T - \lambda I$ is invertible. This means that $\lambda \notin \sigma(T)$, which is equivalent to saying that $\lambda \in \rho(T)$. This establishes the claim.

Remark. We note that Proposition 2.6 also follows immediately from the fact if $\lambda \in \sigma(T)$ then $|\lambda| \leq \|T\|$.

We note that for any $T \in B(H)$, the spectral radius $r(T)$ and the peripheral radius $r_{per}(T)$ of an operator T are equal. If $|\lambda| = r(T)$ for all $\lambda \in \sigma(T)$, then we say that T has a pure peripheral spectrum. The following results shows that any unitary operator T has a pure peripheral spectrum.

Theorem 2.7 If $T \in B(H)$ is unitary, then $\sigma_{per}(T) = \sigma(T)$.

Proof. Since $T \in B(H)$ is unitary, we have $r(T) = 1$. Thus $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \{\lambda \in \mathbb{C} : |\lambda| = r(T)\} = \sigma_{per}(T)$. Equality follows from the fact that $\sigma_{per}(T) \subseteq \sigma(T)$ for any $T \in B(H)$.

Remark. The Gelfand-Beurling formula shows that the spectral radius of an operator $T \in B(H)$ is such that $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ (see [16], pg. 6, [2]) and obviously $r(T) = r(T^*)$. We note that if $r(T) = 0$ then $\sigma(T) = \sigma_p(T) = \{0\}$.

Recall that we have noted in Section 1 that the peripheral point spectrum of an operator can be empty. The following example illustrates this. H

Example 2.8 Let R and L be the right and left shift operators on $H = \ell^2(\mathbb{N})$. Then $r(R) = r(L) = 1, \|R\| = 1$, and $\|L\| = 1$. Clearly $\sigma(R) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ and $\sigma(L) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$. Also,
$\sigma_p(R) = \emptyset, \sigma_p(L) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}, \sigma_c(L) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \sigma_{per}(L)$,

where $\sigma_c(L)$ denotes the continuous spectrum of L.

From this information we conclude that $\sigma_{per-point}(L) = \emptyset$ and $\sigma_{per-point}(R) = \emptyset$.

3. Peripheral Spectrum of some classes of operators

Recall that $T \in B(H)$ is compact if it maps the closed unit ball $B_H(0,1) = \{x \in H : \|x\| \le 1\}$ in H onto a relatively compact subset of H. Equivalently, $T \in B(H)$ is compact if $\overline{T(B_H(0,1))}$ is compact. This definition can be extended to any bounded subset M of H. Equivalently, $T \in B(H)$ is compact if for every bounded sequence $\{x_n\}$ in H, the sequence $\{Tx_n\}$ contains a convergent subsequence.

The structure of the spectrum of compact self-adjoint operators is very simple and so the spectral radius of such operators is always attained.

Theorem 3.1 Let $T \in B(H)$ be a compact operator. If $\lambda \ne 0$, then $\lambda \in \rho(T)$ or $\lambda \in \sigma_p(T)$. Moreover, $\sigma(T)$ is at most countable and 0 is its only possible limit point.

Theorem 3.2 If $T \in B(H)$ is compact, then $r(T) \in \{|\lambda| : \lambda \in \sigma(T)\}$.

Proof. This follows from the compactness of T.

Corollary 3.3 (Frobenius-Perron Theorem) For any positive $T \in B(H)$, $r(T) \in \sigma(T)$.

Proof. T positive and $\lambda \in \sigma(T)$ implies that $\lambda \ge 0$ and so $|\lambda| = \lambda$ (see also [10]). Therefore

$r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}=\sup\{\lambda:\lambda\in\sigma(T)\}\in\sigma(T)$. This proves the claim.

Theorem 3.4 If $A\in B(H)$ is a compact positive operator with $r(A)>0$, then $r(A)\in\sigma_p(A)$.

Proof. Since A is compact and self-adjoint, there exists an orthonormal basis $\{e_1,e_2,...\}$ consisting of eigenvectors of $Ae_i=\lambda_i e_i$, where $\lambda_i\in\mathbb{R}$. Thus either $\sigma(A)=\{0\}$ or $\sigma(A)=\{0,\lambda_1,\lambda_2,...\}$ with $\lim_{n\to\infty}\lambda_n=0$. Since A is positive and $r(A)>0$, we have $|\lambda_i|=\lambda_i>0$ and so $r(A)=\sup\{\lambda_i:\lambda_i\in\sigma(A),i=1,2,...\}\in\sigma_p(A)$. This proves the claim.

Lemma 3.5 If $T\in B(H)$ is a compact self-adjoint operator then $\|T\|$ or $-\|T\|$ is an eigenvalue of A.

Lemma 3.5 shows the existence of a non-zero eigenvalue for any non-zero linear compact and self-adjoint operator.

Corollary 3.6 For every compact self-adjoint operator T on a Hilbert space H, we have $\sigma_p(T)\neq\emptyset$.

Proof. This follows from Lemma 3.5.

Theorem 3.7 If T is a compact positive operator then $\|T\|\in\sigma_{per-point}(T)$.

Proof. T compact and positive together with Lemma 3.5 and the fact that T is normaloid (that is $r(T)=\|T\|$ (see [15]) implies that $\|T\|\in\sigma_{per-point}(T)$.

Note that in Theorem 3.7, the positivity of T is important. For instance, if $T=diag(-1,-1)$, then $\|T\|=1\notin\{-1,-1\}=\sigma_{per-point}(T)$.

4. Operator Equivalence and Peripheral Spectrum

In this section, we study the relationship of the peripheral spectrum for Hilbert space operators in some equivalence relations. These relations include unitary equivalence, similarity, quasisimilarity and metric equivalence.

Remark. Note that, if a map between operators preserves the spectrum, then it preserves the peripheral spectrum. Unitary equivalence and similarity of operators preserve the spectrum of operators. This means that also the peripheral spectrum is invariant under unitary equivalence and similarity. However, equality of peripheral spectrum does not in general imply equality of spectrum or similarity or unitary equivalence of operators (see [26]).

Example 4.1 Let $S=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $T=\begin{bmatrix}1&0\\0&0\end{bmatrix}$. Clearly, $\sigma_{per}(S)=\sigma_{per}(T)$ but $\sigma(S)\neq\sigma(T)$.

Example 4.1 shows that it is possible for operators acting on the same Hilbert space to have different spectrum but equal peripheral spectrum.

Remark. We note that unitary equivalence implies similarity and similarity implies quasisimilarity but the converse is generally not true.

We will give conditions under which classes of operators have the same peripheral spectrum with respect to some operator equivalence relations.

Theorem 4.2 Similarity preserves the spectral radius of operators in Hilbert spaces.

Proof Let $A, B \in B(H)$ and suppose $B = S^{-1}AS$, where S is an invertible operator. Then

$$r(B) = r(S^{-1}AS) = \sup\{|\lambda| : \lambda \in \sigma(S^{-1}AS)\} = \sup\{|\lambda| : \lambda \in \sigma(A)\} = r(A).$$

This proves the claim.

Corollary 4.3 Unitary equivalence preserves the spectral radius of operators.

Theorem 4.4 If $A, B \in B(H)$ are similar, then $\sigma_{per}(A) = \sigma_{per}(B)$.

Proof. Similarity of A and B implies that $\sigma(A) = \sigma(B)$ and hence by definition $r(A) = r(B)$. This also follows from Theorem 4.2. Thus

$$\sigma_{per}(A) = \{\lambda \in \sigma(A) : |\lambda| = r(A)\} = \{\lambda \in \sigma(B) : |\lambda| = r(B)\} = \sigma_{per}(B).$$

Remarks. Note that metric equivalence of operators need not preserve the spectrum of operators (see [21], Proposition 2.16) and hence, need not preserve the peripheral spectrum of operators. On the other hand, equality of spectra or peripheral spectra need not imply metric equivalence of operators. We give conditions under which metric equivalence preserves peripheral spectrum of operators.

Theorem 4.5 If $S, T \in B(H)$ are metrically equivalent then $\sigma_{per}(|S|) = \sigma_{per}(|T|)$.

Proof. This follows from the fact that metric equivalence of T and S implies that $|S| = |T|$ ([21], Theorem 2.14).

It was proved in [21] that metrically equivalent operators have equal spectral radii. However, equality of spectral radii does not necessarily imply equality of spectra and also equality of peripheral spectrum.

Theorem 4.6 If $S, T \in B(H)$ are metrically equivalent positive operators, then $\sigma_{per}(S) = \sigma_{per}(T)$.

Proof. Metric equivalence of $S, T \in B(H)$ implies that $\sigma_{per}(|S|) = \sigma_{per}(|T|)$ by Theorem 4.5. Positivity of S and T is equivalent to $|S| = S$ and $|T| = T$. Therefore $\sigma_{per}(S) = \sigma_{per}(|S|) = \sigma_{per}(|T|) = \sigma_{per}(T)$ as claimed.

Remark. Note that the positivity of the operators in Theorem 4.6 is essential and cannot be dropped. To see this, consider the operators $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. A simple calculation shows that T and S are metrically equivalent but $\sigma_{per}(S) = \{1\} \neq \{-1, 1\} = \sigma_{per}(T)$.

Corollary 4.7 If $A, B \in B(H)$ are metrically equivalent operators, then $\sigma(|A|) \cap \sigma(|B|) \neq \emptyset$.

Proof. By ([21], Theorem 2.14) metric equivalence of A and B implies that $|A| = |B|$ and so $\sigma(|A|) = \sigma(|B|)$ and so $\sigma(|A|) \cap \sigma(|B|) = \sigma(|A|)$, which cannot be empty.

Remark. We note also that equality of spectrum which implies equality of peripheral spectrum need not imply metric equivalence. To see this, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Clearly A and B have equal spectrum, equal spectral radii and equal peripheral spectra but are not metrically equivalent operators.

Theorem 4.8 If $S, T \in B(H)$ are metrically equivalent compact operators, then $\sigma_{per}(S) = \sigma_{per}(T)$.

Remark. Note that compactness of the operators in Theorem 4.8 is essential and cannot be dropped. To see this, consider the unilateral shift U and the identity operator I acting on $H = \ell^2(\mathbb{N})$. Clearly, these operators are metrically equivalent but they are both not compact and so the conclusion that $\sigma_{per}(U) = \sigma_{per}(I)$ is false. In this example, note that $\sigma_{per}(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\} \supset \{1\} = \sigma_{per}(I)$.

Remark. It is known that quasisimilar operators need not have equal spectra ([6], [14]) and hence need not have equal peripheral spectrum. We give conditions under which some classes of quasisimilar operators have equal peripheral spectrum.

Theorem 4.9 If $S, T \in B(H)$ are quasisimilar normal operators, then $\sigma_{per}(S) = \sigma_{per}(T)$.

Proof. By the Fuglede-Putnam commutativity theorem ([8], Lemma 4.1) quasisimilar normal operators S and T are unitarily equivalent. This means that $\sigma(S) = \sigma(T)$ and therefore $\sigma_{per}(S) = \sigma_{per}(T)$.

In 1975 Radjabalipour [22] observed that every subnormal operator is hyponormal. It was shown in ([13], [14]) that quasisimilar hyponormal operators need not be unitarily equivalent and that they need not be similar (see [6]). However, Clary ([6], Theorem 2) has proved that quasisimilar hyponormal operators have equal spectra. Using this fact, we state the following result.

Theorem 4.10 Quasisimilar hyponormal operators have equal peripheral spectrum.

Proof. By ([6], Theorem 2) quasisimilar hyponormal operators have equal spectrum and hence have equal peripheral spectrum.

Let $T \in B(H)$. Define $\sigma_0(T) = \{\lambda \in \sigma(T) : |\lambda| < r(T)\}$, called the interior spectrum of T. It is shown (see [12]) that $\sigma(T) = \sigma_{per}(T) \cup \sigma_0(T)$. We note that one of these parts (and not both) may be empty. However, since by Theorem 2.1 the peripheral spectrum is never empty (see also [20]), we conclude that it is only the interior spectrum $\sigma_0(T)$ that can be empty. For instance, $\sigma_0(I) = \varnothing$, where I is the identity operator.

Remark. The number of elements in the peripheral spectrum of T is called the index of imprimitivity of T (see [10], [1]) and is denoted by $I_m(T)$. Since unitary equivalence and similarity preserve peripheral spectrum, they also preserve the index of imprimitivity of operators. However, metric equivalence need not preserve the index of imprimitivity. For instance, the operators $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ are metrically equivalent but $I_m(A) = 1 \neq 2 = I_m(B)$.

Let $A \in B(H)$. Define by $S(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \in [-\infty, \infty]$, the spectral bound of A. In finite dimensional Hilbert spaces, the spectral bound $S(A)$ of A is the largest real part of an eigenvalue of A and in infinite dimensional Hilbert spaces, the spectral bound $S(A)$ of A turns out to be the largest real part of a spectral value of A.

We note that $S(A)$ may or may not belong to the spectrum of A.

Theorem 4.11 Let $A \in B(H)$. Then $S(A) \leq r(A)$.

Proof. For any $\lambda \in \sigma(T)$, we have that $\lambda \leq |\lambda|$ and that $\operatorname{Re}\lambda \leq |\lambda|$. Therefore, by definition

$$S(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} \leq \sup\{|\lambda| : \lambda \in \sigma(A)\} = r(A).$$

This proves the claim.

Theorem 4.12 If $S(A) \in \mathbb{R}$, then the peripheral spectrum of A is given by

$$\sigma_{per}(A) = \sigma(A) \cap \{S(A) + i\mathbb{R}\}.$$

Example. Consider the diagonal matrix $A = \begin{bmatrix} 1+i & 0 \\ 0 & 2-i \end{bmatrix}$ for an operator acting on $H = \mathbb{C}^2$. Clearly, $\sigma(A) = \{1+i, 2-i\}$. A simple computation shows that $S(A) = 2 \leq \sqrt{5} = r(A)$. Note also by Theorem 4.12 that $\sigma_{per}(A) = \{1+i, 2-i\} \cap \{S(A) + i\mathbb{R}\} = \{1+i, 2-i\} \cap \{2 + i\mathbb{R}\} = \{2-i\}$..

Note that in this example, $S(A) \in \sigma(A)$.

Lemma 4.13 If $A \in B(H)$ is a positive operator, then $S(A) \in \sigma(A)$.

Proof. Follows from Corollary 3.3.

Remark. We note that the converse of Lemma 4.13 need not be true in general.

Theorem 4.14 Let $A \in B(H)$ be a positive operator. Then $r(A) = S(A)$.

Proof. Recall that any positive operator is self-adjoint and hence the spectral values are real numbers. Positivity of A implies that $\operatorname{Re}\lambda = \lambda = |\lambda| \geq 0$ and so

$$r(A) \ = \ sup\{|\lambda| : \lambda \in \sigma(A)\} = \ sup\{\lambda : \lambda \in \sigma(A)\} = \ sup\{Re \ \lambda : \lambda \in \sigma(A)\} = S(A).$$

A number $\lambda \in \mathbb{C}$ is called a dominant spectral value of an operator A if $\lambda \in \sigma(A)$ and if $\operatorname{Re}\mu < \operatorname{Re}\lambda$ for all other spectral values μ of A.

Note that if λ is a dominant spectral value of A, then $\operatorname{Re}\lambda = S(A)$. However, the converse implication is, in general, false. To see this, let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\sigma(A) = \{-i, i\}$, $r(A) = 1$, $\operatorname{Re}\lambda = S(A) = 0$, but 0 is not a dominant spectral value of A.

Indeed, if A is positive then $r(A)$ is a dominant eigenvalue of A. That is, it is strictly larger than the modulus of any other eigenvalue λ of A, with $\lambda \neq r(A)$.

5. Conclusion

Finding the peripheral spectrum of an operator T in infinite or finite Hilbert spaces yields a technique for estimating for the "top" spectral value or eigenvalue of T, respectively. The peripheral spectrum is important for the asymptotic behaviour of the iterates T^n of T for large $n \in \mathbb{N}$, which is applicable in Markov and ergodic analysis. It is useful in studying the asymptotic behaviour of positive $C_0 -$ semigroups $(e^{tT})_{t>0}$ -here researchers gather information about the spectrum $\sigma(T)$ of T, particularly those spectral values with maximal real part and relate them to the asymptotic behaviour of the semigroup $(e^{tT})_{t>0}$. Information about the spectrum of T can be used to obtain information about the long-term behaviour of the semigroup $(e^{tT})_{t>0}$.

Convergence speed of successive approximations to solutions of linear operator equations are determined by certain properties of the peripheral spectrum of the operator involved. This convergence speed is characterized by the growth of the (Fredholm) resolvent when approaching the peripheral spectrum.

The peripheral point spectrum is important for the asymptotic behaviour of the iterates of T (see [3], [4], [9]). In finite dimensional Hilbert spaces, the spectral bounds of positive operators and their relation to the spectral radius, the peripheral spectrum, the peripheral radius are crucial mathematical threshold parameters in population models that are formulated as systems of ordinary differential equations, where the sign of the spectral bound of the operator matrix determines whether, at low density, the population becomes extinct or grows to safety (see [24]).

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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