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Original Article

Study of W_4-Curvature Tensor in Para-Kenmotsu Manifolds

Stephen K. Moindi[1], Bernard M. Nzimbi[2]

[1,2] *Department of Mathematics, University of Nairobi, Kenya.*

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Abstract *- In this paper, we study the notions of W_4-flat para-Kenmotsu, semi-symmetric para-Kenmotsu and symmetric and recurrent relationships with respect to the W_4-curvature tensor.*

Keywords *- Flatness, Recurrent, Semi-symmetric, Symmetric.*

1. Introduction

Sato [19] defined the notion of almost para-contact Riemannian manifold. Adati and Matsumoto [1] defined and studied para-Sasakian and sp-Sasakian which were taken as a special kind of almost-contact Riemannian manifolds. Earlier, Kenmotsu[5] defined a class of almost-contact Riemannian manifolds. Sinha and Prasad [23] have defined a class of almost para-contact metric manifolds, namely para-Kenmotsu and special para-Kenmotsu manifolds. In a recent paper, Satyanarayana and Prasad [20] have proved that if in a para-Kenmotsu manifold $(M_n, g)(n > 3)$, the relation

$$R(X,Y).C = 0$$

holds, where C is the conformal curvature tensor of the manifold and R is the Riemannian curvature tensor, and where $R(X,Y)$ is taken to be the derivation of the tensor algebra at each point of the manifold for the tangent vectors X and Y, then the manifold is conformally flat.

In this paper, we will consider the case

$$R(X,Y).W_4 = 0$$

in a para-Kenmotsu manifold. Let M_n be an $n-$dimensional manifold equipped with structure tensors (ϕ,ξ,η) where ϕ is tensor type (1,1), ξ is a vector field and η is a 1-form such that

$$\eta(\xi) = 1 \tag{1.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \overline{X} = \phi(X) \tag{1.2}$$

Then M_n is called an almost para-contact manifold. If g is the Riemannian metric such that for all vector fields X and Y on M_n

$$g(X,\xi) = \eta(X) \tag{1.3}$$

$$\phi(\xi) = 0, \eta(\phi X) = 0, rank(\phi) = n - 1 \tag{1.4}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.5}$$

then the manifold M_n is said to admit an almost para-contact Riemannian structures (ϕ,ξ,η,g) (see [22], [10], [3]).
A manifold of dimension n with Riemannian metric g admitting a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η satisfying (1.1) and (1.3) along with

$$(\nabla_X\eta)Y = (\nabla_Y\eta)X = 0 \tag{1.6}$$

$$(\nabla_X \nabla_Y \eta)Z = [-g(X,Z) + \eta(X)\eta(Z)]\eta(Y) + [-g(X,Y) + \eta(X)\eta(Y)]\eta(Z) \quad (1.7)$$

$$\nabla_X \xi = \phi^2(X) = X - \eta(X)\xi \quad (1.8)$$

is called a para-Kenmotsu manifold or a p-Kenmotsu manifold. A para-Kenmotsu manifold has been studied by many authors (see [23], [25], [21]). Let (M_n, g) be an n-dimensional manifold admitting a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η satisfying

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y) \quad (1.9)$$

$$g(X,\xi) = \eta(X) \text{ and } (\nabla_X \eta)Y = \varphi(\overline{X}, Y) \quad (1.10)$$

where φ is an associate of ϕ is called a special p-Kenmotsu manifold or in short sp-Kenmotsu manifold [23]. It is known (see [23], [21]) that in a p-Kenmotsu manifold M_n the following relations hold:

$$S(X,\xi) = -(n-1)\eta(X) \quad (1.11)$$

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y,Z)) - g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \quad (1.12)$$

$$R(X,\xi) = -1 \quad (1.13)$$

$$R(X,\xi,\xi) = -X + \eta(X)\xi \quad (1.14)$$

$$R(X,\xi,X) = \xi \quad (1.15)$$

$$R(\xi,X,\xi) = X \quad (1.16)$$

$$R(X,Y,\xi) = \eta(X)Y - \eta(Y)X \quad (1.17)$$

where X is orthogonal to ξ and where S is the Ricci tensor and R the Riemannian curvature;

$$S(X,Y) = Ric(X,Y) = g(\phi X, Y) = -(n-1)g(X,Y) \quad (1.18)$$

2. W_4-Curvature Tensor in para-Kenmotsu Manifolds

Pokhariyal and Mishra [16] defined W_4-curvature tensor as

$$W_4(X,Y,Z,T) = Ric(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(X,Y)Ric(T,Z)]$$

This tensor has received a great deal of attention and has been studied by a number of researchers (see [12], [11], [6], [7]).

Definition 2.1 A *Riemannian manifold M_n is said to flat if* $R(X,Y)Z = 0$.

Definition 2.2 A *para-Kenmotsu manifold is said to be W_4-flat if the W_4-curvature Tensor vanishes identically, that is* $W_4(X,Y)Z = 0$ (see [8]).

Theorem 2.3 A *W_4-flat para-Kenmotsu manifold is a flat manifold.*

Proof. If a para-Kenmotsu manifold is flat, then $R = 0$ in

$$W_4(X,Y,Z,T) = Ric(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(X,Y)Ric(T,Z)].$$

Hence if a para-Kenmotsu manifold is W_4-flat, then we have

$$0 = Ric(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(X,Y)Ric(T,Z)]$$

or

$$Ric(X,Y,Z,T) = \frac{1}{n-1}[g(X,Y)Ric(T,Z) - g(X,Z)Ric(Y,T)].$$

Using $S(X,Y) = -(n-1)g(X,Y) = Ric(X,Y)$, we have

$$R(X,Y,Z,T) = \frac{1}{n-1}[-(n-1)g(T,Z)g(X,T) + (n-1)g(Y,T)g(X,Z)]$$

$$= \frac{n-1}{n-1}[-g(X,Y)g(T,Z) + g(X,Z)g(Y,T)]$$

$$= \mathrm{g(X,Z)g(Y,T)\text{-}g(X,Y)g(T,Z)} .$$

But in a para-Kenmotsu manifold,
$R(X,Y,Z,T) = g(X,Z)g(Y,T) - g(Y,Z)g(X,T)$.
Since
$g(X,Z)g(Y,T) - g(X,Y)g(Z,T) \neq g(X,Z)g(Y,T) - g(Y,Z)g(X,T)$,

this implies that this is only possible if $R(X,Y,Z,T) = 0$. . This proves the claim.

Corollary 2.4 A *W_4-flat para-Kenmotsu manifold is neither Einstein nor η -Einstein manifold.*

3. W_4-Semi-Symmetric para-Kenmotsu Manifold

De and Guha [4] gave the definition of semi-symmetric as $R(X,Y)R(Z,U)V = 0$.

The concept of semi-symmetric para-Kenmotsu manifolds with respect to some curvature tensor has been studied by several authors (see [24], [21], [17], [13]).

Definition 3.1 *A para-Kenmotsu manifold is said to be* W_4*-semi-symmetric if*

$$R(X,Y)W_4(Z,U)V = 0.$$

Theorem 3.2 *A* W_4*-semi-symmetric para-Kenmotsu Manifold is a* W_4*-flat manifold*

Proof. If a para-Kenmotsu Manifold is a W_4-semi-symmetric manifold then

$$\mathrm{R(X,Y)W_4(Z,U)V} = 0.$$

So

$$R(X,Y)W_4(Z,U)V = g(X,W_4(Z,U)V)Y - g(Y,W_4(Z,U)V)X = 0,$$

$$\Rightarrow g(X,W_4(Z,U)V)Y - g(Y,W_4(Z,U)V)X = 0,$$

$$\Rightarrow W_4(X,Z,U,V)Y - W_4(Y,Z,U,V)X = 0,$$

$$\Rightarrow g(W_4(X,Z,U,V)Y,\xi) - g(W_4(Y,Z,U,V)X,\xi) = 0,$$

$$\Rightarrow W_4(X,Z,U,V)\eta(Y) - W_4(Y,Z,U,V)\eta(X) = 0.$$

We note that this is only possible if $W_4(X,Z,U,V) = 0$ and $W_4(Y,Z,U,V) = 0$, since $\eta(X) \neq 0$ and $\eta(\mathrm{Y}) \neq 0$ and thus follows the theorem.

4. W_4-Symmetric Para-Kenmotsu Manifold

Definition 4.1 *A para-Kenmotsu manifold is said to be* W_4*-symmetric if*

$$\nabla_U W_4(X,Y)Z = W_4{}'(U,X,Y)Z = 0. \quad (4.1)$$

For more on this definition (see [9], [14]).

Theorem 4.2 *A* W_4*-symmetric and* W_4*-semi-symmetric para-Kenmotsu manifold is a flat manifold*

Proof. From Theorem 3.2, we found out that a W_4-semi-symmetric para-Kenmotsu manifold is a W_4-flat manifold. So, if a para-Kenmotsu manifold W_4-symmetric then

$$\nabla_U W_4(X,Y)Z = R(X,Y)W_4(Z,U)V - W_4(R(X,Y)Z,U)V - W_4(Z,R(X,Y)U)V - W_4(Z,U)R(X,Y)V = 0 \tag{4.2}$$

Computing each of the four terms in (4.2) separately and subjecting them to equivalent conditions gives

$$\nabla_U W_4(X,Y)Z = R(X,Y)W_4(Z,U)V - W_4(R(X,Y)Z,U)V - W_4(Z,R(X,Y)U)V - W_4(Z,U)R(X,Y)V = 0 \tag{4.3}$$

$$\Rightarrow g(R(X,Y)W_4(Z,U)V,\xi) = g(W_4'(X,Z,U,V)Y,\xi) - g(W_4'(Y,Z,U,V)X,\xi)$$

$$=\eta(W_4'(X,Z,U,V)Y) - \eta(W_4'(Y,Z,U,V)X)$$

$$= W_4'(X,Z,U,V)\eta(Y) - (W_4'(Y,Z,U,V)\eta(X),$$

where

$$R'(X,Y,Z,U) = g(R(X,Y)Z,U)$$

and

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z = g(X,Z)\eta(Y) - g(Y,Z)\eta(X).$$

Again

$$W_4(R(X,Y)Z,U)V = R(R(X,Y)Z,U)V + \frac{1}{n-1}[g(R(X,Y)Z,V)\phi U - g(R(X,Y)Z,U)\phi V]$$

$$\Rightarrow W_4'(R(X,Y)Z,U,V,\xi) = R'(R(X,Y)Z,U,V,\xi) + \frac{1}{n-1}[g(R(X,Y)Z,V)S(U,\xi) - g(R(X,Y)Z,U)S(V,\xi)]$$

$$= R'(R(X,Y)Z,U,V,\xi) + \frac{n-1}{n-1}[-\eta(U)R'(V,X,Y,Z) + \eta(V)R'(U,X,Y,Z)]$$

$$= R'(V,X,Y,Z)\eta(U) - g(R(X,Y)Z,g(U,V)) - \eta(U)R'(V,X,Y,Z) + \eta(V)R'(U,X,Y,Z)$$

$$= \eta(V)R'(U,X,Y,Z) - g(U,V)\eta(R(X,Y)Z) \tag{4.4}$$

$$W_4'(Z,R(X,Y)U)V = R(Z,R(X,Y)U,V) + \frac{1}{n-1}[g(Z,V)\phi R(X,Y)U - g(Z,R(X,Y)U)\phi V]$$

$$\Rightarrow W_4'(Z,R(X,Y)U,V,\xi) = R'(Z,R(X,Y)U,V,\xi) + \frac{1}{n-1}[g(Z,V)S(R(X,Y)U,\xi) - R'(Z,X,Y,U)S(V,\xi)]$$

$$= g(Z,V)\eta(R(X,Y)U) - \eta(Z)R'(V,X,Y,U) + \frac{n-1}{n-1}[-g(Z,V)R'(X,Y,U,\xi) + \eta(V)R'(Z,X,Y,U)]$$

$$= \eta(V)R'(Z,X,Y,U) - \eta(Z)R'(V,X,Y,U) \tag{4.5}$$

$$g(W_4(Z,U)R(X,Y)V,\xi) = W_4'(Z,U,R(X,Y)V,\xi)$$

$$= R'(Z,U,R(X,Y)V,\xi) + \frac{1}{n-1}[g(Z,R(X,Y)V)S(U,\xi) - g(Z,U)S(R(X,Y)V,\xi)]$$

$$= \eta(U)R'(Z,X,Y,V) - \eta(Z)\,R'(U,X,Y,V) + \frac{n-1}{n-1}[-\eta(U)R'(Z,X,Y,U) + g(Z,U)\eta(R(X,Y)V)]$$

$$= g(Z,U)\eta(R(X,Y)V) - \eta(Z)R'(U,X,Y,V). \tag{4.6}$$

Next, putting together (4.3), (4.4), (4.5) and (4.6) we have,

$$W_4{}'(X,Z,U,V)Y - W_4{}'(Y,Z,U,V)X + \eta(V)R'(U,X,Y,Z) - g(U,V)R'(X,Y,Z,\xi) + \eta(V)R'(Z,X,Y,U) - \eta(Z)R'(V,X,Y,U) + g(Z,U)R'(X,Y,V,\xi) - \eta(Z)R'(U,X,Y,V) = 0. \tag{4.7}$$

Since the para-Kenmotsu manifold is symmetric, the $W_4{}'$s vanish and also the coefficients of $\eta(V)$ and $\eta(Z)$ vanish because of symmetry leaving us only with

$$g(Z,U)R'(X,Y,V,\xi) - g(U,V)R'(X,Y,Z,\xi) = 0. \tag{4.8}$$

Since $g(Z,U) \neq g(U,V) \neq 0$, this implies that $R' = 0$ and thus follows the theorem.

5. W_4-Recurrent para-Kenmotsu Manifold

In this section we study the recurrent properties of the W_4-curvature tensor in a para-Kenmotsu manifold M_n. We consider a para-Kenmotsu manifold M_n which is W_4-recurrent. Pokhariyal [15] gave the definition of W_4-recurrent as

$$\nabla_U W_4(X,Y)Z = B(U)W_4(X,Y)Z, \tag{5.1}$$

where B is a non-zero 1-form.

Theorem 5.1 *A* W_4*-recurrent para-Kenmotsu manifold with* $R(X,Y)W_4(Z,U)V = 0$ *and* $g(Z,U)R'(X,Y,Z,\xi) - g(U,V)R'(X,Y,Z,\xi) = 0$ *is a* W_4*-symmetric and a semi-symmetric space.*

Proof. From (5.1), we have

$$\nabla_U W_4(X,Y)Z = B(U)W_4(X,Y)Z,$$

$$\Rightarrow \nabla_X W_4(Z,U)V = R(X,Y)W_4(Z,U)V - W_4(R(X,Y)Z,U)V - W_4(Z,R(X,Y)U)V - W_4(Z,U)R(X,Y)V. \tag{5.2}$$

But we are given

$$R(X,Y)W_4(Z,U)V = 0. \text{ (semi-symmetric space)}$$

This implies that we are left to show that the relation is symmetric under the stated conditions. Therefore, (5.2) becomes

$$\nabla_U W_4(X,Y)Z = B(U)W_4(X,Y)Z = 0,,$$

and from the definition (4.1), the claim is proved.

6. Conclusion

We have proved that a W_4 −flat para-Kenmotsu manifold is flat and that a W_4 −semi-symmetric and W_4 −symmetric para-Kenmotsu manifold is W_4 −flat. This means that a W_4 −semi-symmetric and W_4 −symmetric manifold is flat. We have also proved that a W_4 −recurrent para-Kenmotsu manifold is W_4 −symmetric and semi-symmetric provided that $R(X,Y)W_4(Z,U)V = 0$ *and* $g(Z,U)R'(X,Y,Z,\xi) - g(U,V)R'(X,Y,Z,\xi) = 0$. Therefore, we conclude that a W_4 −recurrent para-Kenmotsu manifold satisfying the above condition is a flat manifold.

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