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Original Article

Implication of Continued Fraction and Metallic Ratios to Resolve Binary Quadratic Diophantine Equations

V. Pandichelvi[1], S. Saranya[2]

[1,2] PG & Research Department of Mathematics, Urumu Dhanalakshmi College, Tamil nadu , India.
(Affiliated to Bharathidasan University).

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Abstract - *In this manuscript, the set of non-zero integer solutions (x, y) to some Diophantine equations of the forms $x^2 - rxy - y^2 = \pm 1$ where $4 \le r \le 9$ by relating continued fraction and metallic ratios are investigated. Furthermore, the collection of all such pair (x, y) are confirmed by MATLAB programs. .*

Keywords *- Diophantine equation, Metallic ratio, Continuous Fraction.*

1. Introduction

Solving Binary quadratic Diophantine equations in number theory is a vast topic of research. In [1-11,20], the author explains the basics of Number theory. Some interesting ways of solving the Diophantine equations are dealt in [12-19,21,22,24,25]. In [23], the author solved the Diophantine equation by using Bronze ratio. In this communication, the non-trivial integer solutions (x, y) to the Diophantine equations $x^2 - rxy - y^2 = \pm 1$ where $4 \le r \le 9$ by applying continued fraction and metallic ratios are probed. Also, the invention of (x, y) sustaining these equations are tested by MATLAB programs.

2. Resolving Binary Quadratic Diophantine Equations by using Continued Fraction and Copper Ratio

The main determination of this section is to resolve two kinds of Diophantine equations

$$x^2 - 4xy - y^2 = \pm 1 \text{ where } x, y \in Z^+. \tag{1}$$

To resolve (1), let us initiate with the ratio $\frac{\sqrt{20}-4}{2}$.

This ratio is the reciprocal of Copper ratio $\frac{\sqrt{20}+4}{2}$ due to fact that

$$\frac{\sqrt{20}-4}{2} \times \frac{\sqrt{20}+4}{2} = 1 \tag{2}$$

By using continued fraction, the ratio $\frac{\sqrt{20}-4}{2}$ can modified as follows.

$$\frac{\sqrt{20}-4}{2} = \frac{1}{\frac{\sqrt{20}+4}{2}} = \frac{1}{4+\left(\frac{\sqrt{20}-4}{2}\right)} = \frac{1}{4+\frac{1}{4+\left(\frac{\sqrt{20}-4}{2}\right)}} etc$$

$$\Rightarrow \frac{\sqrt{20}+4}{2} = 4 + \frac{\sqrt{20}-4}{2} = 4 + \frac{1}{4+\frac{1}{4+\frac{1}{4+\frac{1}{4+\cdots}}}} = [4; \bar{4}]. \tag{3}$$

Now, the required set of solutions to equation (1) can be estimated from the representation of continued fraction (3). For this, let us calculate the sequence of rational numbers from the successive convergent sequence (c_n) as given below.

$$c_0 = \frac{4}{1} = [4; 4], c_1 = \frac{17}{4} = [4; 4,4], c_2 = \frac{72}{17}[4; 4,4,4], c_3 = \frac{305}{72}[4; 4,4.4,4], c_4 = \frac{1292}{305}[4; 4,4,4,4,4] \; etc \tag{4}$$

Here, the numerators and denominators characterize the values of x and y respectively in equation (1). Note that, the sequence $(c_{2n}), n \in W$ affords the collection of positive integer solutions $(x, y) = \{(4,1); (72,17); (1292,305); etc\}$ to the specific form of second degree Diophantine equation $x^2 - 4xy - y^2 = -1$ containing two variables and the sequence (c_{2n+1}), $n \in W$ contributes the set of non-negative integer solutions $(x, y) = \{(17,4); (305,72); (5473,1292); etc\}$ to the binary

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quadratic Diophantine equation $x^2 - 4xy - y^2 = 1$. It is perceived that all the values of x and y sustaining the recurrence relations $x_{n+2} = 4x_{n+1} + x_n,\ y_{n+2} = 4y_{n+1} + y_n$ where $n \geq 1$.

Remarks:

Following the same procedure as explained in section 2, the non-zero integer solutions to various binary quadratic Diophantine equations by using continued fraction and other metallic ratios are illustrated below.

1. Replace the Copper ratio by the Nickel ratio $\frac{\sqrt{29}+5}{2}$ in section 2, the members in sequence $(k_n), n \in W$ are given by $k_0 = \frac{5}{1}, k_1 = \frac{26}{5}, k_2 = \frac{135}{26}, k_3 = \frac{701}{135}, k_4 = \frac{3640}{701}, k_5 = \frac{18901}{3640}, etc.$
   Hence, $(x, y) = \{(5,1), (135,26), (3640,701)\ etc\}$ denotes the infinite set of positive integer solutions to the Diophantine equation $x^2 - 5xy - y^2 = -1$ and $(x, y) = \{(26,5), (701.135), (18901,3640) etc\}$ states the set of integer solutions to the Diophantine equation $x^2 - 5xy - y^2 = 1$. The recurrence relations for x and y are mentioned by $x_{n+2} = 5x_{n+1} + x_n$, $y_{n+2} = 5y_{n+1} + y_n, n \geq 1$.
2. Instead of the Copper ratio, take the Aluminium ratio $\frac{\sqrt{40}+6}{2}$ in section 2, the elements in the sequence $(a_n), n \in W$ are attained by $a_0 = \frac{6}{1}, a_1 = \frac{37}{6}, a_2 = \frac{228}{37}, a_3 = \frac{1405}{228}, a_4 = \frac{8658}{1405}, a_5 = \frac{53353}{8658} etc.$ Then, the Diophantine equation $x^2 - 6xy - y^2 = -1$ is satisfied by all the pairs $(x, y) = \{(6,1), (228,37), (8658,1405) etc\}$ and $x^2 - 6xy - y^2 = 1$ is fulfilled by the set $(x, y) = \{(37,6), (1405,228), (53353,8658), etc\}$. Such values of x and y satisfy the recurrence relations $x_{n+2} = 6x_{n+1} + x_n$, $y_{n+2} = 6y_{n+1} + y_n, n \geq 1$.
3. Consider the Iron ratio $\frac{\sqrt{53}+7}{2}$ in the place of Copper ratio in section 2, the elements in the sequence (l_n), $n \in W$ are received by $i_0 = \frac{7}{1}, i_1 = \frac{50}{7}, i_2 = \frac{357}{50}, i_3 = \frac{2549}{357}, i_4 = \frac{18200}{2549}\ etc.$
   Hence, $(x, y) = \{(7,1), (357,50), (18200,2549) etc\}$ and $(x, y) = \{(50,7)(2549,357) etc\}$ are collections of solutions to the Diophantine equations $x^2 - 7xy - y^2 = -1$ and $x^2 - 7xy - y^2 = 1$ respectively. Moreover, the solutions x and y please the identities $x_{n+2} = 7x_{n+1} + x_n,\ y_{n+2} = 7y_{n+1} + y_n$ for $n \geq 1$.
4. Elect Tin ratio $\frac{\sqrt{68}+8}{2}$ for Copper ratio in the process which is explained in section 2, few numbers in the sequence (t_n), $n \in W$ are evaluated by $t_0 = \frac{8}{1}, t_1 = \frac{65}{8}, t_2 = \frac{528}{65}, t_3 = \frac{4289}{528}, t_4 = \frac{34840}{4289}\ etc.$
   Then, the succeeding two equations $x^2 - 8xy - y^2 = -1$ and $x^2 - 8xy - y^2 = 1$ are satisfied by $(x, y) = \{(8,1), (528,65), (34840,4289) etc\}$ and $(x, y) = \{(65,8), (528,65) etc\}$ respectively. For $n \geq 1$, $x_{n+2} = 8x_{n+1} + x_n$, and $y_{n+2} = 8y_{n+1} + y_n$ highlighted recurrence relations corresponding to the solutions of the above two equations.
5. Select Lead ratio $\frac{\sqrt{85}+9}{2}$ in the place of Copper ratio in section 2 , the sequence $(l_n), n \in W$ is discovered by $l_0 = \frac{9}{1}$, $l_1 = \frac{82}{9}, l_2 = \frac{747}{82}, l_3 = \frac{6805}{747}, l_4 = \frac{61992}{6805}\ etc.$
   Then, the Diophantine equations $x^2 - 9xy - y^2 = -1$ and $x^2 - 9xy - y^2 = 1$ are gratified by $(x, y) = \{(9,1), (747,82), (51992,6805)\ etc\}$ and $(x, y) = \{(82,9), (6805,747) etc\}$ respectively. The relations between x and y are identified by $x_{n+2} = 9x_{n+1} + x_n,\ y_{n+2} = 9y_{n+1} + y_n$ for $n \geq 1$.

The following MATLAP program helps to find all possible values of (x, y) satisfying the Diophantine equations $x^2 - rxy + y^2 = \pm 1, 4 \leq r \leq 9$.

```
clear, clc;

x(1) = input('\n Enter the initial number');

y(1) = input('\n Enter the initial denominator');

c(1) = sym(x(1) /y(1));

for i = 2: 5;

c(i) = sym(k + 1/c(i − 1));

end
c'
```

3. Conclusion

In this paper, the possibility of non-zero integer solutions (x, y) to the Diophantine equations $x^2 - rxy - y^2 = \pm 1$ where $4 \leq r \leq 9$ are evaluated by implementing continued fraction and metallic ratios and the choices of (x, y) satisfying all these equations are cross checked by MATLAB programs. One search solutions to these types of equations by using congruence relations and properties of divisibility.

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