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Original Article

Lie Group Analysis of a KDV-Burgers Equation

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Abstract- *We study a kdv-burgers equation by Lie group analysis. We obtain Lie point symmetries and use them to carry out symmetry reductions. The arising systems are investigated for exact solutions. Solitons are constructed by a linear span of time and space translation symmetries. We also compute conservation laws using multiplier approach.*
Keywords: *Group-invariant, KdV burgers, Lie group analysis, Solitons, Symmetry reductions*

1. Introduction

In 1895, Korteweg and de Vries [20] derived the ubiquitous Korteweg de Vries (KdV) equation to describe propagation of weakly nonlinear waves. The Kortweg-de Vries Burgers equation has been used in several different fields to describe various physical phenomena of interest and arises in several physical contexts, e.g. the propagation of undular bores in shallow water [12], the flow of liquids containing gas bubbles [30], the propagation of waves in an elastic tube filled with a viscous fluid [11], and weakly nonlinear plasma waves with certain dissipative effects [25]. It can also be used as a nonlinear model in crystal lattice theory, the theory of ferroelectricity, nonlinear circuit theory and turbulence. The kdV-Burgers equation (KdVB),[25]

$$\Delta \equiv u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{1}$$

where t and x represent time and spatial independent variables respectively. The constants α, β and γ are nonlinear, dissipative and dispersive parameters respectively. Observe that the cases when $\beta = 0$ [18] and $\gamma = 0$ [17] have been studied. The KdVB equation was derived by Su and Gardner [29] appears in the study of the weak effects of dispersion, dissipation, and nonlinearity in waves propagating in a liquid-filled elastic tube. Recently, the nonlinear fractional partial differential equations, such as fractional KdV-Burgers equation [32], fractional Schrödinger-Korteweg-de Vries equations [6] and fractional Burgers' equations [1], were also presented to describe many important phenomena and dynamic processes in physics. Some theoretical issues concerning the KdVB equation, such as the traveling wave solution, have received considerable attention [4]. A number of exact solitary wave solutions to KdVB equations have been found in the past few years. The exact solutions of a compound KdVB equation were obtained by using a homogeneous balance method in [31]. By using the special truncated expansion method, Hassan[7] constructed solitary wave solutions for the compound KdVB equation and discussed the generalized two-dimensional KdVB equation. The Exp-function method is applied to obtain generalized solitary solutions and periodic solutions for the KdVB equation in [28]. In the past several decades, many authors have paid attention to studying the numerical methods for solving KdVB equations. Soliman extended the variational iterations method to solve the KdVB equations [27]. Shi,Xu, and Guo [26], have recently computed numerical solutions to KdvB by the compact-type CIP technique. In this paper, we study the KdvB by Lie group analysis.

2. Preliminaries

This section presents a prelude that is used in what comes after.

2.1. *Local Lie groups*

[20] We will consider the transformations

$$T_\epsilon : \quad \bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \tag{2}$$

in the Euclidean space $\mathbb{R}^n$ of $x = x^i$ independent variables and $\mathbb{R}^m$ of $u = u^\alpha$ dependent variables. The continuous parameter ϵ ranges from a neighbourhood $\mathcal{N}' \subset \mathcal{N} \subset \mathbb{R}$ of $\epsilon = 0$ for φ^i and ψ^α differentiable and analytic in the parameter ϵ.

Definition 1 *Let $\mathcal{G}$ be a set of transformations in (2) . Then $\mathcal{G}$ is a local Lie group if:*

(i). Given $T_{\epsilon_1}, T_{\epsilon_2} \in \mathcal{G}$, for $\epsilon_1, \epsilon_2 \in \mathcal{N}' \subset \mathcal{N}$, then $T_{\epsilon_1}T_{\epsilon_2} = T_{\epsilon_3} \in \mathcal{G}$, $\epsilon_3 = \phi(\epsilon_1, \epsilon_2) \in \mathcal{N}$ (Closure).

(ii). There exists a unique $T_0 \in \mathcal{G}$ if and only if $\epsilon = 0$ such that $T_\epsilon T_0 = T_0 T_\epsilon = T_\epsilon$(Identity).

(iii). There exists a unique $T_{\epsilon^{-1}} \in \mathcal{G}$ for every transformation $T_\epsilon \in \mathcal{G}$, where $\epsilon \in \mathcal{N}' \subset \mathcal{N}$ and $\epsilon^{-1} \in \mathcal{N}$ such that $T_\epsilon T_{\epsilon^{-1}} = T_{\epsilon^{-1}} T_\epsilon = T_0$ (Inverse).

Remark 1 *The condition (i) is sufficient for associativity of $\mathcal{G}$.*

2.2. *Prolongations*

Consider the system,

$$\Delta_\alpha\left(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}\right) = \Delta_\alpha = 0, \tag{3}$$

where u^α are dependent variables with partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \ldots, u_{(\pi)} = \{u_{i_1\ldots i_\pi}^\alpha\}$, of the first, second, …, up to the πth-orders. We shall denote by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \tag{4}$$

the total differentiation operator with respect to the variables x^i and δ_i^j, the Kronecker delta. Then

$$D_i(x^j) = \delta_i^j, ', \; u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(D_i(u^\alpha)), \quad \ldots, \tag{5}$$

where u_i^α defined in (5) are differential variables [10].

(i). **Prolonged groups** Let $\mathcal{G}$ given by

$$\bar{x}^i = \varphi^i(x^i, u^\alpha, \epsilon), \quad \varphi^i\Big|_{\epsilon=0} = x^i, \quad \bar{u}^\alpha = \psi^\alpha(x^i, u^\alpha, \epsilon), \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \tag{6}$$

where $\Big|_{\epsilon=0}$ means evaluated on $\epsilon = 0$.

Definition 2 *The construction of $\mathcal{G}$ in (6) is equivalent to the computation of infinitesimal transformations*

$$\begin{aligned} \bar{x}^i &\approx x^i + \xi^i(x^i, u^\alpha)\epsilon, \quad \varphi^i\Big|_{\epsilon=0} = x^i, \\ \bar{u}^\alpha &\approx u^\alpha + \eta^\alpha(x^i, u^\alpha)\epsilon, \quad \psi^\alpha\Big|_{\epsilon=0} = u^\alpha, \end{aligned} \tag{7}$$

obtained from (2) by a Taylor series expansion of $\varphi^i(x^i, u^\alpha, \epsilon)$ and $\psi^i(x^i, u^\alpha, \epsilon)$ in ϵ about $\epsilon = 0$ and keeping only the terms linear in ϵ, where

$$\xi^i(x^i, u^\alpha) = \frac{\partial \varphi^i(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}, \quad \eta^\alpha(x^i, u^\alpha) = \frac{\partial \psi^\alpha(x^i, u^\alpha, \epsilon)}{\partial \epsilon}\Big|_{\epsilon=0}. \tag{8}$$

Remark 2 *By using the symbol of infinitesimal transformations, X, (7) becomes*

$$\bar{x}^i \approx (1+X)x^i, \quad \bar{u}^\alpha \approx (1+X)u^\alpha, \tag{9}$$

where

$$X = \xi^i(x^i, u^\alpha)\frac{\partial}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial}{\partial u^\alpha}, \tag{10}$$

is the generator $\mathcal{G}$ in (6).

Remark 3 *The change of variables formula*

$$D_i = D_i(\varphi^j)\bar{D}_j, \tag{11}$$

is employed to construct transformed derivatives from (2). The $\bar{D}_j$ is total differentiation $\bar{x}^i$. As a result

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \; \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha). \tag{12}$$

If we apply the change of variable formula given in (11) on $\mathcal{G}$ given by (6), we get

$$D_i(\psi^\alpha) = D_i(\varphi^j), \; \bar{D}_j(\bar{u}^\alpha) = \bar{u}_j^\alpha D_i(\varphi^j). \tag{13}$$

If we expand (13), we obtain

$$\left(\frac{\partial \varphi^j}{\partial x^i} + u_i^\beta \frac{\partial \varphi^j}{\partial u^\beta}\right)\bar{u}_j^\beta = \frac{\partial \psi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \psi^\alpha}{\partial u^\beta}. \tag{14}$$

The $\bar{u}_i^\alpha$ can be written as functions of $x^i, u^\alpha, u_{(1)}$, meaning that,

$$\bar{u}_i^\alpha = \Phi^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon), \quad \Phi^\alpha\Big|_{\epsilon=0} = u_i^\alpha. \tag{15}$$

Definition 3 *The transformations in (6) and (15) give the first prolongation group $\mathcal{G}^{[1]}$.*

Definition 4 *Infinitesimal transformation of the first derivatives is*

$$\bar{u}_i^\alpha \approx u_i^\alpha + \zeta_i^\alpha \epsilon, \quad \text{where} \quad \zeta_i^\alpha = \zeta_i^\alpha(x^i, u^\alpha, u_{(1)}, \epsilon). \tag{16}$$

Remark 4 *In terms of infinitesimal transformations, $\mathcal{G}^{[1]}$ is given by (7) and (16).*

(ii). **Prolonged generators**

Definition 5 *By the relation (13) on $\mathcal{G}^{[1]}$ from 3, we obtain [10]*

$$D_i(x^j + \xi^j\epsilon)(u_j^\alpha + \zeta_j^\alpha\epsilon) = D_i(u^\alpha + \eta^\alpha\epsilon), \quad \text{which gives} \tag{17}$$

$$u_i^\alpha + \zeta_j^\alpha\epsilon + u_j^\alpha\epsilon D_i\xi^j = u_i^\alpha + D_i\eta^\alpha\epsilon, \tag{18}$$

and thus

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{19}$$

is the first prolongation formula.

Remark 5 *Analogously, one constructs higher order prolongations [8],*

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i\kappa}^\alpha D_j(\xi^\kappa), \quad \ldots, \quad \zeta_{i_1,\ldots,i_\kappa}^\alpha = D_{i_\kappa}(\zeta_{i_1,\ldots,i_{\kappa-1}}^\alpha) - u_{i_1,i_2,\ldots,i_{\kappa-1}j}^\alpha D_{i_\kappa}(\xi^j). \tag{20}$$

Remark 6 *The prolonged generators of the prolongations $\mathcal{G}^{[1]}, \ldots, \mathcal{G}^{[\kappa]}$ of the group $\mathcal{G}$ are*

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha}, \quad \ldots, X^{[\kappa]} = X^{[\kappa-1]} + \zeta_{i_1,\ldots,i_\kappa}^\alpha \frac{\partial}{\partial \zeta_{i_1,\ldots,i_\kappa}^\alpha}, \; \kappa \geq 1, \tag{21}$$

for the group generator X in (10).

2.2.1. ***Group invariants***

Definition 6 *A function $\Gamma(x^i, u^\alpha)$ is said to be an invariant of $\mathcal{G}$ of in (2) if*

$$\Gamma(\bar{x}^i, \bar{u}^\alpha) = \Gamma(x^i, u^\alpha). \tag{22}$$

Theorem 1 *A function $\Gamma(x^i, u^\alpha)$ is an invariant of the group $\mathcal{G}$ given by (2) if and only if it solves the following first-order linear PDE: [10]*

$$X\Gamma = \xi^i(x^i, u^\alpha)\frac{\partial \Gamma}{\partial x^i} + \eta^\alpha(x^i, u^\alpha)\frac{\partial \Gamma}{\partial u^\alpha} = 0. \tag{23}$$

From Theorem (1), we have the following result.

Theorem 2 *The Lie group $\mathcal{G}$ in (2) [8] has precisely $n-1$ functionally independent invariants and one can take as the basic invariants, the left-hand sides of the first integrals*

$$\psi_1(x^i, u^\alpha) = c_1, \dots, \psi_{n-1}(x^i, u^\alpha) = c_{n-1}, \tag{24}$$

of the characteristic equations for (23):

$$\frac{\mathrm{d}x^i}{\xi^i(x^i, u^\alpha)} = \frac{\mathrm{d}u^\alpha}{\eta^\alpha(x^i, u^\alpha)}. \tag{25}$$

2.2.2. *Symmetry groups*

Definition 7 *We define the vector field X (10) as a Lie point symmetry of (3) if the determining equations*

$$X^{[\pi]}\Delta_\alpha\Big|_{\Delta_\alpha = 0} = 0, \quad \alpha = 1, \dots, m, \quad \pi \geq 1, \tag{26}$$

are satisfied for the π-th prolongation of X, namely $X^{[\pi]}$.

Definition 8 *The Lie group $\mathcal{G}$ is a symmetry group of (3) if (3) is form-invariant, that is*

$$\Delta_\alpha\left(\bar{x}^i, \bar{u}^\alpha, \bar{u}_{(1)}, \dots, \bar{u}_{(\pi)}\right) = 0. \tag{27}$$

Theorem 3 *The Lie group $\mathcal{G}$ (2) can be constructed from the infinitesimal transformations in (6) by integrating the Lie equations*

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\epsilon} = \xi^i(\bar{x}^i, \bar{u}^\alpha), \quad \bar{x}^i\Big|_{\epsilon=0} = x^i, \quad \frac{\mathrm{d}\bar{u}^\alpha}{\mathrm{d}\epsilon} = \eta^\alpha(\bar{x}^i, \bar{u}^\alpha), \quad \bar{u}^\alpha\Big|_{\epsilon=0} = u^\alpha. \tag{28}$$

2.3. ***Lie algebras***

Definition 9 *A vector space $\mathcal{V}_r$ of operators [?] X (10) is a Lie algebra if for any $X_i, X_j \in \mathcal{V}_r$,*

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{29}$$

is in $\mathcal{V}_r$ for all $i, j = 1, \dots, r$.

Remark 7 *The commutator is bilinear, skew symmetric and admits to the Jacobi identity [10].*

Theorem 4 *The set of solutions of (26) forms a Lie algebra[9].*

2.3.1. *Exact solutions*

The methods of (G'/G)-expansion method [20], Extended Jacobi elliptic function expansion [21] and Kudryashov [17] are usually applied after symmetry reductions.

2.4. *Conservation laws*

[9]

2.4.1. *Fundamental operators*

Definition 10 *The Euler-Lagrange operator* $\frac{\delta}{\delta u^\alpha}$ *is*

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} (-1)^\kappa D_{i_1}, \ldots, D_{i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{30}$$

and the Lie- Bäcklund operator in abbreviated form is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \ldots. \tag{31}$$

Remark 8 *The Lie- Bäcklund operator (31) in its prolonged form is*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{\kappa \geq 1} \zeta_{i_1 \ldots i_\kappa} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}, \tag{32}$$

for

$$\zeta^\alpha_i = D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \qquad \ldots, \zeta^\alpha_{i_1 \ldots i_\kappa} = D_{i_1 \ldots i_\kappa}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_\kappa}, \quad j = 1, \ldots, n. \tag{33}$$

and the Lie characteristic function

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{34}$$

Remark 9 *The characteristic form of Lie- Bäcklund operator (32) is*

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1 \ldots i_\kappa}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_\kappa}}. \tag{35}$$

2.5. *The method of multipliers*

Definition 11 *A function* $\Lambda^\alpha \left(x^i, u^\alpha, u_{(1)}, \ldots\right) = \Lambda^\alpha$, *is a multiplier of (3) if [20]*

$$\Lambda^\alpha \Delta_\alpha = D_i T^i, \tag{36}$$

where $D_i T^i$ *is a divergence expression.*

Definition 12 *To find the multipliers* Λ^α, *one solves the determining equations (37) [19],*

$$\frac{\delta}{\delta u^\alpha} (\Lambda^\alpha \Delta_\alpha) = 0. \tag{37}$$

2.6. *Ibragimov's conservation theorem*

The technique [9] enables one to construct conserved vectors associated with each Lie point symmetry of (3).

Definition 13 *The adjoint equations of (3) are*

$$\Delta^*_\alpha\left(x^i, u^\alpha, v^\alpha, \ldots, u_{(\pi)}, v_{(\pi)}\right) \equiv \frac{\delta}{\delta u^\alpha}(v^\beta \Delta_\beta) = 0, \tag{38}$$

for a new dependent variable v^α.

Definition 14 *The Formal Lagrangian* $\mathcal{L}$ *of (3) and its adjoint equations (38) is [10]*

$$\mathcal{L} = v^\alpha \Delta_\alpha(x^i, u^\alpha, u_{(1)}, \ldots, u_{(\pi)}). \tag{39}$$

Theorem 5 *Every infinitesimal symmetry X of (3) leads to conservation laws [8]*

$$D_i T^i\Big|_{\Delta_\alpha=0} = 0, \tag{40}$$

where the conserved vector

$$\begin{aligned} T^i = \xi^i\mathcal{L} + W^\alpha\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_i} - D_j\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}}\right) + D_jD_k\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}}\right) - \ldots\right] + \\ D_j(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ij}} - D_k\left(\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}}\right) + \ldots\right] + D_jD_k(W^\alpha)\left[\frac{\partial\mathcal{L}}{\partial u^\alpha_{ijk}} - \ldots\right]. \end{aligned} \tag{41}$$

3. Results

3.1. *Lie point symmetries of kdv burgers equation*(1)

We start first by computing Lie point symmetries of the one-dimensional heat Equation (1), which admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{42}$$

if and only if

$$X^{[3]}\Delta\Big|_{\Delta=0} = 0. \tag{43}$$

$$\mathrm{X}^{[3]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{222}\frac{\partial}{\partial u_{xxx}}, \tag{44}$$

is the third prolongation of the Lie point symmetry X as defined in (21) and

$$\zeta_1 = D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi), \tag{45}$$

$$\zeta_2 = D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi), \tag{46}$$

$$\zeta_{22} = D_x(\zeta_2) - u_{tx}D_x(\tau) - u_{xx}D_x(\xi), \tag{47}$$

$$\zeta_{222} = D_x(\zeta_{22}) - u_{txx}D_x(\tau) - u_{xxx}D_x(\xi), \tag{48}$$

as defined in (20), and

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots, \tag{49}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \tag{50}$$

Applying the definitions of D_t and D_x given in (49) and (50), we obtain the expanded form of the ζ_s as

$$\begin{aligned}
\zeta_1 &= \eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u), \\
\zeta_2 &= \eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u), \\
\zeta_{22} &= \eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_t(-\tau_{xx}) + u_t u_x(-2\tau_{xu}) + u_t u_{xx}(-\tau_u) \\
&\quad + u_{tx}(-2\tau_x) + u_{xx}(\eta_u - 2\xi_x) + u_x u_{tx}(-2\tau_u) + u_x u_{xx}(-3\xi_u) + u_x^2(\eta_{uu} - 2\xi_{xu}) \\
&\quad + u_t u_x^2(-\tau_{uu}) + u_x^3(-\xi_{uu}) \\
\zeta_{222} &= \eta_{xxx} + u_x(3\eta_{xxu} - \xi_{xxx}) + u_{tx}(-3\tau_{xx}) + u_t(-\tau_{xxx}) + u_x u_{tx}(-6\tau_{xu}) + u_t u_x^2(-3\tau_{xuu}) \\
&\quad + u_t u_{xxx}(-\tau_u) + u_t u_x u_{xx}(-3\tau_{uu}) + u_t u_x(-3\tau_{xxu}) + u_x^3(\eta_{uuu} - 3\xi_{xuu}) + u_t u_{xx}(-4\tau_{xu}) \\
&\quad + u_{xx}u_{tx}(-3\tau_u) + u_{xxx}(\eta_u - 3\xi_x) + u_{txx}(-3\tau_x) + u_{xx}(3\eta_{xu} - 3\xi_{xx}) \\
&\quad + u_x u_{xx}(3\eta_{uu} - 9\xi_{xu}) + u_x u_{txx}(-3\tau_u) + u_x^2 u_{tx}(-3\tau_{uu}) + u_{xx}^2(-3\xi_u) + u_t u_x^3(-\tau_{uuu}) \\
&\quad + u_x u_{xxx}(-4\xi_u) + u_x^2 u_{xx}(-6\xi_{uu}) + u_x^2(3\eta_{xuu} - 3\xi_{xxu}) + u_x^4(-\xi_{uuu}).
\end{aligned} \tag{51}$$

By the definition of $X^{[3]}$,

$$(\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{22}\frac{\partial}{\partial u_{xx}} + \zeta_{222}\frac{\partial}{\partial u_{xx}})\Delta\Big|_{u_{xxx}=-\frac{u_t}{\gamma}-\frac{\alpha}{\gamma}uu_x-\frac{\beta}{\gamma}u_{xx}} = 0 \tag{52}$$

which gives

$$\zeta_1 + \alpha\eta u_x + \alpha\zeta_2 u + \beta\zeta_{22} + \gamma\zeta_{222}\big|_{u_{xxx}=-\frac{u_t}{\gamma}-\frac{\alpha}{\gamma}uu_x-\frac{\beta}{\gamma}u_{xx}} = 0. \tag{53}$$

Substituting for $\zeta_1, \zeta_2, \zeta_{22}$ and ζ_{222} in Equation (53), we obtain the following;

$$\begin{aligned}
&\eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u) + \alpha\eta u_x \\
&+ \alpha u\Big\{\eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u)\Big\} \\
&+ \beta\Big\{\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_t(-\tau_{xx}) + u_t u_x(-2\tau_{xu}) + u_t u_{xx}(-\tau_u) \\
&+ u_{tx}(-2\tau_x) + u_{xx}(\eta_u - 2\xi_x) + u_x u_{tx}(-2\tau_u) + u_x u_{xx}(-3\xi_u) + u_x^2(\eta_{uu} - 2\xi_{xu}) \\
&+ u_t u_x^2(-\tau_{uu}) + u_x^3(-\xi_{uu})\Big\} + \gamma\Big\{\eta_{xxx} + u_x(3\eta_{xxu} - \xi_{xxx}) + u_{tx}(-3\tau_{xx}) + u_t(-\tau_{xxx}) \\
&+ u_x u_{tx}(-6\tau_{xu}) + u_t u_x^2(-3\tau_{xuu}) + u_t u_{xxx}(-\tau_u) + u_t u_x u_{xx}(-3\tau_{uu}) + u_t u_x(-3\tau_{xxu}) \\
&+ u_x^3(\eta_{uuu} - 3\xi_{xuu}) + u_t u_{xx}(-4\tau_{xu}) + u_{xx}u_{tx}(-3\tau_u) + u_{xxx}(\eta_u - 3\xi_x) + u_{txx}(-3\tau_x) + \\
&+ u_{xx}(3\eta_{xu} - 3\xi_{xx}) + u_x u_{xx}(3\eta_{uu} - 9\xi_{xu}) + u_x u_{txx}(-3\tau_u) + u_x^2 u_{tx}(-3\tau_{uu}) + u_{xx}^2(-3\xi_u) \\
&+ u_t u_x^3(-\tau_{uuu}) + u_x u_{xxx}(-4\xi_u) + u_x^2 u_{xx}(-6\xi_{uu}) + u_x^2(3\eta_{xuu} - 3\xi_{xxu}) + u_x^4(-\xi_{uuu})\Big\}_{\Delta=0} \\
&= 0
\end{aligned} \tag{54}$$

Now replacing u_{xxx} by $\frac{-\{u_t+\alpha uu_x+\beta u_{xx}\}}{\gamma}$ in Equation (54) produces

$$\begin{aligned}
&\eta_t + u_t(\eta_u - \tau_t) + u_x(-\xi_t) + u_t u_x(-\xi_u) + u_t^2(-\tau_u) + \alpha\eta u_x \\
&+ \alpha u\{\eta_x + u_x(\eta_u - \xi_x) + u_t(-\tau_x) + u_t u_x(-\tau_u) + u_x^2(-\xi_u)\} \\
&+ \beta\Big\{\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_t(-\tau_{xx}) + u_t u_x(-2\tau_{xu}) + u_t u_{xx}(-\tau_u) \\
&+ u_{tx}(-2\tau_x) + u_{xx}(\eta_u - 2\xi_x) + u_x u_{tx}(-2\tau_u) + u_x u_{xx}(-3\xi_u) + u_x^2(\eta_{uu} - 2\xi_{xu}) \\
&+ u_t u_x^2(-\tau_{uu}) + u_x^3(-\xi_{uu})\Big\} + \gamma\Big\{\eta_{xxx} + u_x(3\eta_{xxu} - \xi_{xxx}) + u_{tx}(-3\tau_{xx}) + u_t(-\tau_{xxx}) \\
&+ u_x u_{tx}(-6\tau_{xu}) + u_t u_x^2(-3\tau_{xuu}) + u_t\Big[\frac{-\{u_t + \alpha u u_x + \beta u_{xx}\}}{\gamma}\Big](-\tau_u) + u_t u_x u_{xx}(-3\tau_{uu}) \\
&+ u_t u_x(-3\tau_{xxu}) + u_x^3(\eta_{uuu} - 3\xi_{xuu}) + u_t u_{xx}(-4\tau_{xu}) + u_{xx}u_{tx}(-3\tau_u) + u_{xx}(3\eta_{xu} - 3\xi_{xx}) \\
&+ \Big[\frac{-\{u_t + \alpha u u_x + \beta u_{xx}\}}{\gamma}\Big](\eta_u - 3\xi_x) + u_{txx}(-3\tau_x) + u_{xx}(3\eta_{xu} - 3\xi_{xx}) \\
&+ u_x u_{xx}(3\eta_{uu} - 9\xi_{xu}) + u_x u_{txx}(-3\tau_u) + u_x^2 u_{tx}(-3\tau_{uu}) + u_{xx}^2(-3\xi_u) + u_t u_x^3(-\tau_{uuu}) \\
&+ u_x\Big[\frac{-\{u_t + \alpha u u_x + \beta u_{xx}\}}{\gamma}\Big](-4\xi_u) + u_x^2 u_{xx}(-6\xi_{uu}) + u_x^2(3\eta_{xuu} - 3\xi_{xxu}) + u_x^4(-\xi_{uuu}) \\
&\Big\} = 0
\end{aligned} \tag{55}$$

or

$$\begin{aligned}
&\eta_t + \alpha u\eta_x + \beta\eta_{xx} + \gamma\eta_{xxx} + \\
&u_t\{3\xi_x - \tau_t - \alpha u\tau_x - \beta\tau_{xx} - \gamma\tau_{xxx}\} \\
&+ u_x\{\alpha\eta - \xi_t + 2\alpha u\xi_x + \beta(2\eta_{xu} - \xi_{xx}) + \gamma(3\eta_{xxu} - \xi_{xxx})\} \\
&+ u_t u_x\{3\xi_u - 3\gamma\tau_{xxu} - 2\beta\tau_{xu}\} \\
&+ u_x^2\{3\alpha u\xi_u + \beta(\eta_{uu} - 2\xi_{xu}) + \gamma(3\eta_{xuu} - 3\xi_{xxu})\} \\
&+ u_{xx}\{\beta\xi_x\} + u_x u_{xx}\{\beta\xi_u + \gamma(3\eta_{uu} - 9\xi_{xu})\} \\
&u_{tx}\{-2\beta\tau_x - 3\gamma\tau_{xx}\} + u_x u_{tx}\{-2\beta\tau_u - 6\gamma\tau_{xu}\} + u_t u_x^2\{-\beta\tau_{uu} - 3\gamma\tau_{xuu}\} \\
&u_x^3\{-\beta\xi_{uu} + \gamma(\eta_{uuu} - 3\xi_{xuu})\} + \\
&\gamma\Big\{u_t u_x u_{xx}(-3\tau_{uu}) + u_t u_{xx}(-4\tau_{xu}) \\
&+ u_{xx}u_{tx}(-3\tau_u) + u_{txx}(-3\tau_x) + u_{xx}(3\eta_{xu} - 3\xi_{xx}) \\
&+ u_x u_{txx}(-3\tau_u) + u_x^2 u_{tx}(-3\tau_{uu}) + u_{xx}^2(-3\xi_u) + u_t u_x^3(-\tau_{uuu}) \\
&+ u_x^2 u_{xx}(-6\xi_{uu}) + u_x^4(-\xi_{uuu})\Big\} = 0
\end{aligned} \tag{56}$$

Since the functions τ, ξ and η depend only on t, x and u and are independent of the derivatives of u, we can then split the above equation on the derivatives of u and obtain

$$\tau_x = \tau_u = \tau_t = \xi_u = \xi_x = \eta_{uu} = 0, \tag{57}$$

$$\alpha\eta - \xi_t + 2\beta\eta_{xu} = 0, \tag{58}$$

$$\eta_t + \alpha u\eta_x + \beta\eta_{xx} + \gamma\eta_{xxx} = 0. \tag{59}$$

From Equation (57), it is evident that

$$\tau = C_1, \tag{60}$$
$$\xi = \xi(t), \tag{61}$$
$$\eta = A(t,x)u + B(t,x). \tag{62}$$

If we the value of η in Equation (59), we obtain

$$\begin{aligned} &A_t(t,x)u + B_t(t,x) + \alpha u\{A_x(t,x)u + B_x(t,x)\} \\ &+ \beta\{A_{xx}(t,x)u + B_{xx}(t,x)\} + \gamma\{A_{xxx}(t,x)u + B_{xxx}(t,x)\} = 0. \end{aligned} \tag{63}$$

Splitting Equation (63) on powers of u yields

$$u^2 : A_x(t,x) = 0, \tag{64}$$
$$u : A_t(t,x) + \alpha B_x(t,x) + \beta A_{xx}(t,x) + \gamma A_{xxx}(t,x) = 0, \tag{65}$$
$$u^0 : B_t(t,x) + \beta B_{xx}(t,x) + \gamma B_{xxx}(t,x) = 0. \tag{66}$$

Now from Equation (64), we have that

$$A(t,x) = A(t), \tag{67}$$

and Equation (65) reduces to

$$A_t(t,x) + \alpha B_x(t,x) = 0. \tag{68}$$

Now using $\eta = A(t)u + B(t,x)$ in Equation (58), we have

$$\alpha\{A(t)u + B(t,x)\} - \xi_t = 0. \tag{69}$$

If we separate Equation (69) on powers of u, we obtain

$$u : A(t) = 0, \tag{70}$$
$$u^0 : \alpha B(t,x) - \xi_t = 0. \tag{71}$$

Since $A(t) = 0$ from Equation (70), it follows from Equation (68) that

$$B_x(t,x) = 0, \tag{72}$$

which implies that

$$B(t,x) = B(t). \tag{73}$$

Using $B(t,x) = B(t)$ in Equation (66), one obtains

$$B(t) = C_2. \tag{74}$$

From Equation (71), we have

$$\xi_t = \alpha C_2, \tag{75}$$

which upon integration yields

$$\xi = \alpha C_2 t + C_3. \tag{76}$$

and finally;

$$\tau = C_1, \tag{77}$$
$$\xi = \alpha C_2 t + C_3, \tag{78}$$
$$\eta = C_2. \tag{79}$$

We have obtained a three-dimensional Lie algebra of symmetries spanned by

$$X_1 = \frac{\partial}{\partial t}, \tag{80}$$

$$X_2 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \tag{81}$$

$$X_3 = \frac{\partial}{\partial x}. \tag{82}$$

Remark 10 *The kdv burgers Equation (1) has a three-dimensional Lie algebra of point symmetries.*

3.2. *Commutator Table for Symmetries*

We evaluate the commutation relations for the symmetry generators. By definition of Lie bracket [21], for example, we have that

$$[X_3, X_1] = X_3X_1 - X_1X_3 = \left(\frac{\partial}{\partial x}\frac{\partial}{\partial t}\right) - \left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = 0. \tag{83}$$

Remark 11 *The remaining commutation relations are obtained analogously. We present all commutation relations in table (1) below.*

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	αX_3	0
X_2	$-\alpha X_3$	0	0
X_3	0	0	0

Table 1. A commutator table for Lie algebra of kdv-burgers equation.

3.3. *Group Transformations*

The corresponding one-parameter group of transformations can be determined by solving the Lie equations [22]. Let T_{ϵ_i} be the group of transformations for each $X_i, i = 1, 2, 3$. We display how to obtain T_{ϵ_i} from X_i by finding one-parameter group for the infinitesimal generator X_1, namely,

$$X_1 = \frac{\partial}{\partial t}. \tag{84}$$

In particular, we have the Lie equations

$$\begin{aligned} \frac{d\bar{t}}{d\epsilon_1} &= 1, \quad \bar{t}\Big|_{\epsilon_1=0} = t, \\ \frac{d\bar{x}}{d\epsilon_1} &= 0, \quad \bar{x}\Big|_{\epsilon_1=0} = x, \\ \frac{d\bar{u}}{d\epsilon_1} &= 0, \quad \bar{u}\Big|_{\epsilon_1=0} = u. \end{aligned} \tag{85}$$

Solving the system (85) one obtains,

$$\bar{t} = t + \epsilon_1, \quad \bar{x} = x, \quad \bar{u} = u, \tag{86}$$

and hence the one-parameter group T_{ϵ_1} corresponding to the operator X_1 is

$$T_{\epsilon_1} : \quad (\bar{t}, \bar{x}, \bar{u}) = (t + \epsilon_1, x, u). \tag{87}$$

All the three one-parameter groups are presented below :

$$\begin{aligned} T_{\epsilon_1}: \quad & (\bar{t}, \bar{x}, \bar{u}) = (t + \epsilon_1, x, u), \\ T_{\epsilon_2}: \quad & (\bar{t}, \bar{x}, \bar{u}) = (t, x + \alpha\epsilon_2 t, u + \epsilon_2), \\ T_{\epsilon_3}: \quad & (\bar{t}, \bar{x}, \bar{u}) = (t, x + \epsilon_3, u). \end{aligned} \tag{88}$$

3.4. *Symmetry transformations*

We now show how the symmetries we have obtained can be used to transform special exact solutions of the kdv- burgers equation into new solutions. The Lie group analysis vouches for fundamental ways of constructing exact solutions of PDEs, that is, group transformations of known solutions and construction of group-invariant solutions. We will illustrate these methods with examples. If $\bar{u} = g(\bar{t}, \bar{x})$ is a solution of equation (1)

$$\phi(t, x, u, \epsilon) = g(f_1(t, x, u, \epsilon), f_2(t, x, u, \epsilon)), \tag{89}$$

is also a solution. The one parameter groups dictate to the following generated solutions:

$$\begin{aligned} T_{\epsilon_1}: u =& g(t + \epsilon_1, x), \\ T_{\epsilon_2}: u =& g(t, x + \alpha\epsilon_2 t) - \epsilon_2, \\ T_{\epsilon_3}: u =& g(t, x + \epsilon_3). \end{aligned} \tag{90}$$

3.5. *Construction of Group-Invariant Solutions*

Now we compute the group invariant solutions of KdV-Burger's equation.

(i). $X_1 = \frac{\partial}{\partial t}$

The associated Lagrangian equations

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}, \tag{91}$$

yield two invariants, $J_1 = x$ and $J_2 = u$. Thus using $J_2 = \Phi(J_1)$, we have

$$u(t, x) = F(x). \tag{92}$$

The derivatives are given by :

$$\begin{aligned} u_t =& 0 \\ u_x =& F'(x), \\ u_{xx} =& F''(x) \\ u_{xxx} =& F'''(x). \end{aligned}$$

If we substitute these derivatives into Equation (1) , we obtain the third nonlinear order ordinary differential equation

$$\alpha F(x)F'(x) + \beta F''(x) + \gamma F'''(x) = 0,$$

and upon integration one obtains

$$\frac{\alpha}{2}F^2(x) + \beta F'(x) + \gamma F''(x) = C_1,$$

where C_1 is an arbitrary constant of integration.

Thus the group-invariant solution associated to the X_1 is

$$u(t, x) = F(x).$$

(ii). $X_2 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$ The associated Lagrangian is given by

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\alpha t} = \frac{\mathrm{d}u}{1}. \tag{93}$$

This gives the constants $J_1 = t$ and $J_2 = u - \frac{x}{\alpha t}$, giving the solution

$$u(t, x) = \phi(t) + \frac{x}{\alpha t}. \tag{94}$$

We obtain the derivatives as follows:

$$u_t = \phi'(t) - \frac{x}{\alpha t^2}, \tag{95}$$

$$u_x = \frac{1}{\alpha t}, \tag{96}$$

$$u_{xx} = u_{xxx} = 0. \tag{97}$$

If we substitute the above derivatives in Equation (1), we obtain the second order ordinary differential equation

$$\phi'(t) + \frac{\phi(t)}{t} = 0, \tag{98}$$

whose integration gives

$$\phi(t) = \frac{C_2}{t}, \tag{99}$$

where C_2 is an arbitrary constant of integration. Hence the group-invariant solution is

$$u(t, x) = \frac{x + C_3}{\alpha t}, \quad t \neq 0 \ \ C_3 = \alpha C_2. \tag{100}$$

(iii). $X_3 = \frac{\partial}{\partial x}$ The Lagrangian system associated with the operator X_3 is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}, \tag{101}$$

whose invariants are $J_1 = t$ and $J_2 = u$. So, $u = \psi(t)$ is the group-invariant solution. The derivatives are

$$u_t = \psi'(t) \tag{102}$$

$$u_x = u_{xx} = u_{xxx} = 0. \tag{103}$$

Substituting of $u = \psi(t)$ into (1) yields

$$\psi'(t) = 0. \tag{104}$$

Equation (104) is a first order linear ODE which is satisfied by the function

$$\psi(t) = C_3. \tag{105}$$

Thus the group-invariant solution for (1)is given by

$$u(t, x) = C_3. \tag{106}$$

3.6. *Soliton*

We obtain a traveling wave solution of the KdV-Burgers Equation(1) by considering a linear combination of the symmetries X_3 and X_1, namely, [20]

$$X = cX_3 + X_1 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad \text{for some constant } c. \tag{107}$$

The characteristic equations are

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{c} = \frac{\mathrm{d}u}{0} \tag{108}$$

We get two invariants, $J_1 = x - ct$ and $J_2 = u$. So the group-invariant solution is

$$u(t, x) = \Phi(x - ct), \tag{109}$$

for some arbitrary function Φ and c the velocity of the wave. The derivatives of $\Phi(x-ct)$ are given by

$$u_t = -c\Phi'(x - ct) \tag{110}$$
$$u_x = \Phi'(x - ct) \tag{111}$$
$$u_{xx} = \Phi''(x - ct) \tag{112}$$
$$u_{xxx} = \Phi'''(x - ct). \tag{113}$$

Substitution of u into (1) yields a third order ordinary differential equation

$$-c\Phi' + \alpha\Phi\Phi' + \beta\Phi'' + \gamma\Phi''' = 0, \tag{114}$$

with constant coefficients. Integration with respect to Φ yields

$$-c\Phi + \alpha\frac{\Phi^2}{2} + \beta\Phi' + \gamma\Phi'' = 0, \tag{115}$$

where 0 has been chosen as a constant of integration. Multiply Equation (115) by $-2\Phi'$ to give

$$2c\Phi\Phi' - \alpha\Phi^2\Phi' - 2\beta\Phi'^2 - 2\gamma\Phi''\Phi' = 0, \tag{116}$$

and integration yields

$$c\Phi^2 - \alpha\frac{\Phi^3}{3} - 2\beta\int(\mathrm{d}\Phi)^2 - 2\gamma\mathrm{d}\Phi = 0, \tag{117}$$

Clearly,

$$u(t, x) = \Phi(x - ct), \tag{118}$$

is a soliton, where $\Phi(x - ct)$ satisfies Equation (117).

4. Conservation laws of equation (1)

We will employ multipliers in the construction of conservation laws.

4.1. *The multipliers*

We make use of the Euler-Lagrange operator defined as defined in [22] to look for a zeroth order multiplier $\Lambda = \Lambda(t, x, u)$. The resulting determining equation for computing Λ is

$$\frac{\delta}{\delta u}\left[\Lambda\{u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx}\}\right] = 0. \tag{119}$$

where

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_t\frac{\partial}{\partial u_t}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}-D_x^3\frac{\partial}{\partial u_{xxx}}\dots \tag{120}$$

Expansion of Equation (119) yields

$$\Lambda_u(u_t+\alpha uu_x+\beta u_{xx}+\gamma u_{xxx})+\alpha u_x\Lambda-D_t(\Lambda)-\alpha D_x(u\Lambda)+\beta D_x^2(\Lambda)-\gamma D_x^3(\Lambda)=0. \tag{121}$$

Invoking the total derivatives

$$D_t = \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tx}\frac{\partial}{\partial u_x}+u_{tt}\frac{\partial}{\partial u_t}+\cdots, \tag{122}$$

$$D_x = \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{tx}\frac{\partial}{\partial u_t}+\dots. \tag{123}$$

on Equation (121) produces

$$\begin{aligned}&\Lambda_t+\alpha u\Lambda_x-\beta\Lambda_{xx}+\gamma\Lambda_{xxx}+(3\gamma\Lambda_{xxu}-2\beta\Lambda_{xu})u_x+(3\gamma\Lambda_{xu})u_{xx}+(3\gamma\Lambda_{uu})u_{xx}u_x\\&+(3\gamma\Lambda_{xuu}-\beta\Lambda_{uu})u_x^2+(\gamma\lambda_{uu})u_x^3=0\end{aligned} \tag{124}$$

Splitting Equation (124) on derivatives of u produces an overdetermined system of four partial differential
equations, namely,

$$u_x : 3\gamma\Lambda_{xxu}-2\beta\Lambda_{xu}=0, \tag{125}$$
$$u_{xx} : 3\gamma\Lambda_{xu}=0, \tag{126}$$
$$u_xu_{xx} : 3\gamma\Lambda_{uu}=0, \tag{127}$$
$$u_x^2 : 3\gamma\Lambda_{xuu}-\beta\Lambda_{uu}=0, \tag{128}$$
$$u_x^3 : \Lambda_{uu}=0, \tag{129}$$
$$\text{rest} : \Lambda_t+\alpha u\Lambda_x-\beta\Lambda_{xx}+\gamma\Lambda_{xxx}=0 \tag{130}$$

By observation, we have the following reduced system which is necessarily sufficient for the preceding one

$$\Lambda_{xu}=0, \tag{131}$$
$$\Lambda_{uu}=0, \tag{132}$$
$$\Lambda_t+\alpha u\Lambda_x-\beta\Lambda_{xx}+\gamma\Lambda_{xxx}=0 \tag{133}$$

By Equations (131-132), we have

$$\Lambda=A(t)u+B(t,x), \tag{134}$$

using Equation (134) in (133) yields

$$A_t(t)u+B_t(t,x)+\alpha uB_x(t,x)-\beta B_{xx}(t,x)+\gamma B_{xxx}(t,x)=0. \tag{135}$$

Separating Equation (135) on powers of u gives

$$u : A_t(t)+\alpha B_x(t,x)=0, \tag{136}$$
$$u^0 : B_t(t,x)-\beta B_{xx}(t,x)+\gamma B_{xxx}(t,x)=0. \tag{137}$$

By Equation (136), we have

$$B_x(t,x)=-\frac{A_t(t)}{\alpha}\implies B_{xx}(t,x)=0=B_{xxx}(t,x)=B_t(t,x). \tag{138}$$

Consequently,

$$B(t,x) = -\frac{A_t(t)}{\alpha}x + C(t). \tag{139}$$

Using $B_t(t,x) = 0$ in Equation (139) yields

$$-\frac{A_{tt}(t)}{\alpha}x + C_t(t) = 0. \tag{140}$$

If we separate Equation (140) on powers of x, we obtain

$$x : \frac{A_{tt}(t)}{\alpha} = 0, \tag{141}$$

$$x^0 : C_t(t) = 0, \tag{142}$$

whose integrations yield

$$A(t,x) = \alpha C_1 t + C_2 \tag{143}$$

$$C(t) = C_3. \tag{144}$$

Using Equation (139),

$$B(t,x) = -C_1 x + C_3. \tag{145}$$

We finally ,

$$\Lambda(t,x,u) = C_1(\alpha t u - x) + C_2 u + C_3. \tag{146}$$

Essentially, we extract the three multipliers

$$\begin{aligned} \Lambda_1 &= 1 \\ \Lambda_2 &= u \\ \Lambda_3 &= \alpha t u - x. \end{aligned} \tag{147}$$

Remark 12 *Recall that a multiplier Λ for Equation(1) has the property that for the density $T^t = T^t(t,x,u)$ and flux $T^x = T^x(t,x,u,u_x)$,*

$$\Lambda\left(u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx}\right) = D_t T^t + D_x T^x. \tag{148}$$

We derive a conservation law corresponding to each of the multipliers.

(i). **Conservation law for the multiplier** $\Lambda_1 = 1$

Expansion of equation (148) gives

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = T^t_t + u_t T^t_u + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{xxx} T^x_{u_{xx}}. \tag{149}$$

Splitting Equation (149) on the third derivative of u yields

$$u_{xxx} \quad : \quad T^x_{u_{xx}} = \gamma, \tag{150}$$

$$\text{Rest} \quad : \quad u_t + \alpha u u_x + \beta u_{xx} = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x + u_{xx} T^x_u. \tag{151}$$

The integration of Equation (150) with respect to u_{xx} gives

$$T^x = \gamma u_{xx} + A(t,x,u,u_x). \tag{152}$$

Substituting the expression of T^x from (152) into Equation (149) we get

$$u_t + \alpha u u_x + \beta u_{xx} = T^t_t + T^t_u u_t + A_x + u_x A_u + u_{xx} A_{u_x} \tag{153}$$

which splits on second derivatives of u, to give

$$u_{xx} \;:\; A_{u_x} = \beta, \tag{154}$$
$$\text{Rest} \;:\; u_t + \alpha u u_x = T^t_t + T^t_u u_t + A_x + u_x A_u. \tag{155}$$

Integration of Equation (154) with respect to u_x yields

$$A = \beta u_x + C(t,x,u). \tag{156}$$

Upon using Equation (156) in Equation (153), we have

$$u_t + \alpha u u_x = T^t_t + T^t_u u_t + C_x + u_x C_u, \tag{157}$$

which split on first derivatives to give

$$u_t \;:\; T^t_u = 1, \tag{158}$$
$$u_x \;:\; C_u = \alpha u, \tag{159}$$
$$\text{Rest} \;:\; C_x + T^t_t = 0. \tag{160}$$

Integration of Equations (158-159) yield

$$T^t_u = u + D(t,x) \tag{161}$$
$$C = \frac{\alpha u^2}{2} + E(t,x). \tag{162}$$

By substituting the obtained functions into Equation (160), we have

$$D_t(t,x) + E_x(t,x) = 0. \tag{163}$$

Since $D(t,x)$ and $E(t,x)$ contribute to the trivial part of the conservation law, we take $D(t,x) = E(t,x) = 0$ and obtain the conserved quantities

$$T^t = u, \tag{164}$$
$$T^x = \alpha \frac{u^2}{2} + \beta u_x + \gamma u_{xx}, \tag{165}$$

from which the conservation law corresponding to the multiplier $\Lambda_1 = 1$ is given by

$$D_t(u) + D_x\left(\alpha \frac{u^2}{2} + \beta u_x + \gamma u_{xx}\right) = 0. \tag{166}$$

(ii). **Conservation law for the multiplier $\Lambda_2 = u$**

Expansion of equation (148) gives

$$u\{u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx}\} = T^t_t + u_t T^t_u + T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{xxx} T^x_{u_{xx}}. \tag{167}$$

Splitting Equation (167) on the third derivative of u yields

$$u_{xxx} \;:\; T^x_{u_{xx}} = \gamma u, \tag{168}$$
$$\text{Rest} \;:\; u\{u_t + \alpha u u_x + \beta u_{xx}\} = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x + u_{xx} T^x_u. \tag{169}$$

The integration of Equation (168) with respect to u_{xx} gives

$$T^x = \gamma u u_{xx} + A(t,x,u,u_x). \tag{170}$$

Substituting the expression of T^x from (170) into Equation (167) we get

$$u\{u_t + \alpha u u_x + \beta u_{xx}\} = T^t_t + T^t_u u_t + \gamma u_x u_{xx} + A_x + u_x A_u + u_{xx} A_{u_x} \tag{171}$$

which splits on second derivatives of u, to give

$$\begin{aligned} u_{xx} &: A_{u_x} = \beta u - \gamma u_x, && (172)\\ \text{Rest} &: u\{u_t + \alpha u u_x\} = T^t_t + T^t_u u_t + A_x + u_x A_u. && (173)\end{aligned}$$

Integration of Equation (172) with respect to u_x yields

$$A = -\gamma \frac{u_x^2}{2} + \beta u u_x + B(t, x, u). \tag{174}$$

Upon using Equation (174) in Equation (171), we have

$$u\{u_t + \alpha u u_x\} = T^t_t + T^t_u u_t + B_x(t, x, u) + u_x B_u(t, x, u), \tag{175}$$

which split on first derivatives to give

$$\begin{aligned} u_t &: T^t_u = u, && (176)\\ u_x &: B_u = \alpha u^2, && (177)\\ \text{Rest} &: B_x + T^t_t - \beta u u_{xx} = 0. && (178)\end{aligned}$$

Integration of Equations (176-177) yield

$$T^t_u = \frac{u^2}{2} + C(t, x) \tag{179}$$

$$C = \frac{\alpha u^3}{3} + D(t, x). \tag{180}$$

By substituting the obtained functions into Equation (178), we have

$$C_t(t, x) + D_x(t, x) - \beta u u = 0. \tag{181}$$

Since $D(t, x) = C(t, x) = 0$ contribute to the trivial part of the conservation law, we take and obtain the conserved quantities

$$T^t = \frac{u^2}{2}, \tag{182}$$

$$T^x = \alpha \frac{u^3}{3} + \gamma u u_{xx} - \gamma \frac{u_x^2}{2} + \beta x u u_{xx}, \tag{183}$$

from which the conservation law corresponding to the multiplier $\Lambda_2 = u$ is given by

$$D_t\left(\frac{u^2}{2}\right) + D_x\left(\alpha \frac{u^3}{3} + \gamma u u_{xx} - \gamma \frac{u_x^2}{2} + \beta x u u_{xx}\right) = 0. \tag{184}$$

(iii). **Conservation law for the multiplier $\Lambda_3 = \alpha t u - x$**

Expansion of equation (148) gives

$$(\alpha t u - x)\{u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx}\} = T^t_t + u_t T^t_u + +T^x_x + u_x T^x_u + u_{xx} T^x_{u_x} + u_{xxx} T^x_{u_{xx}}. \tag{185}$$

Splitting Equation (185) on the third derivative of u yields

$$\begin{aligned} u_{xxx} &: T^x_{u_{xx}} = \gamma(\alpha t u - x), && (186)\\ \text{Rest} &: (\alpha t u - x)\{u_t + \alpha u u_x + \beta u_{xx}\} = T^t_t + T^t_u u_t + T^x_x + T^x_u u_x + u_{xx} T^x_u. && (187)\end{aligned}$$

The integration of Equation (186) with respect to u_{xx} gives

$$T^x = \gamma(\alpha t u - x) u_{xx} + A(t, x, u, u_x). \tag{188}$$

Substituting the expression of T^x from (188) into Equation (185) we get

$$(\alpha tu - x)u_t + \alpha uu_x + \beta u_{xx} = T^t_t + T^t_u u_t + A_x + u_x A_u + u_{xx} A_{u_x} \tag{189}$$

which splits on second derivatives of u, to give

$$u_{xx} \quad : \quad A_{u_x} = \beta(\alpha tu - x) - \gamma(\alpha tu_x - 1), \tag{190}$$

$$\text{Rest} \quad : \quad u_t + \alpha uu_x = T^t_t + T^t_u u_t + A_x + u_x A_u. \tag{191}$$

Integration of Equation (154) with respect to u_x yields

$$A = \beta(\alpha tu - x)u_x - \gamma(\frac{\alpha tu_x^2}{2} - u_x) + C(t, x, u). \tag{192}$$

Upon using Equation (192) in Equation (189), we have

$$(\alpha tu - x)\{u_t + \alpha uu_x\} = T^t_t + T^t_u u_t + C_x + u_x C_u, \tag{193}$$

which split on first derivatives to give

$$u_t \quad : \quad T^t_u = \alpha tu - x, \tag{194}$$

$$u_x \quad : \quad C_u = \alpha^2 u^2 t - \alpha xu - \beta(\alpha tu_x - 1), \tag{195}$$

$$\text{Rest} \quad : \quad C_x + T^t_t = 0. \tag{196}$$

Integration of Equations (194-195) yield

$$T^t = \frac{\alpha tu^2}{2} - xu + D(t, x) \tag{197}$$

$$C = \frac{\alpha^2 u^3 t}{3} - \frac{\alpha xu^2}{2} - \beta(\alpha tu_x - 1)u + E(t, x). \tag{198}$$

By substituting the obtained functions into Equation (196), we have

$$D_t(t, x) + E_x(t, x) - \alpha\beta tuu_{xx} = 0. \tag{199}$$

Now setting

$$D_t(t, x) = \alpha\beta tuu_{xx}, \tag{200}$$

$$E_x(t, x) = 0, \tag{201}$$

gives

$$D(t, x) = \frac{\alpha\beta t^2 uu_{xx}}{2} + F(x), \tag{202}$$

$$E(t, x) = E(t), \tag{203}$$

Since $F(x)$ and $E(t)$ contribute to the trivial part of the conservation law, we take $F(x) = E(t) = 0$ and obtain the conserved quantities

$$T^t = \frac{\alpha tu^2}{2} - xu + \frac{\alpha\beta t^2 uu_{xx}}{2} \tag{204}$$

$$T^x = \gamma(\alpha tu - x)u_{xx} - \gamma(\frac{\alpha tu_x^2}{2} - u_x) + \frac{\alpha^2 tu^3}{3} - \frac{\alpha xu^2}{2} + \beta(\alpha tu - x)u_x - \beta(\alpha tu_x - 1)u. \tag{205}$$

from which the conservation law corresponding to the multiplier $\Lambda_3 = (\alpha tu - x)$ is given by

$$D_t\left(\frac{\alpha tu^2}{2} - xu + \frac{\alpha\beta t^2 uu_{xx}}{2}\right)$$
$$+D_x\left(\gamma(\alpha tu - x)u_{xx} - \gamma(\frac{\alpha tu_x^2}{2} - u_x) + \frac{\alpha^2 tu^3}{3} - \frac{\alpha xu^2}{2} + \beta(\alpha tu - x)u_x - \beta(\alpha tu_x - 1)u\right) = 0.$$

Remark 13 *It can be shown that the two sets of conserved quantities are conservation laws. Given that $\Lambda_1 = 1$ is a multiplier shows that the KdVb equation is itself a conversation law.*

5. Conclusion

In this manuscript, a three-dimensional Lie algebra of Lie point symmetries has been applied to study a KdVB equation. A commutator table has been constructed for the obtained Lie algebra. We have also used symmetry reductions to compute exact group-invariant solutions, including a soliton. Conservation laws have also been derived for the model with the use of zeroth order multipliers.

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