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Original Article

A New Subclass of Multivalent Functions Associated with Fractional Differential Operator

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Abstract - *In the present research paper, we define a new subclass of multivalent function associated with Fractional differential operator, with the help of Fractional differential operator we derive some results for a function belonging to a new subclass of multivalent functions. The results mainly includes Coefficient estimate, Radii of starlikeness, convexity and close to convexity property of a function belonging to new subclass.*

Keywords – *Fractional differential operator, Analytic function, Univalent function, Multivalent function, Coefficient estimate.*

1. Introduction

Let $\mathcal{F}(p)$ be the class of analytic and p- valent function $f(z)$ of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \tag{1.1}$$

where p is some natural number, n is a natural number.

The function $f(z)$ is analytic function and p valent function in the open unit disc $D = \{z : |z| < 1\}$
A function $f(z) \in \mathcal{F}(p)$ is a p valent starlike function of order ω if it satisfy the following condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \omega \quad ,z \in D \ , \ 0 \leq \omega < p \ \ ,p \in \mathbb{N} \tag{1.2}$$

and a function $f(z) \in \mathcal{F}(p)$ is a p valent convex function of order ω if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \omega \ ,z \in D \ , \ 0 \leq \omega < p \ \ ,p \in \mathbb{N} \tag{1.3}$$

To form a new class of multivalent function, we use the following definitions.

Definition 1.1: Let $f(z) \in \mathcal{F}(p)$ and $g(z) \in \mathcal{F}(p)$ defined in the form as

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \qquad and \qquad g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$$

Then their convolution product (Hadamard product) is denoted by $(f * g)(z)$ or $(g * f)(z)$ and the Hadmard product is defined as

$$(f * g)(z) = (g * f)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k \tag{1.4}$$

Definition 1.2: The Fractional derivative of order λ is denoted by D_z^{λ} and defined as

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)d\xi}{(z-\xi)^{\lambda}} \qquad 0 \leq \lambda < 1 \tag{1.5}$$

where the function $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and the multiplicity of $(z-t)^{-\lambda}$ can be removed by taking log (z – t) to be real when $z - t > 0$ $D_Z^{\lambda}\, z^p$ in terms of gamma function is expressed as

$$D_Z^{\lambda}\, z^p = \frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} z^{p-\lambda} \qquad 0 \le \lambda < 1 \tag{1.6}$$

Definition 1.3: The fractional derivative operator of order $m + \lambda$ of a function $f(z)$ is denoted by $D_Z^{m+\lambda} f(z)$ and it is defined as $D_z^{m+\lambda} f(z) = \frac{d^m}{dz^m} D_z^{\lambda} f(z)$

$$D_z^{m+\lambda}\, z^k = \frac{\Gamma(k+1)}{\Gamma(k-(m+\lambda)+1)} z^{k-(m+\lambda)} \qquad 0 \le \lambda < 1 \tag{1.7}$$

Where $m \in \mathbb{N} \cup \{0\}$

A new class of multivalent function form by using Fractional Differential Operator is defined in the following definition.

Definition 1.4: Let $f(z) \in \mathcal{F}(p)$ is a member of new class $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ if $f(z)$ satisfies the following condition

$$Re\left\{\frac{(1-\alpha)\, z\left(D_z^{m+\gamma} f(z)\right)' + \delta z^2\left(D_z^{m+\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{m+\gamma} f(z)\right) + (1-\alpha)\delta\, z\left(D_z^{m+\gamma} f(z)\right)' + \alpha z^2\left(D_z^{m+\gamma} f(z)\right)''}\right\} > \beta \tag{1.8}$$

where $z \in D$, $0 \le \alpha < 1$, $0 \le \beta < p$, $0 \le \gamma < 1$ and $0 \le \delta < 1$

We get previously studied subclass of multivalent and univalent function .These subclasses was studied by Khosravianarb et.al (2017), Altintas et.al (1995).

Particular Cases are as follow

1. If $m = 0, \alpha = 0$ in (1.8) then we get $Re\left\{\frac{z\left(D_z^{\gamma} f(z)\right)' + \delta z^2\left(D_z^{\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{\gamma} f(z)\right) + \delta\, z\left(D_z^{\gamma} f(z)\right)'}\right\} > \beta$. This class was studied by Khosravianarb et al. [24]
2. If $m = 0, \alpha = 0, \gamma = 0$ in (1.8) then we get $Re\left\{\frac{zf'(z) + \delta z^2 f''(z)}{(1-\delta)f(z) + \delta\, z f'(z)}\right\} > \beta$ and this was studied by Altintas et al. [19]
3. On taking $m = 0, \alpha = 0, \gamma = 0, \delta = 0$ in (1.8) then we get $\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$, $0 \le \beta < p$ this represent the class of p valent starlike function of order β.
4. On assuming $m = 0, \alpha = 0, \gamma = 0, \delta = 1$ in (1.8) then we get $\left\{1 + \frac{z f''(z)}{f'(z)}\right\} > \beta$, $0 \le \beta < p$ and this represent the class of p valent convex function of order β.

Results related to new class $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$

2. Coefficient Bound

Theorem 2.1 A function $f(z) \in \mathcal{F}(p)$, $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ then $f(z)$ is a member of class $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty}\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}\, a_k \le 1 \tag{2.1}$$

where $M_{p,k} = \frac{\Gamma(k+1)\Gamma(p-(m+\gamma)+1}{\Gamma(p+1)\Gamma k-(m+\gamma)+1}$

$z \in D$, $0 \le \alpha < 1$, $0 \le \beta < p$, $0 \le \gamma < 1$ and $0 \le \delta < 1$

Proof: Let us consider that $f(z) \in V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ so we have

$$Re\left\{\frac{(1-\alpha)\, z\left(D_z^{m+\gamma} f(z)\right)' + \delta z^2\left(D_z^{m+\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{m+\gamma} f(z)\right) + (1-\alpha)\delta\, z\left(D_z^{m+\gamma} f(z)\right)' + \alpha z^2\left(D_z^{m+\gamma} f(z)\right)''}\right\} > \beta \tag{2.2}$$

Since $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and

$$D_z^{m+\gamma} f(z) = \frac{\Gamma(p+1)}{\Gamma(p-(m+\gamma)+1}z^{p-(m+\gamma)} - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)}{\Gamma k-(m+\gamma)+1}\ a_k\, z^{k-(m+\gamma)} \tag{2.3}$$

So we have

$$\left(D_z^{m+\gamma} f(z)\right)' = \frac{\Gamma(p+1)}{\Gamma(p-(m+\gamma)}z^{p-(m+\gamma)-1} - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)}{\Gamma k-(m+\gamma)}\ a_k\, z^{k-(m+\gamma)-1} \tag{2.4}$$

$$\left(D_z^{m+\gamma} f(z)\right)'' = \frac{\Gamma(p+1)}{\Gamma(p-(m+\gamma)-1}z^{p-(m+\gamma)-2} - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)}{\Gamma k-(m+\gamma)-1}\ a_k\, z^{k-(m+\gamma)-2} \tag{2.5}$$

Using (2.3), (2.4) and (2.5) in (2.2) we get

$$Re\left\{\frac{\frac{\Gamma(p+1)}{\Gamma(p-(m+\gamma)}[A+\delta E_p]z^{p-(m+\gamma)} - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)}{\Gamma k-(m+\gamma)}[A+\delta E_k]a_k z^{k-(m+\gamma)}}{\frac{\Gamma(p+1)}{\Gamma(p-(m+\gamma)+1}\left[B+\left(C+\alpha E_p\right)F_p\right]z^{p-(m+\gamma)} - \sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)}{\Gamma k-(m+\gamma)+1}\left[B+(C+\alpha E_k)F_k\right]z^{p-(m+\gamma)}a_k}\right\} > 0 \tag{2.6}$$

For Convenience, we take

$A = (1-\alpha), B = (1-\delta), C = (1-\alpha)\delta, E_p \ = p-(m+\gamma)-1$
$E_k \ = k-(m+\gamma)-1, F_p \ = p-(m+\gamma), F_K \ = k-(m+\gamma)$

on assuming the value of z to be real and let $|z| \to 1$ then we get

$$\sum_{k=n+p}^{\infty}\frac{\Gamma(k+1)\Gamma p-(m+\gamma)+1}{\Gamma(p+1)\Gamma k-(m+\gamma)+1}\left[\,F_k[A-\beta C+E_k(\delta-\beta\alpha)]-\beta B\right]a_k \ \le \left[\,F_p\left[A-\beta C+E_p(\delta-\beta\alpha)\right]-\beta B\right] \tag{2.7}$$

on simplifying we get,

$$\sum_{k=n+p}^{\infty} M_{p,k}\frac{\left[\,F_k[A(1-\beta\delta)+E_k(\delta-\beta\alpha)]-\beta B\right]}{\left[\,F_p[A(1-\beta\delta)+E_p(\delta-\beta\alpha)]-\beta B\right]}a_k \le 1$$

$$\sum_{k=n+p}^{\infty}\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k}\ a_k \le 1 \tag{2.8}$$

where $M_{p,k} = \frac{\Gamma(k+1)\Gamma p-(m+\gamma)+1}{\Gamma(p+1)\Gamma k-(m+\gamma)+1}$

Conversely: Let the inequality (2.1) is true, now we have to prove that $f(z) \in V_{m,n,p}(\alpha,\beta,\gamma,\delta)$ for this we have to show that

$$Re\left\{\frac{(1-\alpha)\,z\left(D_z^{m+\gamma} f(z)\right)' + \delta z^2\left(D_z^{m+\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{m+\gamma} f(z)\right) + (1-\alpha)\delta\, z\left(D_z^{m+\gamma} f(z)\right)' + \alpha z^2\left(D_z^{m+\gamma} f(z)\right)''}\right\} > \beta$$

By using Lemma [27] if $w = u+iv$ then $Re\ w \ \ge \beta \ \Leftrightarrow |w-(1+\beta)| \le |w+(1-\beta)|$
Let $S = |w-(1+\beta)|$

And

$$w \ = \frac{(1-\alpha)\,z\left(D_z^{m+\gamma} f(z)\right)' + \delta z^2\left(D_z^{m+\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{m+\gamma} f(z)\right) + (1-\alpha)\delta\, z\left(D_z^{m+\gamma} f(z)\right)' + \alpha z^2\left(D_z^{m+\gamma} f(z)\right)''} \tag{2.9}$$

$$S = \left|\frac{(1-\alpha)\,z\left(D_z^{m+\gamma} f(z)\right)' + \delta z^2\left(D_z^{m+\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{m+\gamma} f(z)\right) + (1-\alpha)\delta\, z\left(D_z^{m+\gamma} f(z)\right)' + \alpha z^2\left(D_z^{m+\gamma} f(z)\right)''} - (1+\beta)\right| \tag{2.10}$$

and $T = |w+(1-\beta)|$

$$T = \left|\frac{(1-\alpha)\,z\left(D_z^{m+\gamma} f(z)\right)' + \delta z^2\left(D_z^{m+\gamma} f(z)\right)''}{(1-\delta)\left(D_z^{m+\gamma} f(z)\right) + (1-\alpha)\delta\, z\left(D_z^{m+\gamma} f(z)\right)' + \alpha z^2\left(D_z^{m+\gamma} f(z)\right)''} + (1-\beta)\right| \tag{2.11}$$

From (2.10) and (2.11), $T-S > 0$
i.e. $|w+(1-\beta)| - |w-(1+\beta)| > 0$ which implies $Re(w) > \beta$
so, the inequality (1.8) is true which implies $f(z) \in V_{m,n,p}(\alpha,\beta,\gamma,\delta)$
Hence the proof of theorem 1 is complete here.

Corollary 1: Let the function $f(z) \in V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ then

$$a_k \leq \left\{\frac{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}\frac{1}{M_{p,k}} \tag{2.12}$$

where $k = n + p$, p is some natural number, n is a natural number.

3. Radii of Starlikeness, Convexity and Close to Convexity Property

In this section, we find the result related to Radii of starlikeness, convexity and close to convexity of the function $f(z)$ belonging
to the new subclass $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$

Theorem 3.1 Let the function $f(z)$ defined as $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ is a member of class $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$, Then the given function $f(z)$ is p-valent close to convex of order $\lambda\,; 0 \leq \lambda < p$ in $|z| < r_1'$, where

$$r_1' = \inf_{k \geq n+p}\left\{\left(\frac{p-\lambda}{k}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k}\right\}^{\frac{1}{k-p}} \tag{3.1}$$

Proof: To prove $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ is p-valent close to convex of order $\lambda\,; 0 \leq \lambda < p$ in $|z| < r_1'$ for this it is sufficient to show that show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \lambda \qquad |z| < r_1' \tag{3.2}$$

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k z^{k-1}}{z^{p-1}} - p\right|$$
$$= \left|\frac{\sum_{k=n+p}^{\infty} k a_k z^{k-1}}{z^{p-1}}\right|$$
$$\leq \sum_{k=n+p}^{\infty} k a_k\, |z|^{k-p} \tag{3.3}$$

The inequality (3.2) is less than or equal to $p - \lambda$ if

$$\sum_{k=n+p}^{\infty} \left(\frac{k}{p-\lambda}\right) a_k\, |z|^{k-p} \leq 1 \tag{3.4}$$

Since, the function $f(z) \in V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty}\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k}\, a_k \leq 1$$

The inequality (3.2) is hold true if

$$\left(\frac{k}{p-\lambda}\right)|z|^{k-p} \leq \left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k}$$

or, we have

$$|z|^{k-p} \leq \left(\frac{p-\lambda}{k}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k} \tag{3.5}$$

Thus,

$$|z| < r_1' = \inf_{k \geq n+p}\left\{\left(\frac{p-\lambda}{k}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k}\right\}^{\frac{1}{k-p}}$$

So it is proved that, the given function $f(z)$ is p-valent close to convex of order λ.

Theorem 3.2 Let the function $f(z)$ defined as $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ be in the class $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$, Then the given function $f(z)$ is a p-valent starlike of order λ ; $0 \leq \lambda < p$ in $|z| < r_2'$,where

$$r_2' = \inf_{k \geq n+p}\left\{\left(\frac{p-\lambda}{k-\lambda}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\}M_{p,k}\right\}^{\frac{1}{k-p}} \tag{3.6}$$

Proof: To prove the function $f(z)$ is p-valent starlike of order $\lambda\,; 0 \le \lambda < p$ in $|z| < r_2'$ for this it is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \lambda \qquad |z| < r_2' \tag{3.7}$$

From above inequality

$$\left|\frac{zf'(z)}{f(z)} - p\right| = \left|\frac{z(pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k z^{k-1})}{z^p - \sum_{k=n+p}^{\infty} a_k z^k} - p\right|$$

$$= \left|\frac{\sum_{k=n+p}^{\infty}(k-p)a_k z^k}{z^p - \sum_{k=n+p}^{\infty} a_k z^k}\right|$$

$$\le \frac{\sum_{k=n+p}^{\infty}(k-p)a_k |z|^{k-p}}{1-\sum_{k=n+p}^{\infty} a_k |z|^{k-p}} \tag{3.8}$$

The above inequality (3.7) hold true if

$$\sum_{k=n+p}^{\infty} \frac{(k-\lambda)}{(p-\lambda)} a_k\, |z|^{k-p} \le 1 \tag{3.9}$$

Since $f(z) \in V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}\, a_k \le 1$$

So the inequality (3.7) is hold true if

$$\left(\frac{k-\lambda}{p-\lambda}\right) |z|^{k-p} \le \left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}$$

or, we have

$$|z|^{k-p} \le \left(\frac{p-\lambda}{k-\lambda}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k} \tag{3.10}$$

Thus,

$$|z| < r_2' = \inf_{k \ge n+p} \left\{\left(\frac{p-\lambda}{k-\lambda}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}\right\}^{\frac{1}{k-p}}$$

So it is proved that the given function $f(z)$ is p-valent starlike of order λ.

Theorem 3.3 Let the function $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ be in the class $V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ Then the given function $f(z)$ is a p-valent convex function of order $\lambda\,; 0 \le \lambda < p$ in $|z| < r_3'$, where

$$r_3' = \inf_{k \ge n+p} \left\{\left(\frac{p(p-\lambda)}{k(k-\lambda)}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}\right\}^{\frac{1}{k-p}} \tag{3.11}$$

Proof: To prove $f(z)$ is p-valent convex function of order $\lambda\,; 0 \le \lambda < p$ in $|z| < r_3'$ for this we have to show that

$$\left|\frac{zf''(z)}{f'(z)} + (1-p)\right| \le p - \lambda \qquad |z| < r_3' \tag{3.12}$$

From above inequality

$$\left|\frac{zf''(z)}{f'(z)} + (1-p)\right| = \left|\frac{z(p(p-1)z^{p-2} - \sum_{k=n+p}^{\infty} k(k-1)a_k z^{k-2})}{pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k z^{k-1}} + (1-p)\right|$$

$$= \left|\frac{\sum_{k=n+p}^{\infty} k(k-p)a_k z^{k-p}}{p - \sum_{k=n+p}^{\infty} k\alpha_k z^{k-p}}\right|$$

$$\le \frac{\sum_{k=n+p}^{\infty} k(k-p)a_k |z|^{k-p}}{p - \sum_{k=n+p}^{\infty} k a_k |z|^{k-p}}$$

The inequality (3.12) is true if

$$\sum_{k=n+p}^{\infty} \frac{k(k-\lambda)}{p(p-\lambda)} a_k\, |z|^{k-p} \le 1 \tag{3.13}$$

Since, $f(z) \in V_{m,n,p}(\alpha, \beta, \gamma, \delta)$ if and only if

$$\sum_{k=n+p}^{\infty} \left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}\, a_k \le 1$$

The inequality (3.12) is hold true if

$$\frac{k(k-\lambda)}{p(p-\lambda)}|z|^{k-p} \leq \left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}$$

or, we have

$$|z|^{k-p} \leq \frac{k(k-\lambda)}{p(p-\lambda)}\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}$$

Thus,

$$|z| < r_3' = \inf_{k \geq n+p}\left\{\left(\frac{k(k-\lambda)}{p(p-\lambda)}\right)\left\{\frac{(k-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(k-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}{(p-(m+\gamma))[(1-\alpha)(1-\beta\delta)+(p-(m+\gamma)-1)(\delta-\alpha\beta)]-\beta(1-\delta)}\right\} M_{p,k}\right\}^{\frac{1}{k-p}}$$

So it proved that the given function $f(z)$ is p-valent convex function of order λ.

4. Conclusion

In this research paper we studied new subclass of multivalent functions and their properties.

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