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Original Article

Under Homomorphism and Anti-Homomorphism of a Study on (Q, L)-Fuzzy ℓ-Subsemiring of a ℓ-Semiring

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Abstract - *The goal of this study is to investigate the algebraic nature of $[Q, L]$-fuzzyℓ-subsemirings of a ℓ-semiring. We further studied the fundamental hypothesis under homomorphism and anti-homomorphism by looking at a few features of $[Q, L]$-fuzzy ℓ-subsemirings of ℓ-semiring.*

Keywords - *$[Q, L]$- fuzzy subset, $[Q, L]$-fuzzy ℓ-subsemiring, $[Q, L]$-fuzzy relation, Product of $[Q, L]$-fuzzy subsets, Pseudo $[Q, L]$-fuzzy coset.*

1. Introduction

Several scientists investigated the idea of fuzzy sets after L.A.Zadeh's presentation [5]. Azriel Rosenfeld [2] described a fuzzy group. A fuzzy subgroups finding was reported and detailed by AsokKumer Ray [1]. R. Biswas [15] established the concept of fuzzy subgroups and anti-fuzzy subgroups. Fuzzy homomorphism, anti-homomorphism, and anti-fuzzy groups were proposed by N. Palaniappan and T. Muthuraj [12]. A.Solairaju and R.Nagarajan devised and studied another mathematical design known as Q-fuzzy subgroups [3]. We introduce the concept of $[Q, L]$- fuzzy ℓ-subsemiring of a ℓ-semiring and discuss its implications.

2. Preliminaries

Definition 2.1 Let X be a set that isn't empty. A function$A_\mu: X \to [0, 1]$ a fuzzy subset A_μ of X.

Definition 2.2 Let X be a set that isn't empty. Let $L = (L, \leq)$ be a lattice with 0 as the least element and 1 as the greatest element and Q be a set that isn't empty. A (Q, L)-fuzzy subset A_μ of a function$A_\mu: X \times Q \to L$.

Definition 2.3 Let Ɽ be a ℓ-semiring and Q be a set that isn't empty. A $[Q, L]$-fuzzy subset A_μ of Ɽ is said to be a $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring [QLFLSLSR] of Ɽ if it meets the following requirements:

$$A_\mu(x + y, q) \geq A_\mu(x, q) \wedge A_\mu(y, q)$$
$$A_\mu(xy, q) \geq A_\mu(x, q) \wedge A_\mu(y, q)$$
$$A_\mu(x \vee y, q) \geq A_\mu(x, q) \wedge A_\mu(y, q)$$

$A_\mu(x \wedge y, q) \geq A_\mu(x, q) \wedge A_\mu(y, q)$,for every x and y in R and q in Q.

Example 2.1 If $(Z, +, \bullet, \vee, \wedge)$ is a ℓ-semiring and $Q = \{p\}$, the (Q, L)-Fuzzy Set A_μ of Z is defined as follows:

$$A_\mu(x, q) = \begin{cases} 0.6 & if x \in < 2 > \\ 0.3 & otherwise \end{cases}$$

$A_\mu$is unmistakably a (Q, L)-Fuzzy ℓ-subsemiring of a ℓ-semiring.

Definition 2.4 Let A and B be any two $[Q, L]$-fuzzy subsets of sets $G$and H, respectively. The product of $A$and B, denoted by $A \times B$, is defined as $A \times B = \{< ((x, y), q), (A \times B)_\mu((x, y), q) >/$ for every x in Ɽ and y in H and $q \in Q\}$, where $(A \times B)_\mu((x, y), q) = A_\mu(x, q) \wedge B_\mu(y, q)$.

Definition 2.5 Let Ɽ and Ɽ$^{|}$ be any two ℓ-semirings Q be a set that isn't empty. Let f: Ɽ → Ɽ$^{|}$ be any function and A_μ be a $[Q,L]$-fuzzy ℓ-subsemiring in Ɽ, V_μ be a $[Q,L]$-fuzzy ℓ-subsemiring in f(Ɽ) = Ɽ$^{|}$, defined by $V_\mu(y,q) = \operatorname{Sup}_{x\in f^{-1}(y)} A_\mu(x,q)$, for every x in Ɽ and y in Ɽ$^{|}$and $q \in Q$. Then A is known as a V_μ preimage under f and is indicated by $f^{-1}(V_\mu)$.

Definition 2.6 Consider A_μ to be a $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring Ɽ and a in Ɽ.The pseudo $[Q,L]$-fuzzy coset $(aA_\mu)^{Ƥ}$is thus defined as $(aA_\mu)^{Ƥ}(x,q) = Ƥ(a)A_\mu(x,q)$, for any x in Ɽ and for some $p \in$ Ƥ and $q \in Q$.

Definition 2.7 Let A_μ be a $[Q,L]$-fuzzy subset in a set Ş, the strongest $[Q,L]$-fuzzy relation on Ş, that is a $[Q,L]$-fuzzy relation V with respect to A_μ given by $V_\mu((x,y),q) = A_\mu(x,q) \wedge A_\mu(y,q)$, for every $x, y \in$ Ş and $q \in Q$.

Definition 2.8 Let Ɽ be a ℓ-semiring and Q be a set that isn't empty. A $[Q,L]$-fuzzy subset A_μ of Ɽ is said to be a $[Q,L]$-anti-fuzzy ℓ-subsemiring [QLAFLSSR] of Ɽ if it meets the following requirements:

$$A_\mu(x+y,q) \le A_\mu(x,q) \vee A_\mu(y,q)$$

$A_\mu(xy,q) \le A_\mu(x,q) \vee A_\mu(y,q)$
$A_\mu(x \vee y,q) \le A_\mu(x,q) \vee A_\mu(y,q)$
$A_\mu(x \wedge y,q) \le A_\mu(x,q) \vee A_\mu(y,q)$,for every x and y in Ɽ and $q \in Q$.

Definition 2.9 Let A_μ be a $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring Ɽ. Then ${A_\mu}^0$is defined as ${A_\mu}^0(x,q) = A_\mu(x,q)/A_\mu(0,q)$,for every x in Ɽ and $q \in Q$, where 0 is the identity element of Ɽ.

3. Properties of $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring

Theorem 3.1 If A and B are two $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring R, then their intersection $A \cap B$ is a $[Q,L]$-fuzzy ℓ-subsemiring of R.

Proof. Let x and y belong to R and q in Q, $A = \{<(x,q), A(x,q)>/x$ in R and $q \in Q\}$ and $B = \{<(x,q), B(x,q)>/x$ in R and $q \in Q$ }. Let $C = A \cap B$ and $C = \{<(x,q), C(x,q)>/\ x$ in R and $q \in Q\}$.

$$\begin{aligned} C(x+y,q) &= A(x+y,q) \wedge B(x+y,q) \\ &\ge \{A(x,q) \wedge A(y,q)\} \wedge \{B(x,q) \wedge B(y,q)\} \\ &\ge \{A(x,q) \wedge B(x,q)\}\{A(y,q) \wedge B(y,q)\} \\ &= C(x,q) \wedge C(y,q). \end{aligned}$$

Therefore, $C(x+y,q) \ge C(x,q) \wedge C(y,q)$, for all x and y in R and $q \in Q$.

$$\begin{aligned} C(xy,q) &= A(xy,q) \wedge B(xy,q) \\ &\ge \{A(x,q) \wedge A(y,q)\} \wedge \{B(x,q) \wedge B(y,q)\} \\ &\ge \{A(x,q) \wedge B(x,q)\} \wedge \{A(y,q) \wedge B(y,q)\} \\ &= C(x,q) \wedge C(y,q). \end{aligned}$$

Therefore, $C(xy,q) \ge C(x,q) \wedge C(y,q)$.

$$\begin{aligned} C(x \vee y,q) &= A(x \vee y,q) \wedge B(x \vee y,q) \\ &\ge \{A(x,q) \wedge A(y,q)\} \wedge \{B(x,q) \wedge B(y,q)\} \\ &\ge \{A(x,q) \wedge B(x,q)\}\{A(y,q) \wedge B(y,q)\} \\ &= C(x,q) \wedge C(y,q). \end{aligned}$$

Therefore, $C(x \vee y,q) \ge C(x,q) \wedge C(y,q)$.

$$\begin{aligned} C(x \wedge y,q) &= A(x \wedge y,q) \wedge B(x \wedge y,q) \\ &\ge \{A(x,q) \wedge A(y,q)\} \wedge \{B(x,q) \wedge B(y,q)\} \\ &\ge \{A(x,q) \wedge B(x,q)\}\{A(y,q) \wedge B(y,q)\} \\ &= C(x,q) \wedge C(y,q). \end{aligned}$$

Therefore, $C(x \wedge y,q) \ge C(x,q) \wedge C(y,q)$,for all x and y in R and $q \in Q$.
Hence $A \cap B$ is a $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring R.

Theorem 3.2 The intersection of a family of $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring R is a $[Q,L]$-fuzzy ℓ-subsemiring of R.

Proof. Let $\{A_i\}_{i\in I}$ be a family of $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$ and$A = \bigcap_{i\in I} A_i$ Then for x and y belongs to R and $q \in Q$, we have

$A(x+y,q) = \inf_{i\in I} A(x+y,q) \geq \inf_{i\in I}\{A_i(x,q) \wedge A_i(y,q)\}$
$= \inf_{i\in I}\left(A_i(x,q) \wedge \inf_{i\in I} A_i(y,q)\right)$
$= A(x,q) \wedge A(y,q).$
Therefore, $A(x+y,q) \geq A(x,q) \wedge A(y,q)$, for all x and y in R and $q \in Q$.
$A(xy,q) = \inf_{i\in I} A(xy,q) \geq \inf_{i\in I}\{A_i(x,q) \wedge A_i(y,q)\}$
$= \inf_{i\in I}\left(A_i(x,q) \wedge \inf_{i\in I} A_i(y,q)\right)$
$= A(x,q) \wedge A(y,q).$
Therefore, $A(xy,q) \geq A(x,q) \wedge A(y,q)$, for all x and y in R and $q \in Q$.
$A(x \vee y,q) = \inf_{i\in I} A(x \vee y,q) \geq \inf_{i\in I}\{A_i(x,q) \wedge A_i(y,q)\}$
$= \inf_{i\in I}\left(A_i(x,q) \wedge \inf_{i\in I} A_i(y,q)\right)$
$= A(x,q) \wedge A(y,q).$
Therefore, $A(x \vee y,q) \geq A(x,q) \wedge A(y,q)$, for all x and y in R and $q \in Q$.
$A(x \wedge y,q) = \inf_{i\in I} A(x \wedge y,q) \geq \inf_{i\in I}\{A_i(x,q) \wedge A_i(y,q)\}$
$= \inf_{i\in I}\left(A_i(x,q) \wedge \inf_{i\in I} A_i(y,q)\right)$
$= A(x,q) \wedge A(y,q).$
Therefore, $A(x \wedge y,q) \geq A(x,q) \wedge A(y,q)$, for all x and y in R and $q \in Q$.
Hence the intersection of a family of $[Q,L]$- fuzzy ℓ-subsemiring of a ℓ-semiring$R$ is a $[Q,L]$- fuzzy ℓ-subsemiring of R.

Theorem 3.3 If A and B be a$[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring $R$and H, respectively, then $A \times B$ is a is a $[Q,L]$-fuzzy ℓ-subsemiring of $R \times H$.
Proof.$A$and$B$ be a $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring $R$and H respectively. Let x_1 and x_2 be in R, y_1 and y_2 be in H and $q \in Q$. Then (x_1,y_1) and (x_2,y_2) are in $R \times H$ and $q \in Q$.Now,
$A \times B[(x_1,y_1)+(x_2,y_2),q] = A \times B\big((x_1+x_2, y_1+y_2),q\big)$
$= A(x_1+x_2,q) \wedge B(y_1+y_2,q)$
$\geq \{A(x_1,q) \wedge A(x_2,q)\} \wedge \{B(y_1,q) \wedge B(y_2,q)\}$
$= \{A(x_1,q) \wedge B(y_1,q)\} \wedge \{A(x_2,q) \wedge B(y_2,q)\}$
$= A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big).$
Therefore, $A \times B[(x_1,y_1)+(x_2,y_2),q] \geq A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big).$
$A \times B[(x_1,y_1)(x_2,y_2),q] = A \times B\big((x_1x_2, y_1y_2),q\big)$
$= A(x_1x_2,q) \wedge B(y_1y_2,q)$
$\geq \{A(x_1,q) \wedge A(x_2,q)\} \wedge \{B(y_1,q) \wedge B(y_2,q)\}$
$= \{A(x_1,q) \wedge B(y_1,q)\} \wedge \{A(x_2,q) \wedge B(y_2,q)\}$
$= A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big).$
Therefore, $A \times B[(x_1,y_1)(x_2,y_2),q] \geq A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big).$
$A \times B[(x_1,y_1) \vee (x_2,y_2),q] = A \times B\big((x_1 \vee x_2, y_1 \vee y_2),q\big)$
$= A(x_1 \vee x_2,q) \wedge B(y_1 \vee y_2,q)$
$\geq \{A(x_1,q) \wedge A(x_2,q)\} \wedge \{B(y_1,q) \wedge B(y_2,q)\}$
$= \{A(x_1,q) \wedge B(y_1,q)\} \wedge \{A(x_2,q) \wedge B(y_2,q)\}$
$= A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big).$
Therefore, $A \times B[(x_1,y_1) \vee (x_2,y_2),q] \geq A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big)$ for all x and y in R and $q \in Q$.
$A \times B[(x_1,y_1) \wedge (x_2,y_2),q] = A \times B\big((x_1 \wedge x_2, y_1 \wedge y_2),q\big)$
$= A(x_1 \wedge x_2,q) \wedge B(y_1 \wedge y_2,q)$
$\geq \{A(x_1,q) \wedge A(x_2,q)\} \wedge \{B(y_1,q) \wedge B(y_2,q)\}$
$= \{A(x_1,q) \wedge B(y_1,q)\} \wedge \{A(x_2,q) \wedge B(y_2,q)\}$
$= A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big).$
Therefore, $A \times B[(x_1,y_1) \wedge (x_2,y_2),q] \geq A \times B\big((x_1,y_1),q\big) \wedge A \times B\big((x_2,y_2),q\big)$ for all x and y in R and $q \in Q$.
Hence $A \times B$ is a $[Q,L]$-fuzzy ℓ-subsemiring of$R \times H$.

Theorem 2.4 Let A be a $[Q,L]$-fuzzy subset of a ℓ-subsemiring of a ℓ-semiring R and V be the strongest $[Q,L]$-fuzzy relation of R. Then A is a $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring R if and only if V is an $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring $R \times R$.

Proof. Suppose that A is an $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$.Then for any$x = (x_1, x_2)$and$y = (y_1, y_2)$are in $R \times R$.Wehave

(i) $V\big((x+y),q\big) = V\big[\big((x_1,x_2)+(y_1,y_2),q\big)\big]$
$= V(x_1+y_1, x_2+y_2, q)$
$= A((x_1+y_1,q) \wedge A\big((x_2+y_2,q)\big)$
$\geq \{A(x_1,q) \wedge A(y_1,q)\} \wedge \{A(x_2,q) \wedge A(y_2,q)\}$
$= \{A\,(x_1,q) \wedge A(x_2,q)\} \wedge \{A(y_1,q) \wedge A(y_2,q)\}$
$= V((x_1,x_2,q) \wedge V\big((y_1,y_2,q)\big)$
$= V(x,q) \wedge V(y,q)$.

Therefore, $V\big((x+y,q)\big) \geq V(x,q) \wedge V(y,q)$, for all x and y in $R \times R$and $q \in Q$.

(ii) $V\big((xy),q\big) = V[(x_1,x_2)(y_1,y_2),q]$
$= V(x_1y_1, x_2y_2, q)$
$= A\big((x_1y_1,q) \wedge A(x_2y_2,q)\big)$
$\geq \{A(x_1,q) \wedge A(y_1,q)\} \wedge \{A(x_2,q) \wedge A(y_2,q)\}$
$= \{A\,(x_1,q) \wedge A(x_2,q)\} \wedge \{A(y_1,q) \wedge A(y_2,q)\}$
$= V((x_1,x_2,q) \wedge V\big((y_1,y_2,q)\big)$
$= V(x,q) \wedge V(y,q)$.

Therefore, $V\big((xy,q)\big) \geq V(x,q) \wedge V(y,q)$, for all x and y in $R \times R$.

(iii) $V\big((x \vee y),q\big) = V[(x_1,x_2) \vee (y_1,y_2),q]$
$= V(x_1 \vee y_1, x_2 \vee y_2, q)$
$= A\big((x_1 \vee y_1,q) \wedge A(x_2 \vee y_2,q)\big)$
$\geq \{A(x_1,q) \wedge A(y_1,q)\} \wedge \{A(x_2,q) \wedge A(y_2,q)\}$
$= \{A\,(x_1,q) \wedge A(x_2,q)\} \wedge \{A(y_1,q) \wedge A(y_2,q)\}$
$= V((x_1,x_2,q) \wedge V\big((y_1,y_2,q)\big)$
$= V(x,q) \wedge V(y,q)$.

Therefore, $V\big((x \vee y,q)\big) \geq V(x,q) \wedge V(y,q)$, for all x and y in $R \times R$.

(iv) $V\big((x \wedge y),q\big) = V[(x_1,x_2) \wedge (y_1,y_2),q]$
$= V(x_1 \wedge y_1, x_2 \wedge y_2, q)$
$= A\big((x_1 \wedge y_1,q) \wedge A(x_2 \wedge y_2,q)\big)$
$\geq \{A(x_1,q) \wedge A(y_1,q)\} \wedge \{A(x_2,q) \wedge A(y_2,q)\}$
$= \{A\,(x_1,q) \wedge A(x_2,q)\} \wedge \{A(y_1,q) \wedge A(y_2,q)\}$
$= V((x_1,x_2,q) \wedge V\big((y_1,y_2,q)\big)$
$= V(x,q) \wedge V(y,q)$.

Therefore, $V\big((x \wedge y,q)\big) \geq V(x,q) \wedge V(y,q)$, for all x and y in $R \times R$.

This proves that V is an $[Q,L]$-fuzzy ℓ-subsemiring of $R \times R$.

Conversely assume that V is an $[Q,L]$-fuzzy ℓ-subsemiring of $R \times R$, then for any $x = (x_1,x_2)$ and $y = (y_1,y_2)$ are in $R \times R$, we have

(i) $A(x_1+y_1,q) \wedge A(x_2+y_2,q) = V\big((x_1+y_1, x_2+y_2),q\big)$
$= V[(x_1,x_2)+(y_1,y_2),q]$
$= V(x+y,q) \geq V(x,q) \wedge V(y,q)$
$= V\big((x_1,x_2),q\big) \wedge V\big((y_1,y_2),q\big)$
$= \big\{\{A(x_1,q) \wedge A(x_2,q)\} \wedge \{A(y_1,q) \wedge A(y_2,q)\}\big\}$.

If $A\big((x_1+y_1),q\big) \leq A\big((x_2+y_2),q\big), A(x_1,q) \leq A(x_2,q), A(y_1,q) \leq A(y_2,q)$, we get, $A\big((x_1+y_1),q\big) \geq A(x_1,q) \wedge A(y_1,q)$, for all $x_1$and y_1 in R.

(ii) $A(x_1y_1,q) \wedge A(x_2y_2,q) = V\big((x_1y_1, x_2y_2),q\big)$
$= V[(x_1,x_2)(y_1,y_2),q]$
$= V(xy,q) \geq V(x,q) \wedge V(y,q)$
$= V\big((x_1,x_2),q\big) \wedge V\big((y_1,y_2),q\big)$

$= \{\{A(x_1, q) \wedge A(x_2, q)\} \wedge \{A(y_1, q) \wedge A(y_2, q)\}\}$.

If $A((x_1y_1), q) \leq A((x_2y_2), q), A(x_1, q) \leq A(x_2, q), A(y_1, q) \leq A(y_2, q)$, we get, $A((x_1y_1), q) \geq A(x_1, q) \wedge A(y_1, q)$, for all x_1 and y_1 in R.

(iii) $A(x_1 \vee y_1, q) \wedge A(x_2 \vee y_2, q) = V((x_1 \vee y_1, x_2 \vee y_2), q)$
$= V[(x_1, x_2) \vee (y_1, y_2), q]$
$= V(x \vee y, q) \geq V(x, q) \wedge V(y, q)$
$= V((x_1, x_2), q) \wedge V((y_1, y_2), q)$
$= \{\{A(x_1, q) \wedge A(x_2, q)\} \wedge \{A(y_1, q) \wedge A(y_2, q)\}\}$.

If $A((x_1 + y_1), q) \leq A((x_2 + y_2), q), A(x_1, q) \leq A(x_2, q), A(y_1, q) \leq A(y_2, q)$, we get, $A((x_1 \vee y_1), q) \geq A(x_1, q) \wedge A(y_1, q)$, for all x_1 and y_1 in R.

(iv) $A(x_1 \wedge y_1, q) \wedge A(x_2 \wedge y_2, q) = V((x_1 \wedge y_1, x_2 \wedge y_2), q)$
$= V[(x_1, x_2) \wedge (y_1, y_2), q]$
$= V(x \wedge y, q) \geq V(x, q) \wedge V(y, q)$
$= V((x_1, x_2), q) \wedge V((y_1, y_2), q)$
$= \{\{A(x_1, q) \wedge A(x_2, q)\} \wedge \{A(y_1, q) \wedge A(y_2, q)\}\}$.

If $A((x_1 + y_1), q) \leq A((x_2 + y_2), q), A(x_1, q) \leq A(x_2, q), A(y_1, q) \leq A(y_2, q)$, we get, $A((x_1 \wedge y_1), q) \geq A(x_1, q) \wedge A(y_1, q)$, for all x_1 and y_1 in R.

Therefore A is an $[Q, L]$-fuzzy ℓ-subsemiring of R.

Theorem 3.5 If A is an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$, then $H = \{x/x \in R, q \in Q: A(x, q) = 0\}$ is either empty or is a ℓ-subsemiring of R.

Proof. On the off chance that no component fulfills this condition, H is empty. If $x, y \in H$ and $q \in Q$, the

(i) $A(x + y, q) \geq A(x, q) \wedge A(y, q) = 0 \wedge 0 = 0$. Therefore, $A(x + y, q) = 0$.

(ii) $A(xy, q) \geq A(x, q) \wedge A(y, q) = 0 \wedge 0 = 0$. Therefore, $A(xy, q) = 0$.

(iii) $A(x \vee y, q) \geq A(x, q) \wedge A(y, q) = 0 \wedge 0 = 0$. Therefore, $A(x \vee y, q) = 0$.

(iv) $A(x \wedge y, q) \geq A(x, q) \wedge A(y, q) = 0 \wedge 0 = 0$. Therefore, $A(x \wedge y, q) = 0$.

We get $x + y, xy, x \vee y, x \wedge y$in H. Therefore, H is a ℓ-subsemiring of R.

Hence H is either empty or is a ℓ-subsemiring of R.

In the following theorem ° is the composition operation of functions:

Theorem 3.6 Let A be an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$H$ and f is an isomorphism from a ℓ-semiring$R$ onto H. Then $A \circ f A°f$ is an $[Q, L]$-fuzzy ℓ-subsemiring of R.

Proof. Let x and y in R and A be an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$H$. Then we have,

(i) $(A°f)(x + y, q) = A(f(x + y), q)$
$= A(f(x) + f(y), q)$
$\geq A(f(x, q)) \wedge A(f(y, q))$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$

which implies that $(A°f)(x + y, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

(ii) $(A°f)(xy, q) = A(f(xy), q)$
$= A(f(x, q) + f(y, q))$
$\geq A(f(x, q)) \wedge A(f(y, q))$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$

which implies that $(A°f)(xy, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

(iii) $(A°f)(x \vee y, q) = A(f(x \vee y), q)$
$= A(f(x, q) \vee f(y, q))$
$\geq A(f(x), q) \wedge A(f(x), q)$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$

which implies that $(A°f)(x \vee y, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

(iv) $(A°f)(x \wedge y, q) = A(f(x \wedge y), q)$
$= A(f(x, q) \wedge f(y, q))$
$\geq A(f(x), q) \wedge A(f(x), q)$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$

which implies that $(A°f)(x \wedge y, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.
Therefore $(A°f)$ is an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$.

Theorem 3.7 Let A be a $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$H$ and f is an anti-isomorphism from ℓ-subsemiring of a ℓ-semiring$R$ onto H. Then $A°f$ is a $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$.

Proof. Let x and y in R and A be an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$H$. Then we have,

(i) $(A°f)(x + y, q) = A(f(x + y), q)$
$= A\big(f(y, q) + f(x, q)\big)$
$\geq A\big(f(x, q)\big) \wedge A\big(f(y, q)\big)$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$
which implies that $(A°f)(x + y, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

(ii) $(A°f)(xy, q) = A(f(xy), q)$
$= A\big(f(y, q)f(x, q)\big)$
$\geq A\big(f(x, q)\big) \wedge A\big(f(y, q)\big)$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$
which implies that $(A°f)(xy, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

(iii) $(A°f)(x \vee y, q) = A(f(x \vee y), q)$
$= A\big(f(y, q) \vee f(x, q)\big)$
$\geq A(f(x), q) \wedge A(f(y), q)$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$
which implies that $(A°f)(x \vee y, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

(iv) $(A°f)(x \wedge y, q) = A(f(x \wedge y), q)$
$= A\big(f(y, q) \wedge f(x, q)\big)$
$\geq A(f(x), q) \wedge A(f(y), q)$
$\geq (A°f)(x, q) \wedge (A°f)(y, q)$
which implies that $(A°f)(x \wedge y, q) \geq (A°f)(x, q) \wedge (A°f)(y, q)$for all x and y in R and $q \in Q$.

Therefore $A°f$ is an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$.

Theorem 3.8 Let A be an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$, then the pseudo $[Q, L]$-fuzzy coset$(aA)^p$is an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$, for a in R.

Proof. Let A be an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$.For every x and y in R, we have,

(i) $((aA)^p)(x + y, q) = p(a)A(x + y, q)$
$\geq p(a)\{(A(x, q) \wedge A(y, q)\}$
$= \{p(a)A(x, q) \wedge p(a)A(y, q)\}$
$= \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$
Therefore, $((aA)^p)(x + y, q) \geq \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$ for all x and y in R and $q \in Q$.

(ii) $((aA)^p)(xy, q) = p(a)A(xy, q)$
$\geq p(a)\{(A(x, q) \wedge A(y, q)\}$
$= \{p(a)A(x, q) \wedge p(a)A(y, q)\}$
$= \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$
Therefore, $((aA)^p)(xy, q) \geq \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$ for all x and y in R and $q \in Q$.

(iii) $((aA)^p)(x \vee y, q) = p(a)A(x \vee y, q)$
$\geq p(a)\{(A(x, q) \wedge A(y, q)\}$
$= \{p(a)A(x, q) \wedge p(a)A(y, q)\}$
$= \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$
Therefore, $((aA)^p)(x \vee y, q) \geq \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$ for all x and y in R and $q \in Q$.

(iv) $((aA)^p)(x \wedge y, q) = p(a)A(x \wedge y, q)$
$\geq p(a)\{(A(x, q) \wedge A(y, q)\}$
$= \{p(a)A(x, q) \wedge p(a)A(y, q)\}$
$= \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$
Therefore, $((aA)^p)(x \wedge y, q) \geq \{((aA)^p)(x, q) \wedge ((aA)^p)(y, q)\}$ for all x and y in R and $q \in Q$.

Hence $(aA)^p$ is an $[Q, L]$-fuzzy ℓ-subsemiring of a ℓ-semiring R.

Theorem 3.9 Let R and $R^{|}$be any two ℓ-semirings. The homomorphic image of an $[Q, L]$-fuzzy ℓ-subsemiring of R is an $[Q, L]$-fuzzy ℓ-subsemiringof $R^{|}$.

Proof. Let R and $R^{|}$ be any two ℓ-semirings. Let $f: R \to R^{|}$ be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and y in R. Let $V = f(A)$, where A is an $[Q, L]$-fuzzy ℓ-subsemiring of R. We have to prove that V is an $[Q, L]$-fuzzy ℓ-subsemiring of $R^{|}$. Now, for $f(x), f(y)$ in $R^{|}$,

(i) $V\big(f(x,q) + f(y,q)\big) = V(f(x+y), q)$
$\geq A\big((x+y), q\big)$
$\geq A(x,q) \wedge A(y,q)$
which implies that $V(f(x) + f(y), q) \geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$for all x and y in R and $q \in Q$.

(ii) $V(f(x)f(y), q) = V(f(xy), q)$
$\geq A\big((xy), q\big)$
$\geq A(x,q) \wedge A(y,q)$
which implies that $V(f(x)f(y), q) \geq V(f(x), q) \wedge V(f(y), q)$for all x and y in R and $q \in Q$.

(iii) $V\big(f(x,q) \vee f(y,q)\big) = V(f(x \vee y), q)$
$\geq A\big((x \vee y), q\big)$
$\geq A(x,q) \wedge A(y,q)$
which implies that $V(f(x) + f(y), q) \geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$for all x and y in R and $q \in Q$.

(iv) $V\big(f(x,q) \wedge f(y,q)\big) = V(f(x \wedge y), q)$
$\geq A\big((x \wedge y), q\big)$
$\geq A(x,q) \wedge A(y,q)$
which implies that $V(f(x) \wedge f(y), q) \geq V(f(x), q) \wedge V(f(y), q)$for all x and y in R and $q \in Q$.

Hence V is an $[Q, L]$-fuzzy ℓ-subsemiringof $R^{|}$.

Theorem 3.10 Let R and $R^{|}$ be any two ℓ-semirings. The homomorphicpreimage of an $[Q, L]$-fuzzy ℓ-subsemiring of R' is an $[Q, L]$-fuzzy ℓ-subsemiring of R.

Proof. Let R and R' be any two ℓ-semirings. Let $f: R \to R^{|}$be a homomorphism. Then, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)\, f(y)$, for all x and y in R. Let $V = f(A)$, where V is an $[Q, L]$-fuzzy ℓ-subsemiringof $R^{|}$. We have to prove that A is an $[Q, L]$- fuzzy ℓ-subsemiring of R. Let x and y in R.Then,

(i) $A(x+y, q) = V(f(x+y), q)$
$= V(f(x) + f(y), q)$
$\geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$= A(x,q) \wedge A(y,q)$
which implies that $A(x+y, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

(ii) $A(xy, q) = V(f(xy), q)$
$= V\big(f(x,q)f(y,q)\big)$
$\geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$= A(x,q) \wedge A(y,q)$
which implies that $A(xy, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

(iii) $A(x \vee y, q) = V(f(x \vee y), q)$
$= V\big(f(x,q) \vee f(y,q)\big)$
$\geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$= A(x,q) \wedge A(y,q)$
which implies that $A(x \vee y, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

(iv) $A(x \wedge y, q) = V(f(x \wedge y), q)$
$= V(f(x) \wedge f(y), q)$
$\geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$= A(x,q) \wedge A(y,q)$
which implies that $A(x \wedge y, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

Hence A is an (Q, L)-fuzzy ℓ-subsemiring of R.

Theorem 3.11 Let R and $R^{|}$ be any two ℓ-semirings. The anti-homomorphic image of an $[Q, L]$-fuzzy ℓ-subsemiring of R is an $[Q, L]$-fuzzy ℓ-subsemiringof $R^{|}$.

Proof: Let R and $R^{|}$ be any two ℓ-semirings. Let $f: R \to R^{|}$be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y)\, f(x)$, for all $x, y\ \in R$.Let $V = f(A)$, where A is an (Q, L)-fuzzy ℓ-subsemiring of R. We have to prove that V is an (Q, L)-fuzzy ℓ-subsemiring of $R^{|}$. Now, for $f(x), f(y)$ in $R^{|}$,

(i) $V\big(f(x,q) + f(y,q)\big) = V(f\big((y+x), q\big)$
$\geq A(y+x, q)$
$\geq A(y,q) \wedge A(x,q)$
$= A(x,q) \wedge A(y,q)$
which implies that $V(f(x) + f(y), q) \geq V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$.

(ii) $V\big(f(x,q) f(y,q)\big) = V(f\big((yx), q\big)$
$\geq A(yx, q)$
$\geq A(y,q) \wedge A(x,q)$
$= A(x,q) \wedge A(y,q)$
which implies that $V(f(x)f(y), q) \geq V(f(x), q) \wedge V(f(y), q)$.

(iii) $V\big(f(x,q) \vee f(y,q)\big) = V(f\big((y \vee x), q\big)$
$\geq A(y \vee x, q)$
$\geq A(y,q) \wedge A(x,q)$
$= A(x,q) \wedge A(y,q)$
which implies that $V(f(x) \vee f(y), q) \geq V(f(x), q) \wedge V(f(y), q)$ for all x and y in R and $q \in Q$.

(iv) $V\big(f(x,q) \wedge f(y,q)\big) = V(f\big((y \wedge x), q\big)$
$\geq A(y \wedge x, q)$
$\geq A(y,q) \wedge A(x,q)$
$= A(x,q) \wedge A(y,q)$
which implies that $V(f(x) \wedge f(y), q) \geq V(f(x), q) \wedge V(f(y), q)$ for all x and y in R and $q \in Q$.

Hence V is an $[Q, L]$-fuzzy ℓ-subsemiringof $R^{|}$.

Theorem 3.12 Let R and $R^{|}$ be any two ℓ-semirings. The anti-homomorphicpreimage of an $[Q, L]$-fuzzy ℓ-subsemiring of $R^{|}$ is an $[Q, L]$-fuzzy ℓ-subsemiring of R.

Proof. Let R and $R^{|}$ be any two ℓ-semirings. Let $f: R \to R^{|}$ be an anti-homomorphism. Then, $f(x+y) = f(y) + f(x)$ and $f(xy) = f(y)\, f(x)$, for all x and y in R. Let $V = f(A)$, where V is an $[Q, L]$-fuzzy ℓ-subsemiringof $R^{|}$. We have to prove that A is an $[Q, L]$-fuzzy ℓ-subsemiring of R. Let x and y in R.Then

(i) $A\big((x+y), q\big) = V(f(x+y), q)$
$= V\big(f(y,q) + f(x,q)\big)$
$\geq V\big(f(y,q)\big) \wedge V\big(f(x,q)\big)$
$= V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$=\ A(x,q) \wedge A(y,q)$
which implies that $A(x+y, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

(ii) $A\big((xy), q\big) = V(f(xy), q)$
$= V\big(f(y,q) f(x,q)\big)$
$\geq V\big(f(y,q)\big) \wedge V\big(f(x,q)\big)$
$= V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$=\ A(x,q) \wedge A(y,q)$
which implies that $A(xy, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

(iii) $A\big((x \vee y), q\big) = V(f(x \vee y), q)$
$= V\big(f(y,q) \vee f(x,q)\big)$
$\geq V\big(f(y,q)\big) \wedge V\big(f(x,q)\big)$
$= V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$=\ A(x,q) \wedge A(y,q)$
which implies that $A(x \vee y, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

(iv) $A\big((x \wedge y), q\big) = V(f(x \wedge y), q)$

$= V\big(f(y,q) \wedge f(x,q)\big)$
$\geq V\big(f(y,q)\big) \wedge V\big(f(x,q)\big)$
$= V\big(f(x,q)\big) \wedge V\big(f(y,q)\big)$
$= A(x,q) \wedge A(y,q)$

which implies that $A(x \wedge y, q) \geq A(x,q) \wedge A(y,q)$for all x and y in R and $q \in Q$.

Hence A is an (Q,L)-fuzzy ℓ-subsemiring of R.

Theorem 3.13 Let A be an $[Q,L]$-fuzzy ℓ-subsemiring of a ℓ-semiring$R$. A^+be a fuzzy set in R defined by $A^+(x,q) = A(x,q) + 1 - A(0)$, for all x in R and q in Q. Then $A^+$is an (Q,L)- fuzzy ℓ-subsemiring of a ℓ-semiring$R$.

Proof. Let x and y in R and $q \in Q$.We have,

(i) $A^+(x+y,q) = A(x+y,q) + 1 - A(0)$
$\geq \{A(x,q) \wedge A(y,q)\} + 1 - A(0)$
$= \{A(x,q) + 1 - A(0)\} \wedge \{A(y,q) + 1 - A(0)\}$
$= A^+(x,q) \wedge A^+(y,q)$

which implies that$A^+(x+y,q) \geq A^+(x,q) \wedge A^+(y,q)$ for all x, y in R and q in Q.

(ii) $A^+(xy,q) = A(xy,q) + 1 - A(0)$
$\geq \{A(x,q) \wedge A(y,q)\} + 1 - A(0)$
$= \{A(x,q) + 1 - A(0)\} \wedge \{A(y,q) + 1 - A(0)\}$
$= A^+(x,q) \wedge A^+(y,q)$

Therefore, $A^+(xy,q) \geq A^+(x,q) \wedge A^+(y,q)$ for all x, y in R and q in Q.

Hence A^+ is an (Q,L)-fuzzy ℓ-subsemiring of a ℓ-semiring$R$.

4. Conclusion

In the study of the structure of an fuzzy algebraic system, we notice that Q-fuzzy with special properties always play an important role. In this paper, we define (Q,L)-fuzzy ℓ-subsemirings of a ℓ-semiring and investigate some important results. We hope that the research along this direction can be continued, and in fact, this work would serve as a foundation for further study of the theory of semiring, it will be important to complete more hypothetical exploration to set up an overall structure for the commonsense application.

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