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Original Article

Inf-invex Alternative Theorem with Applications to Vector-Valued Games

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Abstract *- In this paper, a vector-valued nonlinear constrained game is studied. It is shown that the solution of this game can be obtained by finding the properly efficient solutions to a symmetric dual pair of multiobjective nonlinear programming problems in which the multiplier vector corresponding to the objective is a vector-valued function of two variables. An inf-invex alternative theorem of Gordan type is used as a tool to prove the equivalence between the constrained vector-valued game and the symmetric dual pair.*

Keywords *- Alternative theorem, Inf-invexity, Symmetric duality, Vector-valued games.*

1. Introduction

Game theory is the most extensively used methodology in decision making problems, when the decision makers have conflicting interests. Due to its myriad applications [1-8], many researchers suggested methods for solving game problem with the technique of certain mathematical programming problem [9-11]. These results were further extended to solve nonlinear constrained game and its equivalence with symmetric dual nonlinear programming problems [12-16].

If a game problem has multiple participants with divergent interests then the goals of an individual participant cannot be described in terms of single index, which inevitably, leads to the use of vectorial objective functions. Such specific feature results in multiple criteria game problems where every player wants to optimize his own vectorial criterion (payoff function) or two-person games in which one player wishes to minimize and the other wishes to maximize the same vectorial function. These multiple criteria games analyse group decision problems when the decision makers consider several criteria, each of which depends on the decision of all players [17-18].

Motivated by such situations, in this paper, we study the following two-person vector-valued nonlinear constrained game $G = (X, Y, f)$ where

$X = \{x \in R^n_+ : p_k(x) \geq 0, k = 1,2, \ldots, s\}$

$Y = \{y \in R^m_+ : q_r(y) \leq 0, r = 1,2, \ldots, v\}$

$f: X \times Y \to R^l$

where $p_k: R^n \to R\ for\ k = 1,2, \ldots, s$ and $q_r: R^m \to R\ for\ r = 1,2, \ldots, t$ are differentiable functions. X and Y represent the strategy spaces of players I and II respectively. $f(x, y)$ represents the pay-off to Player II from Player I when Player I selects strategy x and Player II selects strategy y .

The problem of solving vector-valued nonlinear constrained game $G = (X, Y, f)$ was earlier studied by Corley [12] who established the necessary and sufficient conditions for the solution of such a game. Chandra and Durga Prasad [13] partially generalized the results of constrained scalar valued games [14,16] to a certain convex-concave vector-valued game. They established its relation with a pair of multiobjective programming problems involving nonlinear functions and remarked that the exact equivalence reported in scalar case does not seem to go through in vector valued games due to the certain difficulties with the matching of scalars in the resulting pair of multiobjective programs. Here, we overcome this difficulty by establishing the exact equivalence of the vector-valued game introduced above with a special symmetric dual pair of nonlinear programming problems in which the multiplier vector corresponding to the objectives is a vector-valued function $\lambda(x, y)$ of two variables $x \epsilon X, y \epsilon Y$ rather than a constant vector λ. The principal tool used is Gordan theorem of alternative proved for inf-invex functions of two variables.

2. Definitions

Let R^n be the n-dimensional Euclidean space and R^n_+ be its non-negative orthand. Let X and Y be nonempty sets in R^n and R^m respectively. Let $\psi(x, y)$ be differentiable function of two variables $x \epsilon X\ and\ y \epsilon Y$, and $\eta_1: X \times X \to R^n$ and $\eta_2: Y \times Y \to R^m$. Let $\nabla_x \psi(x, y)$ denotes the partial derivative of ψ with respect to its first component x and $\nabla_y \psi(x, y)$ denotes the partial derivative of ψ with respect to its second component y.

The definition of inf-invexity introduced by Caristi et al. [19]is extended to a function of two variables as follows.

Definition 2.1. $\psi(.,y)$ is said to be inf-invex at $x^0 \epsilon X$, for fixed y, with respect to η_1 and X_0 (a subset of X), if

$$\inf_{x\epsilon X_0}\left(\psi(x,y)-\psi(x^0,y)\right) \geq \inf_{x\epsilon X_0}\langle \eta_1(x,x^0), \nabla_x \psi(x^0,y)\rangle.$$

$\psi(x,.)$ is said to be inf-invex at $y^0 \epsilon Y$, for fixed x, with respect to η_2 and Y_0 (a subset of Y), if

$$\inf_{y\epsilon Y_0}\left(\psi(x,y)-\psi(x,y^0)\right) \geq \inf_{y\epsilon Y_0}\langle \eta_2(y,y^0), \nabla_y \psi(x,y^0)\rangle.$$

Definition 2.2. $\psi(.,y)$ is said to be inf-pseudoinvex at $x^0 \epsilon X$, for fixed y, with respect to η_1 and X_0 , if

$$\inf_{x\epsilon X_0}\langle \eta_1(x,x^0), \nabla_x \psi(x^0,y)\rangle \geq 0 \Rightarrow \inf_{x\epsilon X_0}\left(\psi(x,y)-\psi(x^0,y)\right) \geq 0.$$

$\psi(x,.)$ is said to be inf-pseudoinvex at $y^0 \epsilon Y$, for fixed x, with respect to η_2 and Y_0 , if

$$\inf_{y\epsilon Y_0}\langle \eta_2(y,y^0), \nabla_y \psi(x,y^0)\rangle \geq 0 \Rightarrow \inf_{y\epsilon Y_0}\left(\psi(x,y)-\psi(x,y^0)\right) \geq 0.$$

3. Alternative Theorem

The Gordan's theorem of alternative is an essential tool in optimization to establish the equivalence between the solvability of two systems of inequalities [20]. In this section, we first prove the following version of the Gordan's theorem of the alternative for inf-invex functions of two variables.

Theorem 3.1. Let $g_j : X \times Y \to R, j \in J = \{1,2, \ldots m\}$ be inf-invex, for fixed $y' \in Y$, with respect to η.

$$\text{If } \max_{j\in J}\{g_j(x,y')\} \text{attains a minimum on X,} \tag{1}$$

then either

(i) there exists $x \in X$ such that $g_j(x,y') < 0, j \in J$, or

(ii) there exist scalars $\lambda_j(y') \geq 0, j \in J$, not all zero, dependent on y' , such that

$$\sum_{j\in J} \lambda_j(y') g_j(x,y') \geq 0, \forall x \in X ,$$

but never both.

Proof: Suppose that system (i) has a solution $x' \in X$. Then for any scalars $\lambda_j(y') \geq 0, j \in J$, dependent on y', not all zero,

$$\sum_{j\in J} \lambda_j(y') g_j(x',y') < 0.$$

Hence (ii) cannot have a solution.

Conversely, assume that (i) has no solution $\in X$. Let

$$(P)_{y'} \min_{x\in X} \max_{j\in J}\{g_j(x,y')\}$$

attains its minimum at $x' \in X$.

The equivalent auxiliary problem to problem $(P)_{y'}$ is given by

$(EP)_{y'}$ Minimize w

$$\text{subject to } g_j(x,y') - w \leq 0, j \in J, \tag{2}$$

$$(x,y',w) \in X \times Y \times R.$$

By necessary optimality conditions [15], if (x,y',w') is an optimal solution for $(EP)_{y'}$, then there exist multipliers $\lambda(y') \in R^m$, dependent on y' such that

$$\sum_{j\in J} \lambda_j(y') \nabla_x g_j(x',y') = 0, \tag{3}$$

$$\lambda_j(y')\big(g_j(x',y')-w'\big)=0, j\in J, \quad (4)$$

$$g_j(x',y')\le w', j\in J, \quad (5)$$

$$\sum_{j\in J}\lambda_j(y')=1. \quad (6)$$

Moreover $w'\ge 0$. Otherwise system (i) will have a solution in X.

Also, $\eta(x,x')$ is the common function for the inf-invexity of $g_j(.,y'), j\in J$, for fixed y' , with respect to X, that is,

$$\inf_{x\in X}\Big(g_j(x,y')-g_j(x',y')\Big)\ge \inf_{x\in X}\langle \eta(x,x'),\nabla_x g_j(x',y')\rangle, j\in J.$$

Since $\lambda_j(\bar{y})\ge 0, j\in J$, we have

$$\inf_{x\in X}\left(\sum_{j\in J}\lambda_j(y')g_j(x,y')-\sum_{j\in J}\lambda_j(y')g_j(x',y')\right)\ge \inf_{x\in X}\langle \eta(x,x'),\sum_{j\in J}\lambda_j(y')\nabla_x g_j(x',y')\rangle.$$

Using condition (3), the inequality gives

$$\sum_{j\in J}\lambda_j(y')g_j(x,y')\ge \sum_{j\in J}\lambda_j(y')g_j(x',y'),$$

that is, $\sum_{j\in J}\lambda_j(y')g_j(x,y')\ge 0$ (using (4) and (6)).

Hence system (ii) has a solution.

Remark 3.1 [21]. In the alternative (ii) of Theorem 1, we use scalars $\lambda_j's$ dependent on $\bar{y}$ instead of using fixed scalars $\lambda_j's$.

4. Equivalence of Constrained Vector-Valued Game and Symmetric Dual Pair

In this section, in order to prove the equivalence of the constrained vector-valued game to a symmetric dual pair of non-linear programming problems, we associate the vector-valued game $G=\big(X,Y,f(x,y)\big)$ to the scaler valued game $G'=\big(X,Y,\lambda^T(x,y)f(x,y)\big)$. The multiplier vector $\lambda(x,y)=X\times Y\to R_+^l$ used for achieving this goal is a vector-valued differentiable function of two variables $x\in X, y\in Y$ such that $a_i\le \lambda_i(x,y)\le b_i, i=1,2,\dots,l$ where a_i and b_i are specified constants. Associated with game $G'=\big(X,Y,\lambda^T(x,y)f(x,y)\big)$ are the two programming problems $\min_{x\in X}\max_{y\in Y}\lambda^T(x,y)f(x,y)$ and $\max_{y\in Y}\min_{x\in X}\lambda^T(x,y)f(x,y)$ which will be reduced to the following pair of problems (P1) and (D1) respectively.

(P1) Minimize $\lambda(x,y)^T f(x,y)$

subject to $\nabla_y[\lambda(x,y)^T f(x,y)-\sum_{r=1}^t \mu_r q_r(y)]\le 0,$

$y^T\nabla_y[\lambda(x,y)^T f(x,y)-\sum_{r=1}^t \mu_r q_r(y)]\ge 0,$

$\sum_{r=1}^t \mu_r q_r(y)\ge 0,$

$p_k(x)\ge 0, k=1,2,\dots,s,$

$x\ge 0,\mu\ge 0.$

(D1) Maximize $\lambda(u,v)^T f(u,v)$

subject to $\nabla_x[\lambda(u,v)^T f(u,v)-\sum_{k=1}^s \gamma_k p_k(u)]\ge 0,$

$u^T\nabla_x[\lambda(u,v)^T f(u,v)-\sum_{k=1}^s \gamma_k p_k(u)]\le 0,$

$\sum_{k=1}^s \gamma_k p_k(u)\le 0,$

$q_k(v)\le 0, r=1,2,\dots,t,$

$v\ge 0,\gamma\ge 0.$

Corresponding to these two problems, we have the following vector-valued problems (P2) and (D2).

(P2) Minimize $f(x,y)=[f^1(x,y),\dots,f^p(x,y)]$

subject to $\nabla_y[\lambda(x,y)^T f(x,y) - \sum_{r=1}^{t} \mu_r q_r(y)] \leq 0,$ (7)

$y^T \nabla_y[\lambda(x,y)^T f(x,y) - \sum_{r=1}^{t} \mu_r q_r(y)] \geq 0,$ (8)

$\sum_{r=1}^{t} \mu_r q_r(y) \geq 0,$ (9)

$p_k(x) \geq 0, k = 1,2,\dots,s,$ (10)

$x \geq 0, \mu \geq 0.$ (11)

Let W_1 denotes the feasible solution set of (P2).

(D2) Maximize $f(u,v) = [f^1(u,v),\dots,f^p(u,v)]$

subject to $\nabla_x[\lambda(u,v)^T f(u,v) - \sum_{k=1}^{s} \gamma_k p_k(u)] \geq 0,$ (12)

$u^T \nabla_x[\lambda(u,v)^T f(u,v) - \sum_{k=1}^{s} \gamma_k p_k(u)] \leq 0,$ (13)

$\sum_{k=1}^{s} \gamma_k p_k(u) \leq 0,$ (14)

$q_k(v) \leq 0, r = 1,2,\dots,t,$ (15)

$v \geq 0, \gamma \geq 0.$ (16)

Let W_2 denote the set of feasible solutions of (D2).

We now establish weak duality theorem for the dual pair (P2) and (D2) under the following assumptions:

(A1) For each $x^0 \in X, \mu^0 \in R_+^t, -\lambda(x^0,.)^T f(x^0,.) + \sum_{r=1}^{t} \mu_r^0 q_r(.)$ is inf-pseudoinvex with respect to some ξ *and* Y, where $\sup_{v\in Y}[\xi(v,y) + y] \geq 0, \forall y \in Y$ such that $(x^0, y, \mu^0, \lambda(x^0,y)) \in W_1$.

(A2) For each $v^0 \in Y, \gamma^0 \in R_+^s, \lambda(.,v^0)^T f(.,v^0) - \sum_{k=1}^{k} \gamma_k^0 p_k(.)$ is inf-pseudoinvex with respect to some η *and* X, where $\inf_{x\in X}[\eta(x,u) + u] \geq 0, \forall u \in X$ such that $(u, v^0, \gamma^0, \lambda(u,v^0)) \in W_2$.

Remark 4.1. The hypothesis in (A1) that a scale function $\xi(v,y)$satisfies $\sup_{v\in Y}[\xi(v,y) + y] \geq 0, \forall y \in Y$ is worth noticing, since it is more likely to be satisfied in applications than the hypothesis $[\xi(v,y) + y] \geq 0, \forall y \in Y$ made in the various other papers.

Similar remark holds for assumption (A2) also.

Theorem 4.1 (Weak duality). Assume that for all feasible solutions $(x^0, y^0, \mu^0, \lambda(x^0,y^0))$ for (P2) and all feasible $(u^0, v^0, \gamma^0, \lambda(u^0,v^0))$ for (D2), conditions (A1) and (A2) are satisfied and $0 < \lambda(x^0,y^0) \leq \lambda(u^0,v^0)$. Then the following cannot hold:

$f^i(x^0,y^0) \leq f^j(u^0,v^0)$ *for all* $i = 1,2,\dots,p,$ (17)

$f^j(x^0,y^0) < f^j(u^0,v^0)$ *for some* $j \in \{1,2,\dots,p\}.$ (18)

Proof: Suppose contrary to the result of the theorem (17) and (18) hold. Using

$0 < \lambda(x^0,y^0) \leq \lambda(u^0,v^0)$, it follows

$\lambda(x^0,y^0)^T f(x^0,y^0) \leq \lambda(u^0,v^0)^T f(u^0,v^0).$ (19)

Let $(x^0, y^0, \mu^0, \lambda(x^0,y^0)) \in W_1, (u^0, v^0, \gamma^0, \lambda(u^0,v^0)) \in W_2$.

Since $\sup_{v\in Y}[\xi(v,y) + y] \geq 0, \forall y \in \{\text{y} \in \text{Y} | (x^0, y, \mu^0, \lambda(x^0,y)) \in W_1\}$. From (7), we have

$$\sup_{v\in Y}[\xi(v,y^0) + y^0]\nabla_y \left[\lambda(x^0,y^0)^T f(x^0,y^0) - \sum_{r=1}^{t} \mu_r^0 q_r(y^0)\right] \leq 0.$$

Adding the above inequality in (8), we get

$\sup_{v\in Y}[\xi(v,y^0)]\nabla_y[\lambda(x^0,y^0)^T f(x^0,y^0) - \sum_{r=1}^{t} \mu_r^0 q_r(y^0)] \leq 0,$

that is,

$$\inf_{v\in Y}[-\xi(v,y^0)]\nabla_y\left[\lambda(x^0,y^0)^T f(x^0,y^0)-\sum_{r=1}^{t}\mu_r^0 q_r(y^0)\right]\geq 0$$

Using assumption (A1), this gives

$$\inf_{v\in Y}[-\lambda(x^0,v)^T f(x^0,v)+\sum_{r=1}^{t}\mu_r^0 q_r(v)+\lambda(x^0,y^0)^T f(x^0,y^0)-\sum_{r=1}^{t}\mu_r^0 q_r(y^0)]\geq 0\ ,$$

and since $v^0\in Y$, it results

$$[-\lambda(x^0,v^0)^T f(x^0,v^0)+\sum_{r=1}^{t}\mu_r^0 q_r(v^0)+\lambda(x^0,y^0)^T f(x^0,y^0)-\sum_{r=1}^{t}\mu_r^0 q_r(y^0)]\geq 0.$$

Using (11) and (15), it results

$$-\lambda(x^0,v^0)^T f(x^0,v^0)+\lambda(x^0,v^0)^T f(x^0,v^0)\geq 0.$$

A similar argument using condition (A2) with

$\inf_{x\in X}[\eta(x,u)+u]\geq 0,\forall u\in\{\text{u}\in \text{X}|(u,v^0,\gamma^0,\lambda(u,v^0))\in W_2\}$ gives

$$\lambda(x^0,v^0)^T f(x^0,v^0)-\lambda(u^0,v^0)^T f(u^0,v^0)\geq 0.$$

The last two inequalities imply

$$\lambda(x^0,y^0)^T f(x^0,y^0)\geq\lambda(u^0,v^0)^T f(u^0,v^0)$$

which contradicts (19). Hence the result.

In order to establish strong duality theorem between (P2) and (D2), we characterize the properly efficient solutions of vector-valued problems (P2) and (D2) in terms of the optimal solutions of scalar-valued problems (P1) and (D1) under weaker invexity type conditions, that is inf-invexity conditions, using a version of the Gordan's theorem of alternative (Theorem 1) as a principal tool.

Lemma 4.1. Let $f_i(.,y), i=1,2,\ldots,p$ be inf-invex, for fixed $y\in Y$, with respect to $\eta\ and\ X$. Then $(x^0,y^0,\mu^0,\lambda(x^0,y^0))$ is properly efficient for (P2) if and only if $(x^0,y^0,\mu^0,\lambda(x^0,y^0))$ is optimal for (P1) with $0<\lambda(x,y)\leq \lambda(x^0,y^0)\forall(x,y)\in W_1$.

Proof: The part 'if' of the proof of the lemma follows on the similar lines as that of Theorem 1, [22]. The part 'only if' of the proof of the lemma runs on the similar lines as that of Theorem 2, [22] except that we will apply Gordans's theorem of alternative for inf-invexity (Theorem 1).

Theorem 4.2 (Strong duality). Let $(x^0,y^0,\mu^0,\lambda(x^0,y^0))$ be properly efficient solution for (P2). Let the vectors $\nabla_y[\lambda(x^0,y^0)^T f(x^0,y^0)]$ and $\nabla_y[\sum_{r=1}^{t}\mu_r^0 q_r(y^0)]$ be linearly independent and the Hessian matrix $\nabla_{yy}[\lambda(x^0,y^0)^T f(x^0,y^0)-\sum_{r=1}^{t}\mu_r^0 q_r(y^0)]$ be positive or negative. Let P be the matrix $(\nabla_y p_k(x^0), k=1,2,\ldots.,s)\ and\ P\rho\leq 0,\rho\geq 0$ imply $\rho=0$. Assume that the weak duality Theorem 2 holds. Then there exist $\gamma_k^0, k=1,2,\ldots,s$ such that $(x^0,y^0,\gamma^0,\lambda(x^0,y^0))$ is properly efficient for (D2).

Proof: Since $(x^0,y^0,\mu^0,\lambda(x^0,y^0))$is properly efficient solution for (P2), therefore, it follows from Lemma 1 that $(x^0,y^0,\mu^0,\lambda(x^0,y^0))$ is optimal for (P1). Rest of the proof runs on the same lines as that in Agarwal et al. [23].

Theorem 4.3. For the vector-valued constrained game $G=(X,Y,),\min_{x\in X}\max_{y\in Y}f(x,y)$ exists if and only $\max_{y\in Y}\min_{x\in X}f(x,y)$ exists, and when this happens $\min_{x\in X}\max_{y\in Y}f(x,y)=\max_{y\in Y}\min_{x\in X}f(x,y)$.

Proof follows in view of multiobjective duality between (P2) and (D2) and the way these problems are constructed.

5. Applications and Usefulness of the Results

In the game theory, it is typically assumed that each player has only one payoff function and the strategy set of the game is composed of the product of the individual player's strategy. However, in reality, player's strategy sets maybe interactive and each player may have more than one payoff function. Such games are called multicriteria games or games with vector payoffs. Several authors [23-26] have worked for the growth of this class under generalized convexity or invexity conditions on the pay-off functions and the constraints functions. These are used in modelling various real life problems such as in management decisions, politics and various other situations where the players do not have a priori opinion on the relative importance of the components of their payoff vectors. Suppose such a situation apts to arise in connection with military engagements where two members of a combat team are forced to separate and that they cannot communicate with each other, or which their home base, because of the danger of revealing their positions to their enemy. For example, the payoff function offered to the two members of a combat team is a vector valued function $f: X \times Y \to R^2$defined by $f(x,y) = \left(f^1(x,y), f^2(x,y)\right)$ where $f^1(x,y) = sin(x-y), f^2(x,y) = cos(x+y)$.Strategy spaces of two members are respectively defined by two sets $X = \left\{x \in R_+ : \frac{\pi}{2} - x \geq 0\right\}$ and $Y = \left\{y \in R_+ : y - \frac{\pi}{2} \leq 0\right\}$.The game problem presented in this example can be related to the problem of defining the optimal ways of playing the game in a normalised form. The normalising multiplier vector $\lambda(x,y)$ can be any positive vector-valued function of two variables of the form $\lambda(x,y) = \left(\lambda^1(x,y), \lambda^2(x,y)\right)$ with particular values $\lambda^1(x,y) = \sin(x+y), \lambda^2(x,y) = \cos(x-y)$.The solution of the above game can be found to be exactly equivalent to the solution of the pair of corresponding symmetric duel non-linear programming problems (P1) and (D1). Moreover, the assumptions (A1) and (A2) stated in the text are also satisfied with respect to $\xi(y, y^0) = 2y - y^0 \; and \; \eta(x, x^0) = x + x^0$.

6. Conclusion

A class of two-person vector-valued nonlinear constrained game is studied in this paper. The main feature of the paper is to use the Gordan theorem of alternative as a tool to prove the equivalence between the constrained vector-valued game and the symmetric dual pair of non-linear programming problems. The multiplier vector corresponding to the objective in symmetric dual pair is a vector-valued function of two variables instead of a scaler. The results presented in the paper generalises the results already existing in literature in two ways. One is by generalising the class of functions permitted in the problem to the new notion of functions and other is by generalising the optimality conditions by virtue of Gordan's alternative theorem. Moreover, the results presented here can be generalized to n-person vector-valued game problem with vector-payoff functions showing its equivalence with symmetric duel non-linear programming problems.

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