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*Original Article*

# Asymptotic Behavior of the Global Solutions to the Viscous Liquid-Gas Two Phase Flow

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***Abstract*** *- In this paper, based on global strong solution to the viscous liquid-gas two-phase flow of Yu [Journal of Differential Equations 272 (2021), 732–759], we prove that certain Lebesgue and Sobolev norms of the solution approaches zero as time* $t\to\infty$.

***Keywords*** - *Viscous liquid-gas two-phase, flow Asymptotic behavior, global strong solutions.*

## 1. Introduction

The three-dimensional viscous liquid-gas two-phase flow model reads as follows:

$$\begin{cases} m_t+\operatorname{div}(mu)=0, \\ n_t+\operatorname{div}(nu)=0, \\ (mu)_t+\operatorname{div}(mu\otimes u)-\mu\Delta u-(\mu+\lambda)\nabla \operatorname{div} u+\nabla P(m,n)=0, \end{cases} \tag{1}$$

Here $(x,t)\in\mathbb{R}^3\times(0,T]$, $m=\alpha_l\rho_l$ and $n=\alpha_g\rho_g$ represent the mass of liquid and gas; $u=(u^1,u^2,u^3)$ represents the velocity of the liquid and gas; $P(m,n)$ denotes the pressure. The viscosity coefficients $\mu$ and $\lambda$ satisfy the relation：

$$\mu>0, 2\mu+3\lambda\geq 0. \tag{2}$$

The unknown variables $0\leq\alpha_l,\alpha_g\leq 1$ are volume fractions of the liquid and gas and satisfy

$$\alpha_l+\alpha_g=1, \tag{3}$$

moreover, $\rho_l$ and $\rho_g$ are unknown variables which are the densities of liquid and gas which satisfy the fundamental relation:

$$P(m,n)=a_l^2\left(\rho_l-\rho_{l,0}\right)+P_{l,0}=a_g^2\rho_g, \tag{4}$$

where $a_l$, $a_g$ are sonic speeds of the liquid and gas. $P_{l,0}$ and $\rho_{l,0}$ are the given constants.

In the meantime, note that from (3) and (4), the pressure can be expressed as

$$P(m,n)=C^0\left(-b(m,n)+\sqrt{b(m,n)^2+c(m,n)}\right), \tag{5}$$

where $C^0=\frac{1}{2}a_l^2, k_0=\rho_{l,0}-\frac{P_{l,0}}{a_l^2}>0, a_0=(\frac{a_g}{a_l})^2$ and

$$b(m,n)=k_0-m-\left(\frac{a_g}{a_l}\right)^2 n=k_0-m-a_0n,$$

$$c(m,n)=4k_0\left(\frac{a_g}{a_l}\right)^2 n=4k_0a_0n.$$

Furthermore, note

$$P_l(\rho_l)\triangleq a_l^2\left(\rho_l-\rho_{l,0}\right)+P_{l,0} \quad \text{and} \quad P_g(\rho_g)\triangleq a_g^2\rho_g. \tag{6}$$

It follows from (4) that

$$P_l(\rho_l)=P_g(\rho_g)=P(m,n). \tag{7}$$

In order to solve the model (1), We consider the following initial date:

$$(m,n,u)(x,t=0)=\left(m_0(x),n_0(x),u_0(x)\right)=\left(\alpha_{l0}\rho_{l0},\alpha_{g0}\rho_{g0},u_0\right)(x),\quad x\in\mathbb{R}^3, \tag{8}$$

here, the $\alpha_{l0}$ and $\alpha_{g0}$ satisfy the relation:

$$\alpha_{l0}+\alpha_{g0}=1, \tag{9}$$

the $\rho_{l0}$ and $\rho_{g0}$ satisfy

$$\left(\rho_{l0},\rho_{g0},u_0\right)(x)\to\left(\rho_l,\rho_g,0\right),\ \text{as } |x|\to\infty, \tag{10}$$

where $\rho_l$ and $\rho_g$ satisfy the following relation with $P_l$ and $P_g$ :

$$P_l\left(\rho_l\right)=P_g\left(\rho_g\right). \tag{11}$$

Assume that the two-phase flow have the same velocity and pressure, ignoring the influence of convective force and gas in the mixed momentum equation. For more detailed physical background of the system, we can refer to [1-3, 5, 12, 14-16, 21, 25, 27].

Many mathematicians have studied this model and obtained a lot of results. For one-dimensional case, there are a lot of results about the weak solution. we can refer to [7-10, 18, 19, 22, 24, 25] and the references therein. The two-dimensional global weak solutions with small energy have obtained in [25] when the initial density is positive. When the initial masses have positive upper and lower bounds, Cui-Wen-Yin [6] proved the global existence of smooth solution to the 3D case. Hao-Li [5] studied the global existence of strong solutions in Besov spaces, where the possible vacuum state is contained in the equilibrium state of the gas composition in the far field. When the two initial densities contain vacuum, Guo-Yao-Yang [11, 23] established the 3D classical solutions with the small energy. Recently, Yu [26] proved the global strong solution without any constraints between the initial masses of liquid and gas. Up to now, there are no results on the asymptotic behavior of the strong solutions. Thus, we aim to we prove that certain Lebesgue and Sobolev norms of the solution approaches zero as time approaches infinity.

**Notations:** Let us end this introduction by some notations that will be used in all that follows.

Before introducing our results, we will introduce the function spaces used throughout the paper. For integer $k\geq 0$ and $1\leq r\leq\infty$ Sobolev spaces take the following form:

$$L^r=L^r(\mathbb{R}^3),\quad W^{k,r}=L^r\cap D^{k,r},\quad H^k=W^{k,2},\quad \|u\|_{D^{k,r}}\triangleq\left\|\nabla^k u\right\|_{L^r},$$

$$D^{k,r}=\{u\in L^1_{loc}(\mathbb{R}^3)|\ \ \|\nabla^k u\|_{L^r}<\infty\},\quad D^1=\left\{u\in L^6|\ \ \|\nabla u\|_{L^2}<\infty\right\}.$$

Denoted

$$\dot f\triangleq f_t+(u\cdot\nabla)f,\ \ \omega=\nabla\times u, \tag{12}$$

are the material derivative and the vorticity of $u$ respectively. Next, we use

$$F\triangleq(2\mu+\lambda)\operatorname{div}u-P, \tag{13}$$

represent the effective viscous flux, where

$$\begin{aligned}P&\triangleq P(m,n)-P_l\left(\rho_l\right)=\alpha_l\left(P_l\left(\rho_l\right)-P_l\left(\rho_l\right)\right)+\alpha_g\left(P_g\left(\rho_g\right)-P_g\left(\rho_g\right)\right)\\&=P_l\left(\rho_l\right)-P_l\left(\rho_l\right)=P_g\left(\rho_g\right)-P_g\left(\rho_g\right)\end{aligned} \tag{14}$$

due to (3), (6) and (11).

Yu have obtained the result in [(26), Theorem 1.1] as follows:

**Proposition 1.1.** For the constants satisfying and $0\leq,\bar m,\bar n,M<\infty$ and $3<p\leq 6$, we assume initial data satisfy

$$\alpha_{l0}\geq 0,\alpha_{g0}\geq 0,\rho_{l0}\geq 0,\rho_{g0}\geq 0, \tag{15}$$

$$\begin{cases}m_0\leq\bar m,n_0\leq\bar n,m_0,n_0,\sqrt{n_0}\in D^1\cap D^{1,p},\\ \alpha_{l0},\alpha_{g0}\in D^{1,p},m_0-\alpha_{l0}\rho_l,n_0-\alpha_{g0}\rho_g\in L^p,\\ u_0\in D^1\cap D^{2,2},\mu\|\nabla u_0\|_{L^2}^2+(\mu+\lambda)\|\operatorname{div}u_0\|_{L^2}^2=M,\end{cases} \tag{16}$$

$$P_0=P_l\left(\rho_{l0}\right)=P_g\left(\rho_{g0}\right), \tag{17}$$

and

$$-\mu\Delta u_0-(\mu+\lambda)\nabla\operatorname{div}u_0+\nabla P\left(m_0,n_0\right)=\sqrt{m_0}g,\quad\text{for some } g\in L^2.$$

Then, there exists a positive constant $\varepsilon$ depending on $\mu,\lambda,a_0,c^0,k_0,\bar m,\bar n$ and $M$ such that if $C_0\leq\varepsilon$, then the problem (1)-(2)

and (8)-(10) have a global strong solution satisfying

$$0 \le m(x,t) \le 2(\bar{m}+\rho_l)+1, \quad 0 \le n(x,t) \le 2\left(\bar{n}+\rho_g\right)+1, \quad x \in \mathbb{R}^3, t \ge 0, \tag{18}$$

with

$$C_0 \triangleq \int \left[ \frac{1}{2} m_0 |u_0|^2 + m_0 \int_{\rho_l}^{\rho_{l0}} \frac{P_l(s)-P_l\left(\rho_l\right)}{s^2} ds + n_0 \int_{\rho_g}^{\rho_{g0}} \frac{P_g(s)-P_g\left(\rho_g\right)}{s^2} ds \right] dx. \tag{19}$$

Our first result is to establish the asymptotic behavior of strong solutions of (1).

**Theorem 1.2.** Under the Proposition 1.1, the following asymptotic behavior holds

$$\lim_{t\to\infty} \int \left( |\rho_l - \rho_l|^4 + |\rho_g - \rho_g|^4 + m^{\frac{1}{2}} |u|^4 + |\nabla u|^2 \right)(x,t)dx = 0. \tag{20}$$

Motivated by [4], the following result is asymptotic behavior of the gradient of the densities.

**Theorem 1.3.** Under the assumption of Theorem 1.2, moreover assumed that there exists some points $x_1 \in \mathbb{R}^3$ and $x_2 \in \mathbb{R}^3$ such that $\rho_l(x_1)=0$ and $\rho_g(x_2)=0$, if $\rho_l > 0$, $\rho_g > 0$ then

$$\lim_{t\to\infty} \| \nabla \rho_l(\cdot,t)\|_{L^r} = \infty \text{ and } \lim_{t\to\infty} \| \nabla \rho_g(\cdot,t)\|_{L^r} = \infty, \tag{21}$$

for any $r > 3$.

## 2. Proofs of Theorems 1.2 and 1.3

We denote

$$B_1(T) \triangleq \sup_{0\le t\le T} \left( \sigma \| \nabla u\|_{L^2}^2 \right) + \int_0^T \sigma \|\sqrt{m}\dot{u}\|_{L^2}^2 dt, \tag{22}$$

$$B_2(T) \triangleq \sup_{0\le t\le T} \left( \sigma^3 \|\sqrt{m}\dot{u}\|_{L^2}^2 \right) + \mu \int_0^T \sigma^3 \| \nabla \dot{u}\|_{L^2}^2 dt, \tag{23}$$

$$B_3(T) \triangleq \mu \sup_{t\in[0,T]} \| \nabla u\|_{L^2}^2, \tag{24}$$

where $\sigma(t) \triangleq \min\{1,t\}$ and $\dot{u}$ has the form as in (12),

and assume

$$B_1(T) + B_2(T) \le 2C_0^{\frac{1}{4}}, \tag{25}$$

$$B_3(\sigma(T)) \le 3M. \tag{26}$$

First, the following lemma is obtained by the Lemma 2.3 in [26], which will be used in the proof of our main theorems. For the sake of simplicity, specific proof is omitted.

**Lemma 2.1.** For any $p \in [2,6]$ , then there exists a positive constant $C$ depending only on $\mu, \lambda, a_0, c^0, \bar{m}, \bar{n}, \rho_l, \rho_g$ and initial data such that

$$\| \nabla u\|_{L^p} \le C \| m\dot{u}\|_{L^2}^{\frac{3p-6}{2p}} \left( \| \nabla u\|_{L^2} + \|P\|_{L^2} \right)^{\frac{6-p}{2p}} + \|P\|_{L^p}. \tag{27}$$

**Lemma 2.2** (Lemma 3.2 [26]). There holds

$$\sup_{0\le t\le T} \int \left[ \frac{1}{2} m |u|^2 + \alpha_g^2 \left(\rho_g - \rho_g\right)^2 + \alpha_l^2 \left(\rho_l - \rho_l\right)^2 \right] dx + \mu \int_0^T \| \nabla u\|_{L^2}^2 dt \le CC_0. \tag{28}$$

Proof. Multiplying $(1)_3$ by $u$ and integrating the resultant equation over $\mathbb{R}^3$ imply

$$\frac{1}{2} \frac{d}{dt} \int m |u|^2 dx + \mu \| \nabla u\|_{L^2}^2 + (\mu+\lambda) \|\text{div}\, u\|_{L^2}^2 = -\int u \cdot \nabla P dx. \tag{29}$$

It is easy to have

$$\begin{aligned}
&\int \alpha_l u \cdot \nabla P_l(\rho_l)dx \\
&= \int \alpha_l \rho_l u \cdot \nabla \left( \frac{P_l(\rho_l) - P\left(\rho_l\right)}{\rho_l} \right) dx + \int \alpha_l \rho_l u \cdot \nabla \rho_l \frac{P_l(\rho_l) - P\left(\rho_l\right)}{\rho_l^2} dx \\
&= \int m_t \frac{P_l(\rho_l) - P\left(\rho_l\right)}{\rho_l} dx + \int \alpha_l \rho_l u \cdot \nabla \left( \int_{\rho_l}^{\rho_l} \frac{P_l(s) - P\left(\rho_l\right)}{s^2} ds \right) dx \qquad (30)\\
&= \int \alpha_{lt} \rho_l \frac{P_l(\rho_l) - P\left(\rho_l\right)}{\rho_l} dx + \int \alpha_l \rho_{lt} \frac{P_l(\rho_l) - P(\rho_l)}{\rho_l} dx + \int m_t \int_{\rho_l}^{\rho_l} \frac{P_l(s) - P\left(\rho_l\right)}{s^2} dsdx \\
&= \left( \int m \int_{\rho_l}^{\rho_l} \frac{P_l(s) - P\left(\rho_l\right)}{s^2} dsdx \right)_t + \int \alpha_{lt} \left( P_l(\rho_l) - P\left(\rho_l\right) \right) dx.
\end{aligned}$$

Similarly, we obtain

$$\int \alpha_g u \cdot \nabla P_g(\rho_g) dx = \left( \int n \int_{\rho_g}^{\rho_g} \frac{P_g(s) - P\left(\rho_g\right)}{s^2} dsdx \right)_t + \int \alpha_{gt} \left( P_g(\rho_g) - P\left(\rho_g\right) \right) dx. \qquad (31)$$

It follows from (3), (7) and (29)-(31) that

$$\begin{aligned}
\int u \cdot \nabla P dx &= \int u \cdot \nabla \left[ \alpha_l P_l(\rho_l) + \alpha_g P_g(\rho_g) \right] dx \\
&= \int \left[ \alpha_l u \cdot \nabla P_l(\rho_l) + \alpha_g u \cdot \nabla P_g(\rho_g) \right] dx \qquad (32)\\
&= \left( \int m \int_{\rho_l}^{\rho_l} \frac{P_l(s) - P\left(\rho_l\right)}{s^2} dsdx + \int n \int_{\rho_g}^{\rho_g} \frac{P_g(s) - P\left(\rho_g\right)}{s^2} dsdx \right)_t .
\end{aligned}$$

Combining (29) and (32), and integrating the resulting equality over $(0,T)$ leads to

$$\int \left[ \frac{1}{2} m |u|^2 + \alpha_l \rho_l \int_{\rho_l}^{\rho_l} \frac{P_l(s) - P_l\left(\rho_l\right)}{s^2} ds + \alpha_g \rho_g \int_{\rho_g}^{\rho_g} \frac{P_g(s) - P_g\left(\rho_g\right)}{s^2} ds \right] dx + \mu \int_0^T \| \nabla u \|_{L^2}^2 dt \le C_0. \qquad (33)$$

Moreover, it follows from (6) that

$$\begin{aligned}
&\alpha_g \rho_g \int_{\rho_g}^{\rho_g} \frac{P_g(s) - P_g\left(\rho_g\right)}{s^2} ds \\
&= a_g^2 \alpha_g \left( \rho_g \ln \rho_g - \rho_g \ln \rho_g + \rho_g - \rho_g \right) \\
&= a_g^2 \alpha_g \left( \rho_g \int_0^1 \frac{d}{d\tau} \ln(\tau(\rho_g - \rho_g) + \rho_g) d\tau + \int_0^1 \frac{d}{d\tau} (\rho_g + \tau(\rho_g - \rho_g)) d\tau \right) \\
&= a_g^2 \alpha_g \left( \rho_g \int_0^1 \frac{\rho_g - \rho_g}{\tau(\rho_g - \rho_g) + \rho_g} d\tau + \int_0^1 (\rho_g - \rho_g)) d\tau \right) \qquad (34)\\
&= a_g^2 \alpha_g \left( \rho_g - \rho_g \right)^2 \int_0^1 \frac{1-\tau}{\tau\left(\rho_g - \rho_g\right) + \rho_g} d\tau \\
&= a_g^2 \alpha_g^2 \left( \rho_g - \rho_g \right)^2 \int_0^1 \frac{1-\tau}{\tau\left(n - \alpha_g \rho_g\right) + \alpha_g \rho_g} d\tau \\
&\ge a_g^2 \alpha_g^2 \left( \rho_g - \rho_g \right)^2 \int_0^1 \frac{1-\tau}{\sup\limits_{(x,t)\in\mathbb{R}^3\times[0,T]} n} d\tau \ge C \alpha_g^2 \left( \rho_g - \rho_g \right)^2,
\end{aligned}$$

similarly, we obtain

$$\alpha_l \rho_l \int_{\rho_l}^{\rho_l} \frac{P_l(s) - P_l\left(\rho_l\right)}{s^2} ds \geq C\alpha_l^2 \left(\rho_l - \tilde{\rho}_l\right)^2, \tag{35}$$

combining this with (33) proves Lemma 2.2.

**Lemma 2.3.** Suppose $m, n$ satisfy the assumption (18) and (25)-(26), it can obtain

$$0 \leq P_m \leq C, \quad P_n > 0, \quad 0 \leq \sqrt{n} P_n \leq C, \tag{36}$$

where $C = (a_0, c^0, k_0, \hat{m}, \hat{n})$ .

Proof. A straightforward computation shows that

$$0 \leq P_m = c^0 \left( 1 - \frac{b(m,n)}{\sqrt{b^2(m,n) + c(m,n)}} \right) \leq c^0 \leq C,$$

$$\begin{aligned} 0 < P_n &= a_0 c^0 \left( 1 + \frac{k_0 + m + a_0 n}{\sqrt{(k_0 - m - a_0 n)^2 + 4k_0 a_0 n}} \right) \\ &= a_0 c^0 \left( 1 + \left( \frac{k_0^2 + m^2 + (a_0 n)^2 + 2k_0 m + 2k_0 a_0 n + 2a_0 mn}{k_0^2 + m^2 + (a_0 n)^2 - 2k_0 m + 2k_0 a_0 n + 2a_0 mn} \right)^{\frac{1}{2}} \right) \\ &= a_0 c^0 \left( 1 + \left( 1 + \frac{4k_0 m}{(k_0 - m)^2 + (a_0 n)^2 + 2k_0 a_0 n + 2a_0 mn} \right)^{\frac{1}{2}} \right) \\ &\leq a_0 c^0 \left( 1 + \left( 1 + \frac{4k_0 m}{2k_0 a_0 n} \right)^{\frac{1}{2}} \right) = a_0 c^0 \left( 1 + \left( 1 + \frac{2m}{a_0 n} \right)^{\frac{1}{2}} \right), \end{aligned}$$

which together with (25), we have

$$0 \leq \sqrt{n} P_n \leq a_0 c^0 \left( \sqrt{n} + \left( n + \frac{2m}{a_0} \right)^{\frac{1}{2}} \right) \leq C. \tag{37}$$

Consequently, the Lemma 2.3 is proved.

Using the same arguments as Lemma 3.6 and Lemma 3.7 in [26], these estimates are readily available, and details have been omitted for brevity.

**Lemma 2.4.** Let (m, n, u) is a strong solution of (1), (2) and (8)-(10) satisfying assumptions (18) and (25)-(26), it has

$$B_1(T) + B_2(T) \leq C_0^{\frac{1}{4}} \tag{38}$$

and

$$\sup_{0 \leq t \leq \sigma(T)} (\sigma \| m^{\frac{1}{2}} \dot{u} \|_{L^2}^2) + \int_0^{\sigma(T)} \sigma \| \nabla \dot{u} \|_{L^2}^2 dt \leq C. \tag{39}$$

The following result about is absolutely crucial to prove Theorems 1.2 and 1.3

**Lemma 2.5.** There holds for any T>0

$$\int_0^T \sigma^3 \| P \|_{L^4}^4 dt \leq C C_0^{\frac{3}{8}}. \tag{40}$$

Proof. On the one hand, a direct calculation with (13) gives that

$$
\begin{aligned}
-\left(\int P^3 dx\right)_t &= -3\int P^2\left(P_m m_t + P_n n_t\right)dx \\
&= 3\int P^2\left[P_m\left(u\cdot\nabla m + m\,\mathrm{div}\,u\right) + P_n\left(u\cdot\nabla n + n\,\mathrm{div}\,u\right)\right]dx \\
&= 3\int P^2\left(P_m u\cdot\nabla m + P_n u\cdot\nabla n\right)dx + 3\int P^2\left(P_m m + P_n n\right)\mathrm{div}\,u dx \\
&= 3\int P^2 u\cdot\nabla P dx + 3\int P^2\left(P_m m + P_n n\right)\mathrm{div}\,u dx \\
&= -\int \mathrm{div} u P^3 dx + 3\int P^2\left(P_m m + P_n n\right)\mathrm{div}\,u dx \\
&= -\frac{1}{\lambda+2\mu}\int (F+P)P^3 dx + 3\int P^2\left(P_m m + P_n n\right)\mathrm{div}\,u dx \\
&= -\frac{1}{\lambda+2\mu}\int F P^3 dx - \frac{1}{\lambda+2\mu}\int P^4 dx + 3\int P^2\left(P_m m + P_n n\right)\mathrm{div}\,u dx,
\end{aligned}
\tag{41}
$$

which gives

$$
\frac{1}{\lambda+2\mu}\int P^4 dx = \left(P^3 dx\right)_t - \frac{1}{\lambda+2\mu}\int F P^3 dx + 3\int P^2\left(P_m n + P_n n\right)\mathrm{div}\,u dx. \tag{42}
$$

Multiplying (42) by $\sigma^3$ and integrating it over $(0,T)$, and using (38), (28), (39) and (41), we can get

$$
\begin{aligned}
&\int_0^T \sigma^3 \|P\|_{L^4}^4 dt \\
&\le C\sigma^3\|P\|_{L^3}^3 + C\int_0^T \sigma'\|P\|_{L^2}^2 dt + \delta\int_0^T \sigma^3\|P\|_{L^4}^4 dt + C\int_0^T \sigma^3\|\nabla u\|_{L^2}^2 dt + \int_0^T \sigma^3\|F\|_{L^4}^4 dt \\
&\le C\sigma^3\|P\|_{L^2}^2 + \delta\int_0^T \sigma^3\|P\|_{L^4}^4 dt + C\int_0^{\sigma(T)}\|P\|_{L^2}^2 dt + C\int_0^T \sigma^3\left(\|\nabla u\|_{L^2}^4 + \|m^{\frac{1}{2}}\dot{u}\|_{L^2}^4\right)dt \\
&\le \delta\int_0^T \sigma^3\|P\|_{L^4}^4 dt + CC_0 + \int_0^T (\sigma\|\nabla u\|_{L^2}^2)\|\nabla u\|_{L^2}^2 dt + C\int_{\sigma(T)}^T (\sigma^3\|m^{\frac{1}{2}}\dot{u}\|_{L^2}^2)\sigma\|m^{\frac{1}{2}}\dot{u}\|_{L^2}^2 dt \\
&+\int_0^{\sigma(T)}\left(\sigma^3\|m^{\frac{1}{2}}\dot{u}\|_{L^2}^2\right)^{\frac{1}{2}}\left(\sigma\|m^{\frac{1}{2}}\dot{u}\|_{L^2}^2\right)^{\frac{1}{2}}\sigma\|m^{\frac{1}{2}}\dot{u}\|_{L^2}^2 dt \\
&\le \delta\int_0^T \sigma^3\|P\|_{L^4}^4 dt + CC_0^{\frac{3}{8}},
\end{aligned}
\tag{43}
$$

where we deal with the second inequality by using the following estimate

$$
\begin{aligned}
\|P\|_{L^2}^2 &= \|\alpha_l(P_l(\rho_l)-P_l(\rho_l)) + \alpha_g(P_g(\rho_g)-P_g(\rho_g))\|_{L^2}^2 \\
&= \|a_l^2\alpha_l(\rho_l-\rho_l) + a_g^2\alpha_g(\rho_g-\rho_g)\|_{L^2}^2 \\
&\le C(\|\alpha_l(\rho_l-\rho_l)\|_{L^2}^2 + \|\alpha_g(\rho_g-\rho_g)\|_{L^2}^2) \le CC_0.
\end{aligned}
\tag{44}
$$

Choosing enough small $\delta$ in (43), we will have (40).

**Proof of Theorem 1.2.** First, thanks to $P_t = P_m m_t + P_n n_t$ and $\nabla P = P_m\nabla m + P_n\nabla n$, which together with $(1)_1$ and $(1)_2$ yields

$$
P_t + u\cdot\nabla P + (P_m m + P_n n)\mathrm{div}u = 0. \tag{45}
$$

Multiplying (45) by $4|P|^3$ and integrating the resulting equation by parts imply

$$
\left(\|P\|_{L^4}^4\right)_t \le C\|\mathrm{div}u\|_{L^2}^2 + C\|P\|_{L^4}^4,
$$

which together with (28) and (40) yields that

$$
\begin{aligned}
\int_1^\infty\left(\|P\|_{L^4}^4\right)_t dt &\le C\int_1^\infty\|\mathrm{div}u\|_{L^2}^2 dt + C\int_1^\infty \sigma^3\|P\|_{L^4}^4 dt \\
&\le C\int_0^\infty\|\nabla u\|_{L^2}^2 dt + C\int_0^\infty \sigma^3\|P\|_{L^4}^4 dt \\
&\le C.
\end{aligned}
\tag{46}
$$

Combining (40) with (46) leads to

$$\lim_{t\to\infty}\|P\|_{L^4}^4 = 0, \tag{47}$$

which together with (8) gives

$$\lim_{t\to\infty}\| \rho_l - \rho_l\|_{L^4} = 0 \text{ and } \lim_{t\to\infty}\| \rho_g - \rho_g\|_{L^4} = 0. \tag{48}$$

It follows from (3.13) of Lemma 3.3 in [26] that

$$\begin{aligned}&\left(\frac{\mu}{2}\sigma^k\| \nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2}\sigma^k\|\operatorname{div}u\|_{L^2}^2\right)_t + \sigma^k\|\sqrt{m}\dot{u}\|_{L^2}^2 \\ &\le \left(\int \sigma^k \operatorname{div} u P\right)_t + C(\sigma^k + k\sigma^{k-1}\sigma')\| \nabla u\|_{L^2}^2 + C\sigma^k\| \nabla u\|_{L^3}^3 + Ck\sigma^{k-1}\sigma' C_0.\end{aligned} \tag{49}$$

Let $k = 0$ in (49), integrating the resultant equation over and using (27), (28), (38) and (44), one can get

$$\begin{aligned}&\int_1^\infty\left|\left(\frac{\mu}{2}\| \nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2}\|\operatorname{div}u\|_{L^2}^2\right)_t\right|dt \\ &\le C\int_1^\infty (\|\nabla u\|_{L^2}^2 + \| \nabla u\|_{L^3}^3)dt + C \\ &\le C\int_1^\infty (\|\nabla u\|_{L^2}^2 + \| \nabla u\|_{L^2}\| \nabla u\|_{L^4}^2)dt + C \\ &\le C\int_1^\infty (\|\nabla u\|_{L^2}^2 + \| \nabla u\|_{L^2}^4 + \| m\dot{u}\|_{L^2}^3\| \nabla u\|_{L^2} + \| m\dot{u}\|_{L^2}^3\|P\| + \|P\|_{L^4}^4)dt + C \le C.\end{aligned} \tag{50}$$

By (28), we obtain that

$$\int_1^\infty \| \nabla u\|_{L^2}^2 dt \le \int_0^\infty \| \nabla u\|_{L^2}^2 dt \le C,$$

which together with (50) implies

$$\lim_{t\to\infty}\| \nabla u\|_{L^2} = 0, \tag{51}$$

Hence, we can obtain

$$\lim_{t\to\infty}\int m^{\frac{1}{2}} | u |^4 dx = 0, \tag{52}$$

due to (51) and the following fact

$$\int m^{\frac{1}{2}} | u |^4 dx \le C\left(\int m | u |^2 dx\right)^{\frac{1}{2}}\left(\int |u |^6 dx\right)^{\frac{1}{2}} \le C\| \nabla u\|_{L^2}^3. \tag{53}$$

Therefore, combining (48), (51) and (52) gives (20).

**Proof of Theorem 1.3.** Assume the conclusion of Theorem 1.3 is not valid, then there exists $M > 0$ and $\left\{t_{m_i}\right\}_{i=1}^\infty$ and $t_{m_i} \to \infty$, such that $\| \nabla\rho_l(\cdot, t_{m_i})\|_{L^\gamma} \le M$ . Therefore, there exists $C > 0$ independent of $t_{n_j}$ such that for $a = 3\gamma / (3\gamma + 4(\gamma - 3)) \in (0,1)$ ,

$$\| \rho_l(\cdot, t_{m_i}) - \rho_l\|_{C(\overline{\mathbb{R}^3})} \le C\| \nabla\rho_l(\cdot, t_{m_i})\|_{L^\gamma}^a\| \rho_l(\cdot, t_{m_i}) - \rho_l\|_{L^4}^{1-a} \le CM^a\| \rho_l(\cdot, t_{m_i}) - \rho_l\|_{L^4}^{1-a},$$

which together with (20) yields

$$\lim_{t_{m_i}\to\infty}\| \rho_l(\cdot, t_{m_i}) - \rho_l\|_{C(\overline{\mathbb{R}^3})} = 0. \tag{54}$$

On the other hand, since assumptions in Theorem 1.3, there exists a some point $x_1(t) = x_1$ such that

$$\rho_l(x_1(t), t) \equiv 0 \text{ for all } t > 0.$$

As a result, we can have

$$\| \rho_l(\cdot, t_{m_i}) - \rho_l\|_{C(\overline{\mathbb{R}^3})} \ge | \rho_l(x_1(t_{m_i}), t_{m_i}) - | \equiv \rho_l > 0,$$

which contradicts (54). So $(21)_1$ can be obtained. By the same way, $(21)_2$ can be obtained. To sum up, the Theorem 1.3 is proved.

## Conflicts of Interest

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The author(s) declare that they have no competing interests.

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