# Application of Tikhonov Regularization in Generalized Inverse of Adjacency Matrix of Undirected Graph 

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Abstract - In this article, we found the Moore-Penrose generalized inverse of adjacency matrix of an undirected graph, explicitly. We proved that the matrix $R_{\lambda}=\left[r_{i j}\right]$ is nonsingular where $r_{i i}=\frac{1}{\lambda}+\operatorname{deg} v_{i}$ and $r_{i j}=\left|N_{G}\left(v_{i}\right) \cap N_{G}\left(v_{j}\right)\right|$ for $i \neq j$ and, we proved that $A_{G}^{\dagger}=\left[s_{i j}\right]_{1 \leq i, j \leq n}$ where $s_{i j}=s_{j i}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{j}, f_{i}\right\rangle$. The proof of the main result was based on the Tikhonov regularization.

Keywords - Moore-Penrose generalized inverse, Adjacency matrix, Tikhonov regularization, Undirected graph, Nonsingularity.

## I. INTRODUCTION

Let $G=(V, E)$ be an undirected graph of order $n$ and size $m$ where set $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ is a vertex set, and $E$ is an edge set of $G$. The adjacency matrix of $G$ is an $n \times n$ matrix $A_{G}=\left[a_{i j}\right]_{1 \leq i, j \leq n}$ where

$$
a_{i j}= \begin{cases}1 & \text { if } v_{i} v_{j} \in E  \tag{1}\\ 0 & \text { otherwise }\end{cases}
$$

The adjacency matrix $A_{G}$ is a symmetric matrix, that is, $A_{G}=A_{G}^{t}$ where $A_{G}^{t}=\left[a_{j i}\right]_{1 \leq i, j \leq n}$ since $a_{i j}=a_{j i}[4,5,6,7,8]$. The Moore-Penrose generalized inverse of $A_{G}$ is an $n \times n$ matrix $A_{G}^{\dagger}$ where for each $y \in \mathbb{R}^{n}$ we have the following: 1. $\left\|A_{G} A_{G}^{\dagger} y-y\right\| \leq\left\|A_{G} x-y\right\|$ for all $x \in \mathbb{R}^{n}$.
2. If $\left\|A_{G} z-y\right\| \leq\left\|A_{G} x-y\right\|$ for all $x \in \mathbb{R}^{n}$, then $\left\|A_{G}^{\dagger} y\right\| \leq\|z\| .[1,2,3,24,25]$

Here, the norm $\|\cdot\|$ is the standard norm in $\mathbb{R}^{n}$. Another way to define the Moore-Penrose generalized inverse is using the Moore-Penrose equation, see [2,3,9,10,11,12,24,25].

One of the known approaches in finding the Moore-Penrose generalized inverse of any $m \times n$ matrix $A$ with real number entries is decomposing it using Singular-Value decomposition, say $A=U \Sigma V^{t}$ where $U$ is an $m \times m$ orthogonal matrix, $\Sigma$ is an $m \times n$ rectangular diagonal matrix with nonnegative real numbers on the diagonal, and $V$ is an $n \times n$ orthogonal matrix. If $A=U \Sigma V^{t}$ is the Singular-Value decomposition of $A$, then $A^{\dagger}=V \Sigma^{\dagger} U^{t}$ where $\Sigma^{\dagger}$ is the Moore-Penrose generalized inverse of $\Sigma$, which was formed by replacing every nonzero diagonal entry of $\Sigma$ by its reciprocal and transposing the resulting matrix.[3,18,19]

Another way of computing the Moore-Penrose generalized inverse of any bounded linear transformation was based on the Tikhonov regularization. Given a linear transformation $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$, then for $y \in \operatorname{Ran}(T)+\operatorname{Ran}(T)^{\perp}$, the problem

$$
\begin{equation*}
\min _{\substack{x \in \mathbb{X}}}\left(\alpha\|x\|^{2}+\|T x-y\|^{2}\right) \tag{2}
\end{equation*}
$$

has a unique solution $x_{\alpha}$. Furthermore, $x_{\alpha} \rightarrow T^{\dagger} y$ as $\alpha \rightarrow 0^{+} .[1,13,14,15,16,17]$
In this study, we found the Moore-Penrose generalized inverse of the adjacency matrix of the undirected graph $G$ using the Tikhonov regularization. Specifically, we proved that if $A_{G}$ is the adjacency matrix of $G, \lambda>0$ and $R_{\lambda}=\left[r_{i j}\right]_{1 \leq i, j \leq n}$ where

$$
r_{i j}= \begin{cases}\left|N_{G}\left(v_{i}\right) \cap N_{G}\left(v_{j}\right)\right| & \text { if } i \neq j  \tag{3}\\ \frac{1}{\lambda}+\operatorname{deg} v_{i} & \text { if } i=j\end{cases}
$$

then $R_{\lambda}$ is nonsingular, for all $\lambda>0$ and $A_{G}^{\dagger}=\left[s_{i j}\right]_{1 \leq i, j \leq n}$ where

$$
\begin{equation*}
s_{i j}=s_{j i}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{j}, f_{i}\right\rangle . \tag{4}
\end{equation*}
$$

In this paper, $G$ is always an undirected graph, the norm $\|\cdot\|$ is the standard norm in $\mathbb{R}^{n}$, the inner product $\langle\cdot$,$\rangle is the$ standard inner product in $\mathbb{R}^{n}$, and the set $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ is the standard basis of $\mathbb{R}^{n}$. In addition to that, the set of neigbourhood of $v \in V$ is denoted by $N_{G}(v)=\{u: u v \in E\}$.

## II. PRELIMINARY

Let $G=(V, E)$ be an undirected graph of order $n$ with size $m$ where $V=\left\{v_{1}, v_{2}, v_{3}, \ldots, v_{n}\right\}$ and $E$ be the edge set of $G$. For $i=1,2,3, \ldots, n$, set

$$
\delta_{u}(v)=\left\{\begin{array}{l}
1 \text { if } u v \in E  \tag{5}\\
0 \text { if } u v \notin E
\end{array}\right.
$$

and vector $f_{i}=\left(\delta_{v_{i}}\left(v_{1}\right), \delta_{v_{i}}\left(v_{2}\right), \delta_{v_{i}}\left(v_{3}\right), \ldots, \delta_{v_{i}}\left(v_{n}\right)\right) \in \mathbb{R}^{n}$. Then, the following proposition is easy to stablish:
Proposition 2.1. The following hold:
i. If $A_{G}$ is the adjacency matrix of $G$, then for every $x \in \mathbb{R}^{n}$, we have

$$
\begin{equation*}
A_{G} x=\sum_{i=1}^{n}\left\langle x, f_{i}\right\rangle e_{i} . \tag{6}
\end{equation*}
$$

ii. For every $y \in \mathbb{R}^{n}$, we have

$$
\begin{equation*}
A_{G}^{t} y=\sum_{i=1}^{n}\left\langle y, e_{i}\right\rangle f_{i} \tag{7}
\end{equation*}
$$

iii. For $i=1,2,3, \ldots n$, we have $A_{G} e_{i}=f_{i}$.
iv. For all $i, j \in\{1,2,3, \ldots, n\}$, we have $\left\langle f_{i}, f_{j}\right\rangle=\left|N_{G}\left(v_{i}\right) \cap N_{G}\left(v_{j}\right)\right|$. Furthermore, for $i=1,2,3, \ldots, n$ we have $\left\|f_{i}\right\|^{2}=\operatorname{deg} v_{i}$.

Proof: The proof for (i) and (iv) are straightforward. Thus, we will only prove (ii) and (iii).
Proof of (ii): Let $x, y \in \mathbb{R}^{n}$. Then,

$$
\left\langle A_{G}^{t} y, x\right\rangle=\left\langle y, A_{G} x\right\rangle=\left\langle y, \sum_{i=1}^{n}\left\langle x, f_{i}\right\rangle e_{i}\right\rangle=\sum_{i=1}^{n}\left\langle x, f_{i}\right\rangle\left\langle y, e_{i}\right\rangle=\left\langle\sum_{i=1}^{n}\left\langle y, e_{i}\right\rangle f_{i}, x\right\rangle
$$

Therefore,

$$
\begin{equation*}
A_{G}^{t} y=\sum_{i=1}^{n}\left\langle y, e_{i}\right\rangle f_{i} . \tag{8}
\end{equation*}
$$

Proof (iii): Let $i \in\{1,2,3, \ldots n\}$. Since $A_{G}=A_{G}^{t}$, then

$$
A_{G} e_{i}=A_{G}^{t} e_{i}=\sum_{k=1}^{n}\left\langle e_{i}, e_{k}\right\rangle f_{k}=f_{i}
$$

since $\left\langle e_{i}, e_{j}\right\rangle=\delta_{i j}$ where $\delta_{i j}$ is the Kronecker delta.

## III. MAIN RESULTS

It was well-known from the Tikhonov regularization that if $A$ is an $n \times n$ matrix with real entries, then for $y \in \mathbb{R}^{n}$, the problem

$$
\begin{equation*}
\min _{\substack{x \in \mathbb{R}^{2}}}\left(\alpha\|x\|^{2}+\|A x-y\|^{2}\right) \tag{9}
\end{equation*}
$$

has unique solution $x_{\alpha}$. Furthermore, $\left\|x_{\alpha}-A^{\dagger} y\right\| \rightarrow 0^{+}[1,13,14,15,16,17]$. The next theorem was based on the Tikhonov regularization.

Theorem 3.1 Let $G=(V, E)$ be undirected graph of order $n$ and size $m$. For $\lambda>0$, define the $n \times n$ matrix $R_{\lambda}=\left[r_{i j}\right]_{1 \leq i, j \leq n}$ where

$$
r_{i j}= \begin{cases}\left|N_{G}\left(v_{i}\right) \cap N_{G}\left(v_{j}\right)\right| & \text { if } i \neq j  \tag{10}\\ \frac{1}{\lambda}+\operatorname{deg} v_{i} & \text { if } i=j\end{cases}
$$

Then, $R_{\lambda}$ is nonsingular, for all $\lambda>0$ and $A_{G}^{\dagger}=\left[s_{i j}\right]_{1 \leq i, j \leq n}$ where $s_{i j}=s_{j i}=\lim \left\langle R_{\lambda}^{-1} e_{j}, f_{i}\right\rangle$.
Proof: Observe that $\frac{1}{\lambda} I_{n}+A_{G}^{t} A_{G}=\frac{1}{\lambda} I_{n}+A_{G}^{2}=R_{\lambda}$. Since $A_{G}^{2}$ is a positive operator, then $-\frac{1}{\lambda}$ is not an eigenvalue of $A_{G}^{2}$ for all $\lambda>0$, and thus, $R_{\lambda}=\frac{1}{\lambda} I_{n}+A_{G}^{2}$ is nonsingular. Let $y \in \mathbb{R}^{n}$, and let $x_{\lambda}$ be the unique solution to the problem:

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n}}\left(\frac{1}{\lambda}\|x\|^{2}+\left\|A_{G} x-y\right\|^{2}\right) \tag{11}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
\nabla\left(\frac{1}{\lambda}\left\|x_{\lambda}\right\|^{2}+\left\|A_{G} x_{\lambda}-y\right\|^{2}\right)=0 \tag{12}
\end{equation*}
$$

The equation in (12) implies that $\left(\frac{1}{\lambda} I_{n}+A_{G}^{2}\right) x_{\lambda}=A_{G} y$, and therefore, $R_{\lambda} x_{\lambda}=A_{G} y$. Since $R_{\lambda}$ is nonsingular, then $x_{\lambda}=$ $R_{\lambda}^{-1} A_{G} y$. Tikhonov regularization implies that $R_{\lambda}^{-1} A_{G} \rightarrow A_{G}^{\dagger}$ as $\lambda \rightarrow+\infty$, and hence, the matrix representation of $A_{G}^{\dagger}$ with respect to the standard basis $\left\{e_{1}, e_{2}, \ldots, e_{n}\right\}$ is given by $A_{G}^{\dagger}=\left[s_{i j}\right]$ where $s_{i j}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{j}, A_{G} e_{i}\right\rangle=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{j}, f_{i}\right\rangle$. Since $A_{G}$ is symmetric, then $\left(A_{G}^{\dagger}\right)^{t}=\left(A_{G}^{t}\right)^{\dagger}=\left(A_{G}^{\dagger}\right)$. Hence, $s_{i j}=s_{j i}$. .

The next corollary immediately holds.
Corollary 3.2. $A_{G}$ is nonsingular if and only if $\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} f_{j}, f_{i}\right\rangle=\delta_{i j}$, where $\delta_{i j}$ is the Kronecker delta.
Proof: $A_{G}$ is nonsingular if and only if $R_{\lambda}^{-1} A_{G}^{2} \rightarrow I_{n}$ as $\lambda \rightarrow+\infty$ if and only if $\delta_{i j}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} A_{G} e_{j}, A_{G} e_{i}\right\rangle=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} f_{j}, f_{i}\right\rangle$, as desired.

The following are the applications of theorem 3.1 and corollary 3.2.

## Example 1: Generalized Inverse of Star Graph

Let $S_{n}=(V, E)$ be a star graph of order $n$, where $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ and $E=\left\{v_{1} v_{i}: i=2,3,4, \ldots, n\right\}[20,21]$. Then, $A_{S_{n}}^{\dagger}=$ $\frac{1}{n-1} A_{S_{n}}$.
Proof: Observe $N_{S_{n}}\left(v_{i}\right)=\left\{v_{1}\right\}$ for $i=2,3,4, \ldots, n$, and thus, $\left|N_{S_{n}}\left(v_{i}\right) \cap N_{S_{n}}\left(v_{j}\right)\right|=1$ for $i, j \in\{2,3,4, \ldots, n\}, \operatorname{deg}\left(v_{i}\right)=1$ for $i=2,3,4, \ldots, n, \operatorname{deg}\left(v_{1}\right)=n-1$, and $\left|N_{S_{n}}\left(v_{1}\right) \cap N_{S_{n}}\left(v_{j}\right)\right|=0$ for $j=2,3,4, \ldots, n$. Therefore, we have

$$
R_{\lambda}=\left[\begin{array}{ccccccc}
\frac{1}{\lambda}+n-1 & 0 & 0 & 0 & 0 & & 0 \\
0 & \alpha & 1 & 1 & 1 & & 1 \\
0 & 1 & \alpha & 1 & 1 & \cdots & 1 \\
0 & 1 & 1 & \alpha & 1 & & 1 \\
& \vdots & & & & \ddots & \vdots \\
0 & 1 & 1 & 1 & 1 & \cdots & \alpha
\end{array}\right]
$$

where $\alpha=\frac{1}{\lambda}+1$. Observe that

$$
R_{\lambda}^{-1}=\left[\begin{array}{ccccccc}
\frac{1}{\frac{1}{\lambda}+n-1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & \beta & \theta & \theta & \theta & \cdots & \theta \\
0 & \theta & \beta & \theta & \theta & \cdots & \theta \\
0 & \theta & \theta & \beta & \theta & \cdots & \theta \\
0 & \theta & \theta & \theta & \beta & \cdots & \theta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \theta & \theta & \theta & \theta & \cdots & \beta
\end{array}\right]
$$

where $\theta=\frac{1}{-\alpha-n \alpha+3 \alpha+n-2}$ and $\beta=\frac{-\alpha-n+3}{-\alpha^{2}-n \alpha+3 \alpha+n-2}$. Thus, $R_{\lambda}^{-1} e_{1}=\left(\frac{1}{\frac{1}{\lambda}+n-1}, 0,0,0, \ldots, 0,0\right)$, and for $k>1$ we have $R_{\lambda}^{-1} e_{k}=$ $(0, \theta, \ldots, \theta, \beta, \theta, \ldots, \theta)$ where the first coordinate is 0 , the $k$ th coordinate is $\beta$, otherwise $\theta$. Since $f_{1}=(0,1,1,1, \ldots, 1)$ and $f_{k}=(1,0,0, \ldots, 0)$ for $k>1$, then $A_{S_{n}}^{\dagger}=\left[s_{i j}\right]_{1 \leq i, j \leq n}$ where

$$
\begin{gathered}
s_{11}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{1}, f_{1}\right\rangle=\lim _{\lambda \rightarrow+\infty} 0=0 \\
s_{1 k}=s_{k 1}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{1}, f_{k}\right\rangle=\lim _{\lambda \rightarrow+\infty} \frac{1}{\frac{1}{\lambda}+n-1}=\frac{1}{n-1} \quad(k>1) \\
s_{i j}=s_{j i}=\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} e_{i}, f_{j}\right\rangle=\lim _{\lambda \rightarrow+\infty} 0=0 \quad(i, j>1) .
\end{gathered}
$$

Therefore,

$$
A_{S_{n}}^{\dagger}=\left[\begin{array}{ccccccc}
0 & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\
\frac{1}{n-1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{n-1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{n-1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{n-1} & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n-1} & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right]
$$

Example 2: The complete graph of order 4 is the graph $K_{4}=(V, E)$, where $V=\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$ and $E=\left\{v_{i} v_{j}: i, j \in\right.$ $\{1,2,3,4\}\}[22,23]$. Since $N_{K_{4}}\left(v_{k}\right)=\left\{v_{i}: i \neq k\right\}$ for all $k \in\{1,2,3,4\}$, then $\operatorname{deg}\left(v_{k}\right)=3$ and $\left|N_{K_{4}}\left(v_{i}\right) \cap N_{K_{4}}\left(v_{j}\right)\right|=2$ for $i \neq j$. Therefore,

$$
R_{\lambda}=\left[\begin{array}{cccc}
3+\frac{1}{\lambda} & 2 & 2 & 2 \\
2 & 3+\frac{1}{\lambda} & 2 & 2 \\
2 & 2 & 3+\frac{1}{\lambda} & 2 \\
2 & 2 & 2 & 3+\frac{1}{\lambda}
\end{array}\right]
$$

Taking the inverse of $R_{\lambda}$, we have

$$
R_{\lambda}^{-1}=\left[\begin{array}{llll}
\beta & \theta & \theta & \theta \\
\theta & \beta & \theta & \theta \\
\theta & \theta & \beta & \theta \\
\theta & \theta & \theta & \beta
\end{array}\right]
$$

where $\beta=\frac{7+\frac{1}{\lambda}}{\left(3+\frac{1}{\lambda}\right)\left(7+\frac{1}{\lambda}\right)-12}$ and $\theta=\frac{-2}{\left(3+\frac{1}{\lambda}\right)\left(7+\frac{1}{\lambda}\right)-12}$. Thus, $R_{\lambda}^{-1} e_{1}=(\beta, \theta, \theta, \theta), R_{\lambda}^{-1} e_{2}=(\theta, \beta, \theta, \theta), R_{\lambda}^{-1} e_{3}=(\theta, \theta, \beta, \theta)$ and $R_{\lambda}^{-1} e_{4}=(\theta, \theta, \theta, \beta)$. Since $f_{1}=(0,1,1,1), f_{2}=(1,0,1,1), f_{3}=(1,1,0,1), f_{4}=(1,1,1,0)$, then

$$
\begin{equation*}
\left\langle R_{\lambda}^{-1} e_{i}, f_{i}\right\rangle=3 \theta \quad(i=1,2,3,4) \tag{13}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle R_{\lambda}^{-1} e_{i}, f_{j}\right\rangle=\beta+2 \theta \quad(i \neq j) \tag{14}
\end{equation*}
$$

Letting $\lambda \rightarrow+\infty$, then $\theta \rightarrow \frac{-2}{9}$ and $\beta \rightarrow \frac{7}{9}$. Thus, $A_{k_{4}}^{\dagger}=\left[s_{i j}\right]$ where $s_{i i}=3\left(-\frac{2}{9}\right)=-\frac{2}{3}$ for $i=1,2,3,4$ and $s_{i j}=s_{j i}=\frac{7}{9}+$ $2\left(-\frac{2}{9}\right)=\frac{1}{3}$ for $i \neq j$. Therefore,

$$
A_{K_{4}}^{\dagger}=\left[\begin{array}{cccc}
\frac{-2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3}
\end{array}\right]
$$

Example 3: Show using corollary 3.2 that the adjacency matrix of $K_{4}$ is nonsingular, and thus, $A_{K_{4}}^{\dagger}=A_{K_{4}}^{-1}$.
Proof: In view of corollary 3.2, we need to show that

$$
\begin{equation*}
\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} f_{j}, f_{i}\right\rangle=\delta_{i j} \tag{15}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta. Applying example 2 , note that we have $R_{\lambda}^{-1} f_{1}=(3 \theta, \beta+2 \theta, \beta+2 \theta, \beta+2 \theta), R_{\lambda}^{-1} f_{2}=$ $(\beta+2 \theta, 3 \theta, \beta+2 \theta, \beta+2 \theta), R_{\lambda}^{-1} f_{3}=(\beta+2 \theta, \beta+2 \theta, 3 \theta, \beta+2 \theta)$ and $R_{\lambda}^{-1} f_{4}=(\beta+2 \theta, \beta+2 \theta, \beta+2 \theta, 3 \theta)$. Thus,

$$
\left\langle R_{\lambda}^{-1} f_{j}, f_{i}\right\rangle= \begin{cases}3(\beta+2 \theta) & \text { if } i=j  \tag{16}\\ 3 \theta+2(\beta+2 \theta) & \text { if } i \neq j\end{cases}
$$

Letting $\lambda \rightarrow+\infty$, then $3(\beta+2 \theta) \rightarrow 1$ and $3 \theta+2(\beta+2 \theta) \rightarrow 0$, and therefore, $\lim _{\lambda \rightarrow+\infty}\left\langle R_{\lambda}^{-1} f_{j}, f_{i}\right\rangle=\delta_{i j}$, as desired. $\square$

## IV. CONCLUSIONS

If $A_{G}$ is nonsingular, then $A_{G}^{\dagger}=A_{G}^{-1}$. Thus, theorem 3.1 is much difficult to use than by solving the inverse of $A_{G}$. However, it is useful when the adjacency matrix of the graph is singular since the inverse of $A_{G}$ does not exists. Consequently, theorem 3.1 states that we can solve the problem on finding the Moore-Penrose generalized inverse of the adjacency matrix of a graph by finding the inverse of the resolvent $R_{\lambda}=\frac{1}{\lambda} I+A_{G}^{2}$ of $A_{G}^{2}$, which is always nonsingular for $\lambda>0$.

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