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Original Article

Distributions Generated from Functions in Smirnov Spaces as Boundary Values of Holomorphic Functions

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Abstract – *We characterise the distributions generated from the boundary values of functions from Smirnov spaces.*

Keywords – *Boundary values of distributions, Distributions, Smirnof spaces.*

I. INTRODUCTION

We use the following notation and preliminaries. U stands for open unit disc in C and T its' boundary, i.e. $U = \{z \in C||z| < 1\}, T = \partial U,$ *and* Π^+ is the upper half plane, meaning $\Pi^+ = \{z \in C|Imz > 0\}$. For a function f holomorphic on a region Ω we right $f \epsilon H(\Omega)$. For $p \geq 1$, $L^p(\Omega)$ is the space of measurable functions on Ω such that $\int_\Omega |f(x)|^p dx < \infty$; L^p_{loc} is the space of measurable functions on Ω such that for every compact set $K \subset \Omega$ the following holds $\int_K |f(x)|^p dx < \infty$.

Smirnof spaces on U and Π^+ and their properties: Smirnof class $N^+(U)$, is a subclass of Navalina class $N(U)$, which consists of all functions such that

$$\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} log^+ \left| f(re^{i\theta}) \right| d\theta < \infty \ \ (f \in N(U))$$

and in addition

$$\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} log^+ \left| f(re^{i\theta}) \right| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log(1 + \left| f^*(e^{i\theta}) \right|) \, d\theta,$$

where $f^*\left(e^{i\theta}\right) = \lim_{r \to 1} f(re^{i\theta})$ and exists almost everywhere on T. Concerning the upper half space we recall that $f \in N^+(\Pi^+)$ if

$$\sup_{y > 0} \int_{-\infty}^{\infty} \log(1 + |f(x + iy)|) \, dx < \infty \ \ (f \in N(\Pi^+))$$

and if

$$\lim_{y \to 0^+} \int_R \log(1 + |f(x + iy|) \, dx = \int_R \log(1 + |f^*(x)|) \, dx,$$

where $f^*(x) = \lim_{y \to 0^+} f(x + iy)$ which again hold a.e. on R.

Theorem. ([21]) The function f, holomorphic on U, belongs to N^+ if and only if for every $\varepsilon > 0$ there exist $\delta > 0$ such that for every measurable set $E \subset T$, satisfying $m(E) < \delta$ the following holds

$$\int_E log^+ \left| f(re^{i\theta}) \right| d\theta < \varepsilon, \qquad \text{for all } 0 \leq r < 1.$$

We will use a fact which is a charaterization of bounded subsets of the class N^+ which we state in the following.
Theorem. ([25]) L is bounded in $N^+(\Pi^+)$ if and only if

i) There exist $C > 0$ such for all $f \in N^+(\Pi^+)$

$$\int_R \log(1 + |f^*(x)|)dx < C$$

for all $f \in L$.

ii) For every $\varepsilon > 0$, exist $\delta > 0$ such that

$$\int_{\mathrm{E}} \log(1 + |f^*(x)|)\,dx < \varepsilon$$

for all $f \in L$, and every Lebesgue measurable $E \subset R$ satisfying $m(E) < \delta$.

Distributions: $C^\infty(R^n)$ denotes the set of all complex valued functions infinitely differentiable on R^n; $C_0^\infty(R^n)$ is the subset of $C^\infty(R^n)$ which contains compactlly supported functions. Support of the function f denoted with $suppf$ is the cloasure of the set $\{x: f(x) \neq 0\}$ in R^n. $D = D(R^n)$ denotes the space $C_0^\infty(R^n)$ endowed with the topology defined with the convergence: the sequence $\{\varphi_\lambda\}$, of functions $\varphi_\lambda \epsilon D$, converges to $\varphi \epsilon D$ when $\lambda \to \lambda_0$ if and only if there exist compact subset of R^n such that $supp\ \varphi_\lambda \subseteq$ K for all λ, $supp\ \varphi \subseteq$ K, and for every n-tuple α of nonegative integers the sequence $\{D_x^\alpha(\varphi_\lambda(x))\}$ converges to $\{D_x^\alpha(\varphi(x))\}$ uniformly on K when $\lambda \to \lambda_0$. With $D' = D'(R^n)$ is denoted the space of all continuous, linear functionals on D, where the continuity is in the sense: from $\varphi_\lambda \to \varphi$ in D when $\lambda \to \lambda_0$ it follows that $\langle T, \varphi_\lambda \rangle \to \langle T, \varphi \rangle$ in C, when $\lambda \to \lambda_0$. The space D' is called the space of distributions. We use the convention $\langle T, \varphi \rangle = T(\varphi)$ for the value of the functional Tacting on the function φ.
Let $\varphi \epsilon D$ and $f(x) \in L_{loc}^1(R^n)$. Then the functional T_f on D defined with

$$\langle \mathrm{T}_f, \varphi \rangle = \int_{R^n} f(t)\varphi(t)dt, \varphi \epsilon D,$$

is an element in D' and it is called a regular distribution generated with the function f.

II. RESULTS

The following theorem is used as a main motivation for the results Theorem 1 and Theorem 2 in this article.
Theorem. ([16]) Sufficient and necessary condition for the measurable function $\varphi(e^{i\theta})$ defined on T to coincide almost everywhere on T with the boundary value $f^*(e^{i\theta})$ of some function $f(z)$ in $N(U)$, is to exist a sequence of polynomials $\{P_n(z)\}$ such that:

i. $\{P_n(e^{i\theta})\}$ converges to $\varphi(e^{i\theta})$ almost everywhere on T;
ii. $\overline{\lim_{n\to\infty}} \int_0^{2\pi} (log^+ |P_n(e^{i\theta})|) d\theta < \infty$.

Theorem 1. Let $T_{f^*} \in D'$ is generated from the boundary value $f^*(x)$ of a function $f(z)$ in $N^+(\Pi^+)$. There exist sequence of polynomials $\{P_n(z)\}$, $z \in \Pi^+$, and respectivelly $\{T_n\}$, $T_n \in D'$, generated from the boundary values $P_n{}^*(x)$ of the polynomials $P_n(z)$, i.e. $T_n = T_{P_n^*}$ such that:

i. $T_n \to T_{f^*}$ in D' when $n \to \infty$,
ii. $\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} log\ (1 + |P_n^*(x)|)|\varphi(x)|\,dx < \infty$ for every $\varphi \epsilon D$.

Proof.
Let the assumptions of the theorem hold. Then we associate to f the following distribution

$$\langle \mathrm{T}_{f^*}, \varphi \rangle = \int_R \log(1 + |f^*(x)|)\varphi(x)dx, \varphi \epsilon D,$$

and since $f \in N^+(\Pi^+)$, one has $\sup_{y>0} \int_{-\infty}^{\infty} \log(1 + |f(x + iy)|)\,dx < \infty$ and $\lim_{y\to 0^+} \int_R \log(1 + |f(x + iy|)\,dx =$ $\int_R \log(1 + |f^*(x)|)\,dx$. Because of the definition of $N^+(\Pi^+)$, the integral in the previous definition makes sence (it is finite and the function $\log(1 + |f^*(x)|) \in L_{loc}^1(R)$).

Let $y_n > 0, \forall n \epsilon N$, and $\lim_{n\to\infty} y_n = 0$. We define a sequence of complex functions $\{F_n(z)\}$ with

$$F_n(z) = \log(1 + |f(z + iy_n)|).$$

For the defined sequence of complex functions we have

$$F_n(x + iy) = \log(1 + |f(x + iy + iy_n)|) = \log(1 + |f(x + i(y + y_n))|),$$

from where the bounds

$$\lim_{n\to\infty}\lim_{y\to 0^+}\int_R F_n(x + iy)\,dx = \lim_{y\to 0^+}\lim_{n\to\infty}\int_R F_n(x + iy)\,dx = \int_R \log(1 + |f^*(x)|)\,dx \ldots (*)$$

a.e. on R since $f \in N^+$ and the double limit $(n, y) \to (\infty, 0)$ exist. It is obvious that the functions $F_n(z)$ are holomorphic on $\Pi^+ \cup R$. Chose $\varepsilon > 0$. For every $n\epsilon N$ there exits polynomial $P_n(z)$ such that $|F_n(z) - P_n(z)| < \varepsilon/(2Mm(K))$, for all $z \in K_1$, K is arbitrary but fixed compact subset of Π^+. The lest statement follows from Margelijan theorem.

In what follows we prove i. and ii.

i. Let $\varphi \in D$, $supp\ \varphi = K \subset Re\ K_1$. Then

$$|\langle T_n, \varphi\rangle - \langle T_{f^*}, \varphi\rangle| = \left|\int_{-\infty}^{\infty} |P_n^*(x)|\varphi(x)dx - \int_{-\infty}^{\infty} \log(1 + |f^*\,(x)|)\varphi(x)dx\right|$$

$$= \left|\int_{-\infty}^{\infty} [|P_n^*(x)| - \log(1 + |f^*(x)|)]\varphi(x)dx\right| = \left|\int_K [|P_n^*(x)| - \log(1 + |f^*\,(x)|)]\varphi(x)dx\right|$$

$$\le M\int_K \big||P_n^*(x)| - |F_n^*(x)| + |F_n^*(x)| - \log(1 + |f^*\,(x)|)\big|dx$$

$$\le M\int_K \big||P_n^*(x)| - |F_n^*(x)|\big|dx + M\int_K \big||F_n^*(x)| - \log(1 + |f^*\,(x)|)\big|dx$$

$$= M lim_{y\to 0^+}\int_K \big||P_n^*(x)| - |F_n^*(x)|\big|dx + M lim_{y\to 0^+}\int_K \big||F_n^*(x)| - \log(1 + |f\quad (x + iy)|)\big|dx.$$

Taking the limit $n \to \infty$ in the esimates one obtains that the later sum of integrals will be dominated with ε. Namely, the first integral is dominated with $\varepsilon/2$ by Margelijan theorem and the construction of the polynomials P_n. The second integral is dominated with $\frac{\varepsilon}{2}$ because of (*).

The Later calculation implies that $\langle T_n, \varphi\rangle \to \langle T_{f^*}, \varphi\rangle$ when $n \to \infty$ for every, but fixed, $\varphi\epsilon D$, meaning $T_n \to T_{f^*}$ weakly in D'. To prove the convergence in the strong topology it sufficies to prove the same convergence for $\varphi \in B$ for an arbitrary bounded set in D. Choose $B \subset D$, arbitrary bounded set. The condition of boundnes implies that there exist a compact set K such that $supp\ \varphi \in K$, $\big||\varphi|\big|_{D(K)} < M$, for every $\varphi \in B$. Note that the calculations at the beginning of the paragraph hold for every $\varphi \in B$ and the new compact set chosen for the boundness condition. Hence, $T_n \to T_{f^*}$ in D'.

(ii)

$$\int_{-\infty}^{\infty} |log(1 + |P_n^*\,(x)|)||\varphi(x)|dx$$

$$= \int_K [log(1 + |P_n^*\,(x)| - |F_n^*(x)| + |F_n^*(x)|)]|\varphi(x)|dx$$

$$\le \int_K [log(1 + |F_n^*\,(x)|) + |P_n^*(x) - F_n^*(x)|)]|\varphi(x)|dx$$

$$\le \int_K \log(1 + |F_n^*(x)|)|\varphi(x)| + \int_K |P_n^*(x) - F^*(x)|\,|\varphi(x)|dx$$

$$\le M\int_K Log(1 + |F_n^*(x)|)\ \ dx + M\int_K |P_n^*\,(x) - F_n^*(x)|\,dx$$

$$\leq M\int_K \ Log(1+|f^*\ (x+iy_n)|)\ \ dx+\varepsilon/2$$

Arbitrarness of ε implies that $\lim_{n\to\infty}\int_R \ |Log(1+|P_n^*\ (x)||\varphi(x)|dx<C'$ meaning

$lim_{n\to\infty}\ Log(1+|P_n^*\ (x)|\ dx<\infty$, for all $\varphi\epsilon D$.

In the proof of ii. In the previous calculations we have used the inequalities $|a+b|\leq|a|+|b|$, $\log(1+a+b)\leq\log(1+a)+b,\ for\ a,b>0$.

Theorem 2. Let $\varphi_0\in N^+(\Pi^+)$ and $T_{\varphi_0}\in D'$ is generated with the function $\varphi_0$in the sence of the previous theorem. Let there exist sequence of polynomials $P_n(z)$ satisfying the conditions:

i. The sequence of distributions generated by the boundary values $P_n^*(x)\ of\ P_n(z)$ converges to T_{φ_0} in D' when $n\to\infty$;

ii. $lim_{n\to\infty}\int_{-\infty}^{\infty}\log(1+P_n(x+iy))\,|\varphi(x)|\,dx<C<\infty,\forall z=x+iy\in\ \Pi^+,\varphi\epsilon D.$

Then there exist a function $f\in H(\Pi^+)$ such that

$$\int_K \log(1+|f(x+iy)|)dx<C<\infty,\forall\ z=x+iy\in\Pi^+,$$

for every compact $K\subset R$, and

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log(1+|f(x+iy)|)\varphi(x)\,dx=\ \langle \mathrm{T}_{\varphi_0},\varphi\rangle$$

for all $\varphi\in D$.

Proof. Let the assumptions of the theorem are fulfilled. In [3] it is proven that from i., i.e.

$$\lim_{n\to\infty}\int_R \ P_n^*\ (x)\varphi(x)dx=\int_R \ \varphi_0(x)\varphi(x)dx,\varphi\epsilon D,$$

implies the existence of $f\in H(\Pi^+)$ such that the sequence of polynomials converges to f , uniformly on arbitrary compact subsets of Π^+ when $n\to\infty$.

Firstly we will prove that this function f is holomorphic and satisfies the condition

$$\int_K \log(1+|f(x+iy)|)\ dx\leq C$$

for all $z=x+iy\in\Pi^+$and arbitrary compact set $K\subset R$.

Indeed, we use the condition ii., i.e.

$$\limsup_{n\to\infty}\int_K \ \log(1+|P_n(x+iy)|)\,|\varphi(x)|\,dx<C<\infty,\forall\ z=x+iy\in\ \Pi^+,\varphi\epsilon D.$$

There exist $\varphi(x)\epsilon C_0^\infty(R^n)$, $\varphi(x)=1,\forall x\epsilon K$. To obtain the last statement its enough to take characteristic function of the set K and to regularize it. Substitution of such φ in to ii., implies that for every $n\in N$,

$$\int_K \ \log(1+|P_n(x+iy)|)\ dx<C<\infty,\forall\ z=x+iy\in\ \Pi^+.$$

Now,

$$\int_{\mathrm{K}} \ \log(1+|f(x+iy)|)\,dx=\int_{\mathrm{K}} \ \lim_{n\to\infty}\log(1+|P_n(x+iy)|)$$

$$\leq\limsup_{n\to\infty}\int_{-\infty}^{\infty}\log(1+|P_n(x+iy)|)\ dx<C<\infty,$$

i.e.

$\int_{\mathrm{K}} \ \log(1+|f(x+iy)|)\ dx\leq C<\infty$ for arbitrary compact set $K\subset R$ and every $z=x+iy\in\ \Pi^+$.

It remains to be proved that $\lim_{y\to 0^+}\int_{-\infty}^{\infty}\log(1+|f(x+iy)|)\varphi(x)\,dx=\langle \mathrm{T}_{\varphi_0},\varphi\rangle$, for every $\varphi\in D$.

Let $\varphi\epsilon D$ and $supp\varphi=\mathrm{K}\subset R$. Then

$$\lim_{y\to 0^+}\int_R \ \log(1+|f(x+iy)|)\varphi(x)\,dx=\lim_{y\to 0^+}\int_R \ \lim_{n\to\infty}P_n^*(x+iy)\varphi(x)\,dx=$$

$$=\lim_{y\to 0^+}\lim_{n\to\infty}\int_K \ P_n(x+iy)\varphi(x)dx=\lim_{n\to\infty}\lim_{y\to 0^+}\int_K \ P_n(x+iy)\varphi(x)dx=$$

$$= \lim_{n\to\infty} \int_K P_n^*(x)\varphi(x)dx = \int_R \varphi_0(x)\varphi(x)dx = \langle \mathrm{T}_{\varphi_0}, \varphi \rangle,$$

$for\ every\ \varphi \epsilon D.$

The previous equalities are obvious, exept the following

$$\lim_{y\to 0^+} \lim_{n\to\infty} \int_K P_n(x+iy)\varphi(x)dx = \lim_{n\to\infty} \lim_{y\to 0^+} \int_K P_n(x+iy)\varphi(x)dx \dots . (**)$$

for $z = x + iy \in \Pi^+$.

The proof follows with similar technique as the one used in $(*)$.

III. CONCLUSION

One can define associated distribution to arbitrary element of the Smirnof class. That distribution can be obtained as a boundary value of analytic functions on the upper half space.The boundary values are in distributional since, partially we obtain the converse statement by imposing boundnes condition, in some sense, on the analytic functions.

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