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Original Article

New Results on Open Support of Graphs under Addition

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Abstract — *Let $G=(V,E)$ be a graph. For each $v\in V$, the open support of v under addition, denoted by $supp(v)$, is the sum of degrees of vertices adjacent to v. The open support of G under addition is the sum of supports of all its vertices. In this paper, we determined the open support of some special types of graphs. We also defined the open support of an edge under addition and showed its connection with the open support of a G.*

Keywords — *Open support, Neighborhood, Degree, Adjacency matrix, Incidence matrix.*

I. INTRODUCTION

Any mathematical object involving points and connections between them may be called a graph. They may represent physical networks, such as transportation networks, electrical circuits, or organic molecules [12-25]. They are also used in representing less tangible interactions as might occur in databases, sociological relationships, or in the flow of control in a computer program. One of the most basic notation in graph theory is the degree of a vertex, many graph structure properties can be characterized by it. For example, the famous Hand Shaking Principle implies that the sum of degrees of the vertices of a graph is twice the number of edges.

The graph considered in this paper are finite, undirected and simple. Let $G=(V(G),E(G))$ be a graph with n vertices and m edges. The *degree* of a vertex $v\in V$, denoted by $d_G(v)$, is the number of edges of G incident with v. The maximum degree and minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex of a degree 0 in G is called an *isolated vertex* and a vertex of degree 1 is called a *pendent vertex* or an *end vertex* of G. The edge incident with a pendent vertex is a *pendant edge*. The *neighborhood* of a vertex $v\in V(G)$ is $N_G(v)=\{u\in V(G)\ |uv\in E(G)\}$. The open neighbourhood of a set $X\subseteq V(G)$ is $N_G(X):=\cup_{v\in X} N_G(v)$. The vertex delete subgraph $G-v$ is obtained by deleting v together with all the edges incident with v. The following concepts was introduced by Balamurugan et al. [2,3].

Definition 1.1. *Let $G=(V,E)$ be a graph. An open support of a vertex v under addition is defined by $\sum_{u\in N(v)} d(u)$ and it is denoted by $supp(v)$.*

Definition 1.2. *Let $G=(V,E)$ be a graph. An open support of G under addition is defined by $\sum_{v\in V(G)} supp(v)$ and it is denoted by $supp(G)$.*

The open support of paths, cycles and complete (bipartite) graphs are easily determined. In [2,3,9], the open support of some special types of graphs were studied. Interested readers may also see [4,5] for results concerning closed supports of graphs. In this paper, we give the open support of some more graphs. We also define the open support of an edge under addition and show a connection to the open support of a graph.

II. DEFINITIONS

In this section, we give some definitions and notations which can be found in [1,6,7,8,10,11].

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two disjoint graphs. We get some larger graphs from them as follows.

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Cartesian product $G_1 \square G_2$ with vertex set $V_1 \times V_2 := \{(u,v) : u \in V(G_1) \text{ and } v \in V(G_2)\}$ and edge set $\{(u_1,v_1)(u_2,v_2) : u_1 = u_2, v_1v_2 \in E(G_2) \text{ or } u_1u_2 \in E(G_1), v_1 = v_2\}$.

Direct (or tensor) product $G_1 \times G_2$ with vertex set $V_1 \times V_2$ and edge set $\{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$.

Strong (or normal) product $G_1 \boxtimes G_2 := (G_1 \square G_2) \cup (G_1 \times G_2)$.

A path with n vertices is denoted by P_n.

Definition 2.1. *The **planar grid** $P_m \square P_n$ is the cartesian product of paths P_m and P_n.*

Definition 2.2. ***Möbius ladder** is obtained from the planar grid $P_n \square P_2$ by joining the opposite endpoints of the two copies of P_n.*

Definition 2.3. *The **book graph** B_m is the cartesian product $S_m \square P_2$, where S_m is the star with $m+1$ vertices.*

Definition 2.4. *The **helm graph** H_n is obtained from a wheel W_n with center w by attaching a pendant vertex at each vertex except w. The **sun graph** is obtained by deleting w from the helm H_n.*

Definition 2.5. *The **generalized helm** H_n^m is obtained by inserting m vertices to every pendant edge of the helm H_n. The **generalized sun graph** is defined as $S_n^m := H_n^m - w$, where w is the center of H_n^m.*

Definition 2.6. *The k-**blow up** $D_k(G)$ of G is obtained by replacing each $v_i \in V(G)$ by a k-set V_i and add all possible edges between V_i and V_j for each $v_iv_j \in E(G)$.*

III. MAIN RESULTS

In this section, we are going to obtain the open support of some types of graphs especially those were defined in section 2. We begin with the open support of bull graph, which is a vertex-deleted subgraph of sun graph S_3.

Theorem3.1. *For the bull graph $G = S_3 - v$ where v is a pendent vertex of S_3, we have $supp(G) = 24$.*

Proof: Let the vertex set of G be $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and the edge set of G be $E(G) = \{v_1v_2, v_2v_3, v_2v_4, v_3v_4, v_4v_5\}$, see fig.1

Note that $d(v_1) = d(v_5) = 1$, $d(v_2) = d(v_4) = 3$ and $d(v_3) = 2$. It follows that

- $supp(v_1) = d(v_2) = 3$,
- $supp(v_2) = d(v_1) + d(v_3) + d(v_4) = 6$,
- $supp(v_3) = d(v_2) + d(v_4) = 6$,
- $supp(v_4) = d(v_2) + d(v_3) + d(v_5) = 6$,
- $supp(v_5) = d(v_4) = 3$.

Therefore, we conclude that

$$\begin{aligned} supp(G) &= supp(v_1) + supp(v_2) + supp(v_3) + supp(v_4) + supp(v_5) \\ &= 3 + 6 + 6 + 6 + 3 \\ &= 24. \end{aligned}$$

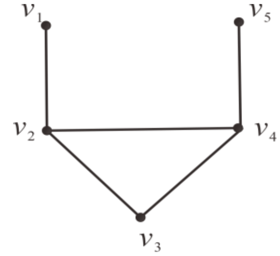

Fig. 1 Bull graph

The following theorem determines the open support of Möbius ladder.

Theorem3.2. *For the Möbius ladder* $G = M_n$*, we have* $supp(G) = 18n$.

Proof: Let the vertex set of G be $V(G) = \{v_i, u_i : 1 \le i \le n\}$ and the edge set of G be $E(G) = \{v_i v_{i+1}, u_i u_{i+1} : 1 \le i \le n-1\} \cup \{v_i u_i : 1 \le i \le n\} \cup \{v_1 u_n, u_1 v_n\}$. See fig 2 for the case $n = 5$.

Clearly, for $1 \le i \le n$ we have $d(v_i) = d(u_i) = 3$. It follows that

$$supp(v_1) = d(v_2) + d(u_1) + d(u_n) = 9$$

and similarly,

$$supp(u_1) = supp(v_n) = supp(u_n) = 9.$$

For $2 \le i \le n-1$, we have

$$supp(v_i) = d(v_{i+1}) + d(v_{i-1}) + d(u_i) = 9$$

and

$$supp(u_i) = d(u_{i+1}) + d(u_{i-1}) + d(v_i) = 9.$$

Therefore, we conclude that

$$supp(G) = \sum_{i=1}^{n} (supp(v_i) + supp(u_i)) = n(9+9) = 18n.$$

Example 3.3. *For the Möbius ladder* M_5 *we have*

$$supp(M_5) = \sum_{i=1}^{5} (supp(v_i) + supp(u_i)) = 5(9+9) = 90.$$

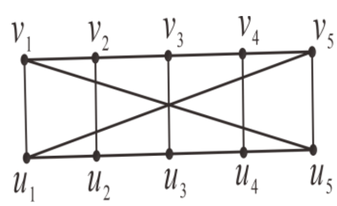

Fig. 2 M_5 **graph**

The following two results consider open supports of graphs which are cartesian product or strong product of two paths.

Theorem3.4. *For the planar grid* $G = P_m \square P_n$*, we have* $supp(G) = 16mn - 14(m+n) + 8$.

Proof: Let the vertex set of G be $V(G) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ and the edge set of G be $E(G) = \{v_{i,j} v_{i,j+1} : 1 \le i \le m, 1 \le j \le n-1\} \cup \{v_{i,j} v_{i+1,j} : 1 \le i \le m-1, 1 \le j \le n\}$. See fig 3 for the case $m = 5$ and $n = 6$.

Clearly, $d(v_{11}) = d(v_{1,n}) = d(v_{m,1}) = d(v_{m,n}) = 2$ and for $2 \le i \le m-1, 2 \le j \le n-1$ we have $d(v_{1,j}) = d(v_{m,j}) = d(v_{i,1}) = d(v_{i,n}) = 3$ and $d(v_{i,j}) = 4$. We divide $V(G)$ into six classes according to the valve of open support of each vertex.

Class 1. $V_1 = \{v_{11}, v_{1,n}, v_{m,1}, v_{m,n}\}$.

For each $v \in V_1$, by symmetry, we have

$$supp(v) = supp(v_{11}) = d(v_{12}) + d(v_{21}) = 3 + 3 = 6.$$

Class 2. $V_2 = \{v_{12}, v_{1,n-1}, v_{21}, v_{2,n}, v_{m-1,1}, v_{m-1,n}, v_{m,2}, v_{m,n-1}\}$.

For each $v \in V_2$, by symmetry, we have

$$supp(v) = supp(v_{12}) = d(v_{11}) + d(v_{13}) + d(v_{22}) = 2 + 3 + 4 = 9.$$

Class 3. $V_3 = \{v_{i,1}, v_{i,n}, v_{1,j}, v_{m,j} : 3 \le i \le m-2, 3 \le j \le n-2\}$.

For each $v \in V_3$, by symmetry, we have

$$supp(v) = supp(v_{i,1}) = d(v_{i-1,1}) + d(v_{i+1,1}) + d(v_{i,2}) = 3+3+4 = 10.$$

Class 4. $V_4 = \{v_{22}, v_{2,n-1}, v_{m-1,2}, v_{m-1,n-1}\}$.

For each $v \in V_4$, by symmetry, we have

$$supp(v) = supp(v_{22}) = d(v_{12}) + d(v_{21}) + d(v_{23}) + d(v_{32}) = 3+3+4+4 = 14.$$

Class 5. $V_5 = \{v_{i,2}, v_{i,n-1}, v_{2,j}, v_{m-1,j} : 3 \le i \le m-2, 3 \le j \le n-2\}$.

For each $v \in V_5$, by symmetry, we have

$$supp(v) = supp(v_{23}) = d(v_{22}) + d(v_{24}) + d(v_{13}) + d(v_{33}) = 4+4+3+4 = 15.$$

Class 6. $V_6 = \{v_{i,j} : 3 \le i \le m-2, 3 \le j \le n-2\}$.

For each $v \in V_6$, by symmetry, we have

$$supp(v) = supp(v_{i,j}) = d(v_{i-1,j}) + d(v_{i+1,j}) + d(v_{i,j-1}) + d(v_{i,j+1}) = 4 \times 4 = 16.$$

In summary, we conclude that

$$\begin{aligned} supp(G) &= \sum_{i=1}^{m}\sum_{j=1}^{n} supp(v_{i,j}) \\ &= 6\times 4 + 9\times 8 + 10[2(n-4)+2(m-4)] + 14\times 4 \\ &\quad +15[2(n-4)+2(m-4)] + 16(m-4)(n-4) \\ &= 16mn - 14(m+n) + 8. \end{aligned}$$

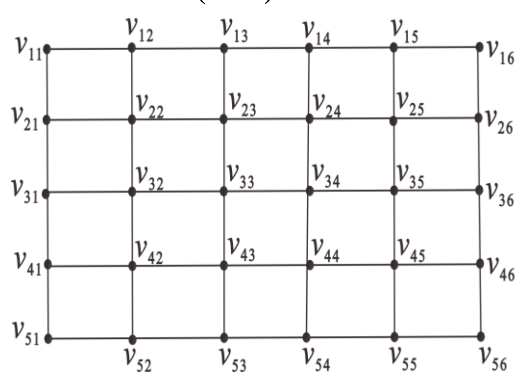

Fig. 3 $P_5 \square P_6$ **graph**

Example 3.5. $supp(P_5 \square P_6) = 16(5)(6) - 14(5+6) + 8 = 334.$

Theorem3.6. *For the graph* $G = P_m\ )\ \ P_n$, *we have* $supp(G) = 64mn - 78(m+n) + 92$.

Proof: Let the vertex set of G be $V(G) = \{v_{i,j} : 1 \le i \le m, 1 \le j \le n\}$ and the edge set of G be $E(G) = \{v_{i,j}v_{i,j+1} : 1 \le i \le m, 1 \le j \le n-1\} \cup \{v_{i,j}v_{i+1,j} : 1 \le i \le m-1, 1 \le j \le n\} \cup \{v_{i,j}v_{i+1,j+1} : 1 \le i \le m-1, 1 \le j \le n-1\} \cup \{v_{i,j}v_{i-1,j+1} : 2 \le i \le m, 1 \le j \le n-1\}$. See fig 4 for the case $m = 5$, $n = 6$.

Note that $d(v_{11}) = d(v_{1,n}) = d(v_{m,1}) = d(v_{m,n}) = 3$, and for $2 \le i \le m-1, 2 \le j \le n-1$, $d(v_{1,j}) = d(v_{m,j}) = d(v_{i,1}) = d(v_{i,n}) = 5$ and $d(v_{i,j}) = 8$. We divide $V(G)$ into six classes according to the valve of open support of each vertex under addition.

Class 1. $V_1 = \{v_{11}, v_{1,n}, v_{m,1}, v_{m,n}\}$.

For each $v \in V_1$, by symmetry, we have

$$supp(v) = supp(v_{11}) = d(v_{12}) + d(v_{21}) + d(v_{22}) = 5+5+8 = 18.$$

Class 2. $V_2 = \{v_{12}, v_{1,n-1}, v_{21}, v_{2,n}, v_{m-1,1}, v_{m-1,n}, v_{m,2}, v_{m,n-1}\}$.

For each $v \in V_2$, by symmetry, we have

$$supp(v) = supp(v_{12}) = d(v_{11}) + d(v_{13}) + d(v_{21}) + d(v_{22}) + d(v_{23}) = 3+5+5+8+8 = 29.$$

Class 3. $V_3 = \{v_{i,1}, v_{i,n}, v_{1,j}, v_{m,j} : 3 \le i \le m-2, 3 \le j \le n-2\}$.

For each $v \in V_3$, by symmetry, we have

$$\begin{aligned} supp(v) &= supp(v_{i,1}) = d(v_{i-1,1}) + d(v_{i+1,1}) + d(v_{i-1,2}) + d(v_{i,2}) + d(v_{i+1,2}) \\ &= 5+5+8+8+8 = 34. \end{aligned}$$

Class 4. $V_4 = \{v_{22}, v_{2,n-1}, v_{m-1,2}, v_{m-1,n-1}\}$.

For each $v \in V_4$, by symmetry, we have

$$supp(v) = supp(v_{22}) = \sum_{j=1}^{3}\sum_{i=1}^{3} d(v_{i,j}) - d(v_{2,2}) = 3 + 4\times 5 + 3\times 8 = 47.$$

Class 5. $V_5 = \{v_{i,2}, v_{i,n-1}, v_{2,j}, v_{m-1,j} : 3 \le i \le m-2, 3 \le j \le n-2\}$.

For each $v \in V_5$, by symmetry, we have

$$\begin{aligned} supp(v) &= supp(v_{23}) = d(v_{12}) + d(v_{13}) + d(v_{14}) + d(v_{22}) + d(v_{24}) + d(v_{32}) + d(v_{33}) + d(v_{34}) \\ &= 3\times 5 + 5\times 8 = 55. \end{aligned}$$

Class 6. $V_6 = \{v_{i,j} : 3 \le i \le m-2, 3 \le j \le n-2\}$.

For each $v \in V_6$, by symmetry, we have

$$supp(v) = supp(v_{i,j}) = \sum_{s=i-1}^{i+1}\sum_{t=j-1}^{j+1} d(v_{s,t}) - d(v_{i,j}) = 8\times 8 = 64.$$

In summary, we conclude that

$$\begin{aligned} supp(G) &= \sum_{i=1}^{m}\sum_{j=1}^{n} supp(v_{i,j}) \\ &= 18\times 4 + 29\times 8 + 34[2(n-4) + 2(m-4)] + 47\times 4 \\ &\quad + 55[2(n-4) + 2(m-4)] + 64(m-4)(n-4) \\ &= 64mn - 78(m+n) + 92. \end{aligned}$$

Example 3.7.

$$\begin{aligned} supp(P_5 \ \ P_6) &= \sum_{i=1}^{5}\sum_{j=1}^{6} supp(v_{i,j}) \\ &= 18\times 4 + 29\times 8 + 34\times 6 + 47\times 4 + 55\times 6 + 64\times 2 \\ &= 64(5)(6) - 78(5+6) + 92 \\ &= 1154. \end{aligned}$$

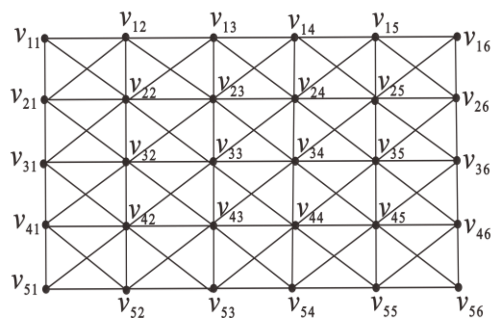

Fig. 4 P_5) P_6 **graph**

We determine the open support of book graphs as follows.

Theorem3.8. *For the book graph* $G = B_m$, *we have* $supp(G) = 2m^2 + 12m + 2$.

Proof: Let the vertex set of G be $V(G) = \{v, u, v_i, u_i : 1 \le i \le m\}$ and the edge set of G be $E(G) = \{uv, u_i v_i, vv_i, uu_i : 1 \le i \le m\}$. See fig 5 for the case $m = 6$.

Note that $d(v) = d(u) = m+1$ and for $1 \le i \le m$, we have $d(v_i) = d(u_i) = 2$. It follows that

$$supp(v) = d(u) + \sum_{i=1}^{m} d(v_i) = m+1+2\times m = 3m+1.$$

Similarly, $supp(u) = 3m+1$.

For $1 \le i \le m$,

$$supp(v_i) = d(v) + d(u_i) = m+1+2 = m+3.$$

Similarly, $supp(u_i) = m+3$ for $1 \le i \le m$.

Thus, we conclude that

$$\begin{aligned} supp(G) &= supp(v) + supp(u) + \sum_{i=1}^{2m} supp(v_i) + \sum_{i=1}^{2m} supp(u_i) \\ &= 3m+1+3m+1+m\times 2\times(m+3) \\ &= 2m^2 + 12m + 2. \end{aligned}$$

Example 3.9. $supp(B_6) = 19\times 2 + 9\times 12 = 2(6)^2 + 12(6) + 2 = 146.$

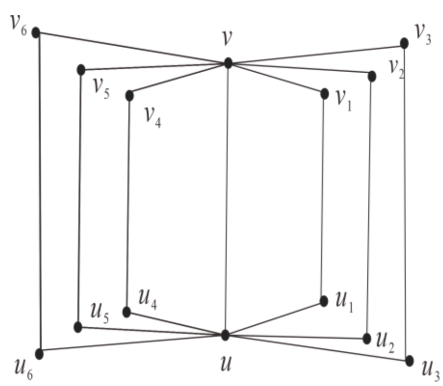

Fig. 5 Book graph B_6

In the following, we consider (generalized) helm and (generalized) sun graphs with some cycle-subgraph C_n, we denote the edge set of C_n by $E(C_n) = \{v_i v_{i+1} : 1 \le i \le n\}$, where $v_{n+1} = v_1$.

Theorem3.10. *For the helm graph* $G = H_n$*, we have* $supp(G) = n^2 + 17n$.

Proof: Let the vertex set of G be $V(G) = \{w\} \cup \{u_i, v_i : 1 \le i \le n\}$ and the edge set of G be $E(G) = \{wv_i, v_iu_i, v_iv_{i+1} : 1 \le i \le n\}$. See fig 6 for the case $n = 7$.

Note that $d(w) = n$ and for $1 \le i \le n$, $d(v_i) = 4$ and $d(u_i) = 1$. It follows that

$$supp(w) = \sum_{i=1}^{n} d(v_i) = 4n.$$

For $1 \le i \le n$,

$$supp(v_i) = d(w) + d(u_i) + d(v_{i+1}) + d(v_{i-1}) = n + 4 + 4 + 1 = n + 9$$

and

$$supp(u_i) = d(v_i) = 4.$$

Thus, we conclude that

$$\begin{aligned} supp(G) &= supp(w) + \sum_{i=1}^{n} \big(supp(v_i) + supp(u_i)\big) \\ &= 4n + n(n + 9 + 4) \\ &= n^2 + 17n. \end{aligned}$$

Example 3.11. $supp(H_7) = 28 + 7(16 + 4) = (7)^2 + 17(7) = 168.$

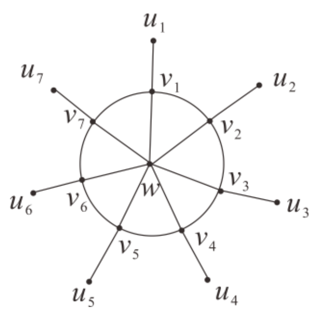

Fig. 6 Helm graph H_7

Theorem3.12. *For the generalized helm graph* $G = H_n^m$*, we have* $supp(G) = n^2 + 4mn + 17n$.

Proof: Let the vertex set of G be $V(G) = \{w\} \cup \{v_i : 1 \le i \le n\} \cup \{v_{i,j} : 1 \le i \le n, 1 \le j \le m+1\}$ and the edge set of G be $E(G) = \{wv_i, v_iv_{i,m+1}, v_iv_{i+1} : 1 \le i \le n\} \cup \{v_{i,j}v_{i,j+1} : 1 \le i \le n, 1 \le j \le m\}$.

Note that $d(w) = n$, and for $1 \le i \le n$, $1 \le j \le m$ we have $d(v_i) = 4$, $d(v_{i,1}) = 1$ and $d(v_{i,j+1}) = 2$. It follows that

$$supp(w) = \sum_{i=1}^{n} d(v_i) = 4n,$$

and for $1 \le i \le n$, $3 \le j \le m$,

- $supp(v_i) = d(w) + d(v_{i+1}) + d(v_{i-1}) + d(v_{i,m+1}) = n + 4 + 4 + 2 = n + 10,$
- $supp(v_{i,1}) = d(v_{i,2}) = 2$,
- $supp(v_{i,2}) = d(v_{i,1}) + d(v_{i,3}) = 3$,
- $supp(v_{i,m+1}) = d(v_{i,m}) + d(v_i) = 2 + 4 = 6$,
- $supp(v_{i,j}) = d(v_{i,j-1}) + d(v_{i,j+1}) = 4$.

Thus, we conclude that

$$supp(G) = supp(w) + \sum_{i=1}^{n}\left(supp(v_i) + supp(v_{i,1}) + supp(v_{i,2}) + supp(v_{i,m+1})\right)$$
$$+\sum_{i=1}^{n}\sum_{j=3}^{m} supp(v_{i,j})$$
$$= 4n + (10 + n + 2 + 3 + 6)n + 4n(m-2)$$
$$= n^2 + 4mn + 17n.$$

Theorem3.13. *For the sun graph* $G = S_n$, *we have* $supp(G) = 10n$.

Proof: Let the vertex set of G be $V(G) = \{u_i, v_i : 1 \le i \le n\}$ and the edge set of G be $E(G) = \{v_i u_i, v_i v_{i+1} : 1 \le i \le n\}$. See fig 7 for the case $n = 7$.

For $1 \le i \le n$, it follows from the fact $d(v_i) = 3$ and $d(u_i) = 1$ we have that

$$supp(v_i) = d(u_i) + d(v_{i+1}) + d(v_{i-1}) = 3+3+1 = 7$$

and

$$supp(u_i) = d(v_i) = 3.$$

Thus, we conclude that

$$supp(G) = \sum_{v \in V(G)} supp(v) = \sum_{i=1}^{n}\left(supp(v_i) + supp(u_i)\right) = (3+7)n = 10n.$$

Example 3.14. $supp(S_7) == 7(7+3) = 7(10) = 70.$

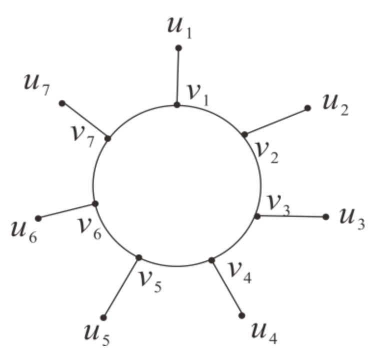

Fig. 7 Sun graph S_7

Theorem3.15. *For the generalized sun graph* $G = S_n^m$, *we have* $supp(G) = 10n + 4mn.$

Proof: Let the vertex set of G be $V(G) = \{v_i, v_{i,j} : 1 \le i \le n, 1 \le j \le m+1\}$ and the edge set of G be $E(G) = \{v_i v_{i,m+1}, v_i v_{i+1} : 1 \le i \le n\} \cup \{v_{i,j} v_{i,j+1} : 1 \le i \le n, 1 \le j \le m\}$.

For $1 \le i \le n$, $1 \le j \le m$, we have $d(v_i) = 3$, $d(v_{i,1}) = 1$ and $d(v_{i,j+1}) = 2$. It follows that

- $supp(v_i) = d(v_{i+1}) + d(v_{i-1}) + d(v_{i,m+1}) = 3+3+2 = 8$,
- $supp(v_{i,1}) = d(v_{i,2}) = 2$,
- $supp(v_{i,2}) = d(v_{i,1}) + d(v_{i,3}) = 3$,
- $supp(v_{i,m+1}) = d(v_{i,m}) + d(v_i) = 2+3 = 5$,
- $supp(v_{i,j}) = d(v_{i,j-1}) + d(v_{i,j+1}) = 4$.

Thus, we conclude that

$$
\begin{aligned}
supp(G) &= \sum_{i=1}^{n}\big(supp(v_i)+supp(v_{i,1})+supp(v_{i,2})+supp(v_{i,m+1})\big)\\
&\quad+\sum_{i=1}^{n}\sum_{j=3}^{m} supp(v_{i,j})\\
&=(8+2+3+5)n+4n(m-2)\\
&=10n+4mn.
\end{aligned}
$$

Next, we consider the open support of blow up graphs of paths and cycles.

Theorem3.16. *For the graph* $G=D_k(P_n)$ *, we have* $supp(G)=2k^3(2n-3)$ *.*

Proof: Let the vertex set of G be $V(G)=\{v_{i,j}:1\le i\le k, 1\le j\le n\}=\bigcup_{j=1}^{n}V_j$ where $V_j=\{v_{i,j}:1\le i\le k\}$. Let the edge set of G be $E(G)=\bigcup_{j=1}^{n-1}E(V_j,V_{j+1})=\bigcup_{j=1}^{n-1}\{v_{s,j}v_{t,j+1}:1\le s\le k, 1\le t\le k\}$. See fig 8 for the case $k=3$, $n=4$.

Note that for $1\le i\le k, 2\le j\le n-1$, $d(v_{i,j})=2k$ and $d(v_{i,1})=d(v_{i,n})=k$. It follows that for $1\le i\le k$ and $3\le j\le n-2$,

- $supp(v_{i,1})=\sum_{r=1}^{k}d(v_{r,2})=2k^2$,
- $supp(v_{i,2})=\sum_{r=1}^{k}\big(d(v_{r,1})+d(v_{r,3})\big)=k(k+2k)=3k^2$,
- $supp(v_{i,j})=\sum_{r=1}^{k}\big(d(v_{r,j-1})+d(v_{r,j+1})\big)=k(2k+2k)=4k^2$,
- $supp(v_{i,n-1})=\sum_{r=1}^{k}\big(d(v_{r,n-2})+d(v_{r,n})\big)3k^2$,
- $supp(v_{i,n})=\sum_{r=1}^{k}d(v_{r,n-1})=2k^2$.

Thus, we conclude that

$$
\begin{aligned}
supp(G) &= \sum_{i=1}^{k}(supp(v_{i,1})+supp(v_{i,n})+supp(v_{i,2})+supp(v_{i,n-1}))\\
&\quad+\sum_{i=1}^{k}\sum_{j=3}^{n-2} supp(v_{i,j})\\
&=k(2k^2+2k^2+3k^2+3k^2)+4k^2\times k(n-4)\\
&=2k^3(2n-3).
\end{aligned}
$$

Example 3.17. $supp(D_3(P_4))=2(3)^3(2\cdot 4-3)=270.$

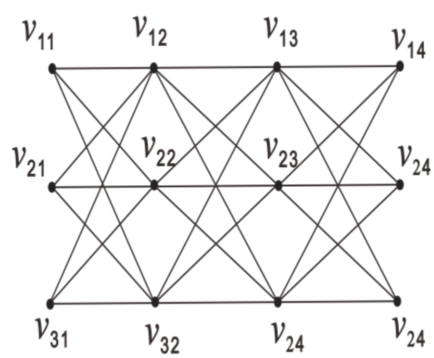

Fig. 8 $D_3(P_4)$ **graph**

Theorem3.18. *For the graph* $G = D_k(C_n)$ *, we have* $supp(G) = 4k^3n.$

Proof: Let the vertex set of G be $V(G) = \{v_{i,j} : 1 \le i \le k, 1 \le j \le n\}$ and the edge set of G be $E(G) = \bigcup_{j=1}^{n} E(V_j, V_{j+1}) = \bigcup_{j=1}^{n} \{v_{s,j}v_{t,j+1} : 1 \le s \le k, 1 \le t \le k\}$. See fig 9 for the case $k = 3$, $n = 4$.

For $1 \le i \le k, 1 \le j \le n$, note that $d(v_{i,j}) = 2k$, we have

$$supp(v_{i,j}) = \sum_{v \in N(v_{i,j})} d(v) = \sum_{r=1}^{k} \left(d(v_{r,j-1}) + d(v_{r,j+1}) \right) = k(2k + 2k) = 4k^2.$$

Thus, we conclude that

$$\begin{aligned} supp(G) &= \sum_{i=1}^{k} \sum_{j=1}^{n} supp(v_{i,j}) \\ &= kn \times 4k^2 \\ &= 4k^3 n. \end{aligned}$$

Example 3.19. $supp(D_3(C_4)) = 36 \times 3 \times 4 = 4(3)^3(4) = 432.$

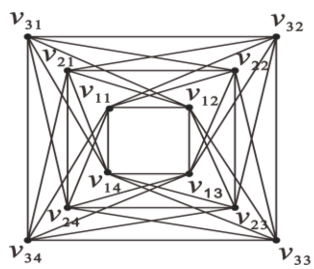

Fig. 9 $D_3(C_4)$ **graph**

The following result, which was essentially proved in [3], is a general result of open support of graphs. We give a new proof through using the algebraic methods.

Theorem3.20. *For any graph* G *, we have* $supp(G) = \sum_{v \in V(G)} d^2(v).$

Proof: Let $G = (V, E)$ be a graph with n vertices and m edges. Let $\mathrm{A}(G)$ be the adjacency matrix of G. We define a "degree-adjacency matrix" $\mathrm{DA}(G)$ which is obtained by multiplying each v-row (the row corresponding to vertex v) of $\mathrm{A}(G)$ the degree $d(v)$.

We count the sum of entries of $\mathrm{DA}(G)$ in two ways and the equating the two counts. On the one hand, the sum of the entries in the column corresponding to vertex v is $\sum_{u \in N(v)} d(u)$, therefore the sum of all the entries in $\mathrm{DA}(G)$ is $\sum_{v \in V(G)} \sum_{u \in N(v)} d(u) = supp(G)$. On the other hand, since the sum of the entries in the row corresponding to vertex v is $d^2(v)$, the sum of all the entries in $\mathrm{DA}(G)$ also equals $\sum_{v \in V(G)} d^2(v)$. Thus, we have $supp(G) = \sum_{v \in V(G)} d^2(v)$.

It follows from Theorem 3.20 immediately that

Corollary 3.21. *The open support of each* d*-regular graph* G *is* $supp(G) = d^2 |G|$.

VI. CONCLUSION

To study the open support of graphs from other perspectives, in this section we define the open support of an edge under addition and use this to get the open support of a graph.

Definition 4.1. *Let* $G = (V, E)$ *be a graph. The* ***open support of an edge*** $e = uv$ *under addition is defined by* $d(u) + d(v)$ *and it is denoted by* $supp(e)$.

Now we prove $\sum_{e\in E(G)} supp(e) = \sum_{v\in V(G)} d^2(v)$. This gives a new understanding for $supp(G)$: The open support of a graph G under addition equals the sum of its edges' open support, namely

$$supp(G) = \sum_{v\in V(G)} supp(v) = \sum_{e\in E(G)} supp(e).$$

Theorem4.2. *For any graph G, we have* $\sum_{e\in E(G)} supp(e) = \sum_{v\in V(G)} d^2(v).$

Proof: Let $G=(V,E)$ be a graph with n vertices and m edges. Let $\mathrm{M}(G)$ be the incidence matrix of G. We define a "degree-incidence matrix" $\mathrm{DM}(G)$, which is obtained by multiplying each v-row (the row corresponding to vertex v) of $\mathrm{M}(G)$ the degree $d(v)$.

We count the sum of matrix $\mathrm{DM}(G)$ in two ways and the equating the two counts. On the one hand, the sum of the entries in the row corresponding to vertex v is $d(v)^2$, therefore $\sum_{v\in V(G)} d^2(v)$ is just the sum of all the entries in $\mathrm{DM}(G)$. On the other hand, the sum of the entries in the column corresponding to edge e is $supp(e)$, the sum of entries in $\mathrm{DM}(G)$ is also equals $\sum_{e\in E(G)} supp(e)$. Thus, we have $\sum_{e\in E(G)} supp(e) = \sum_{v\in V(G)} d^2(v)$.

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