International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373/doi:10.14445/22315373/IJMTT-V68I2P507
Volume 68 Issue 2, 43-46, February, 2022

Original Article

The Stabilization of Sequences from the Collatz Conjecture

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Abstract — *This paper is an analysis of the Collatz conjecture, and the sequences generated through recursive use of the rules used for generating those numbers. Analysis of other embedded sequences will also be looked at that lead to the Binomial Distribution.*

Keywords — *Sequences, Hailstone numbers, Collatz conjecture, Even values, Odd values, Recursion, Recursive, Binomial distribution, Binomial expansion theorem***.**

I. INTRODUCTION

Lothar Collatz was a mathematician from Germany that worked in the mathematical field of numerical analysis, and is known for the "3x + 1 problem" that was first proposed by him in 1937. For many students such as Lothar Collatz, it was common to study at several universities. In 1928, Collatz studied at the University of Greifswald, moved to Munich, and then went to Göttingen before finally ending up in Berlin to pursue doctoral work under the guidance of Alfred Klose. In 1935, Collatz completed his doctorate for his paper on *Das Differenzenverfahren mit höherer Approximation für lineare Differentialgleichurnge (Difference methods with higher approximation for linear differential equations).* The name of Collatz will be known by many people because of his famous "Collatz problem". For a mathematician who produced so much important and fundamental work in the field of mathematics, the legacy of his work is summed up by the famous problem that bares his name. Yet, it is this very problem that has captured the imagination of so many people because of how novel, and simple the problem appears to be. An appearance that hides the complicated nature of the inner workings of the natural numbers [1].

II. PRELIMINARIES

Definition 2.1. Hailstone sequence

A hailstone sequence is generated through positive integers of x in the sequence defined by $\{a_n\}$ where a_n is found as a value applied to x through recursion n times $a_n = f^{\,n}(x)$, $x \in \{1, 2, 3, 4, \ldots\}$ and $n = 0,1,2,\ldots$ where $f^0(x) = x$ and for $n > 0$,

$$f^n(x)=\begin{cases} 3x+1 \, if \; x \bmod 2 = 1 \, (odd) \\ \dfrac{x}{2} \, if \; x \bmod 2 \, (even) = 0 \end{cases} \quad [2].$$

Definition 2.2. Collatz Conjecture

Statement: Any positive integer $x\in \mathbb{Z}^+$, a Hailstone sequence that starts with an initial value of x will eventually reduce to 1.

III. DISCUSSION OF PROBLEM IN THIS ARTICLE AND SOLUTION

Let $\{a_n\}$ be a hailstone sequence, and $x\in \mathbb{Z}+$. Also let $\{2n + 1\}$ define the sequence of odd numbers, and $\{2n + 2\}$ define the sequence of even numbers. For all values of x, a Hailstone sequence that starts with an initial value of x will eventually reduce to 1.

Solution to the Problem-Mathematical derivation of Collatz Theorem

Theorem Number 1. $\forall\, x\in \mathbb{Z}^+$ A_n exists where the set A_n is made up of numbers in Hailstone sequences starting with x.

Theorem Number 2. $\exists\, x\in \{2n + 1\} \ni 3x + 1 = \{2^{2n+2}\}\in \{2^n\}$. $3x + 1 = \{2^{2n+2}\} \rightarrow A_n = \{1\}$.

IV. PROOF OF THE THEOREMS

Theorem Number 1. $\forall\, x \in \mathbb{Z}^+$ A_n exists.

Proof. The set A_n is made up of numbers a_n where a_n is found as a value applied to x through recursion n times $a_n = f^n(x)$, $x \in \mathbb{Z}^+$. According to Definition 2.1, $f^0(x) = x$ and for $n > 0$,

$$f^n(x) = \begin{cases} 3x+1 \; if \; x \bmod 2 = 1\,(odd) \\ \frac{x}{2} \; if \; x \bmod 2\,(even) = 0 \end{cases}.$$

This means that $\forall\, x \in \mathbb{Z}^+$ $f^n(x)$ is a natural number. So if a_n exists, then A_n exists $\forall\, x \in \mathbb{Z}^+$.

Remark 1: In the proof given above, there is no implication that the set A_n has to contain 1 or that the set A_n has to be a finite value. This proof is presenting the fact if the terms a_n do exist, then the set A_n must also exist as a set of positive integers.

Theorem Number 2. $\exists\, x \in \{2n+1\} \ni 3x+1 = \{2^{2n+2}\} \in \{2^n\}$. $(3x+1 = \{2^{2n+2}\} \rightarrow A_n = \{1\})$.
Proof. According to Definition 2.1 if $x \in \{2n+1\}$ then $f^n(x) = 3x+1$. The function $f^n(x) = 3x+1$ turns any odd value into an even value $\{2n+2\}$, thus forcing even values to have to be evaluated by the function $f^n(x) = x/2$.
For the subset of even values $\{2^n\}$ where $n = 0, 1, 2, 3, \ldots$ $2^n = 1, 2, 4, 8, 16, 32, 64, \ldots$

Case 1. Say that $3x + 1 = 2^n$. For the case where $n = 0$ we get $3x + 1 = 1 \rightarrow 3x = 0 \rightarrow x = 0$.
Case 2. For the case where $n = 1$, $3x + 1 = 2 \rightarrow 3x = 1 \rightarrow x = 1/3$.

This means that $3x + 1$ cannot be equal to the set $\{2^n\}$ because $x \in \mathbb{Z}^+$ $(x > 0)$. However, there are values on the set $\{2^n\}$ that can be described by the set $\{2^{2n+2}\}$ for $n = 0, 1, 2, 3, \ldots$ where $2^{2n+2} = 4, 16, 64, 256, \ldots$

Case 3. Say that $3x + 1 = 2^{2n+2}$. For the case where $n = 0$ we get $3x + 1 = 4 \rightarrow 3x = 3 \rightarrow x = 1$.
Case 4. For the case where $n = 1$, $3x + 1 = 16 \rightarrow 3x = 15 \rightarrow x = 5$.
Case 5. For the case where $n = 2$, $3x + 1 = 64 \rightarrow 3x = 63 \rightarrow x = 21$.
Case 6. For the case where $n = 3$, $3x + 1 = 256 \rightarrow 3x = 255 \rightarrow x = 85$.

Postulate 1: The choice of odd value $\{2n+1\}$ or even value $\{2n+2\}$ can be described by *p* for choosing an odd value, and *1 – p* for choosing an even value. From Postulate 1, it is understood that the choice to start with an odd value or an even value is describing a binomial experiment because the choice of choosing an integer for $x > 0$ can only have two possible outcomes. Also, since the choice for choosing to start off with an odd value or an even value is completely random this means that x can be described as a binomial random variable. Finally, all sequences generated by the rules of the Collatz Conjecture can be described in terms of a binomial experiment because:

1. Each sequence that has been tested up to 2^{68} has consisted of n identical trials [4].
2. Each trial (rules of the Collatz Conjecture) results in one of two outcomes (odd-value or even value) [4].
3. The probability of success denoted by *p* is always the same from trial to trial [4].
4. Each trial is independent because the outcome of one trial does not determine the outcome of other trials [4].

This means that it is possible to start describing experimental outcomes of the Collatz Conjecture in terms of the Binomial Expansion Theorem given by Equation 1 in the form provided below.

Equation 1: $$(x+y)^n = \sum_{k=0}^{n} \left(\frac{n!}{k!\,(n-k)!}\right) x^k y^{(n-k)}$$ [5].

From Definition 2.1 if $x \in \{2n+1\}$ then $f^n(x) = 3x+1$. The function $f^n(x) = 3x+1$ turns any odd value into an even value $\{2n+2\}$, thus forcing even values to have to be evaluated by the function $f^n(x) = x/2$. This means that when n is mapped to 2n +2, *Equation 1* turns into the form given below as described by *Equation 2.*

Equation 2: $$(x+y)^{(2n+2)} = \sum_{k=0}^{2n+2} \left(\frac{(2n+2)!}{k!\,(2n+2-k)!}\right) x^k y^{(2n+2-k)}$$ for n = 0, 1, 2, 3,…

Where the quantity $\frac{(2n+2)!}{k!(2n+2-k)!}$ generates all of the coefficients on the sequence $\{2^{2n+2}\}$. And from Postulate 1, the choice of choosing an odd value, or an even value can be described by *p*, and *1 – p* respectively. Now set *x = p*, and *y = 1 – p*, and put into *Equation 2* to get *Equation 3* below.

Equation 3: $(p+1-p)^{(2n+2)}=\sum_{k=0}^{2n+2}\left(\frac{(2n+2)!}{k!(2n+2-k)!}\right)p^k(1-p)^{(2n+2-k)}$ for n = 0, 1, 2, 3,…

Notice from *Equation 3* that $(p+1-p)^{(2n+2)}=(1)^{(2n+2)}=1^{2n} * 1^2=1^{2n}$ for n = 0, 1, 2, 3,…
Which converges to 1 for all powers of 2n. This means that no matter what integer of x > 0 you choose from the beginning, any sequence generated by the rules of the Collatz Conjecture will eventually converge to 1 as shown by *Equation 4* below.

Equation 4: $(p+1-p)^{(2n+2)}=\sum_{k=0}^{2n+2}\left(\frac{(2n+2)!}{k!(2n+2-k)!}\right)p^k(1-p)^{(2n+2-k)}=1^{2n} * 1^2=1^{2n} * 1=1^{2n}=1$

for n = 0, 1, 2, 3,…

V. CONCLUSION

The Collatz Conjecture was first introduced by Lothar Collatz in the year 1937. In this article the Collatz Conjecture is shown to be correct for a Hailstone sequence that becomes stable for specific odd values that set the function $f^n(x) = 3x + 1 = \{2^{2n+2}\}$. All sequences used in writing this paper were generated through the use of a computer program I wrote written in the C language. The source code that I wrote is included in the appendix at the end of this paper.

APPENDIX A

```
#include <stdio.h>
#include <stdlib.h>
#include <unistd.h>
// Typecasting the function
int syracuse(int x);
int main()
{
	int x;
	printf("Please enter a whole value greater than zero: ");  // Asking for input
	scanf("%d", &x);
	syracuse(x);  // Calling the function
	return (0);
}
int syracuse(int x)
{
	printf("%d\n", x);  // Print the generated values
	int y = x;
	//base cases
	if (x <= 0)
	{
		printf("You have entered an invalid number.\n");
		sleep(10);
		system("clear");
		return (0);
	}
	else if(x == 1)
	{
		return 1;
	}
	//Conditions of the Syracuse conjecture
```

```
        if (x % 2 != 0)
        {
                y = (3 * x) + 1;
        }
        else if(x % 2 == 0)
        {
                y = x/2;
        }
        //Recursive Case
        return syracuse(y);
    }
```


ACKNOWLEDGMENT

I would like to acknowledge the efforts of Douglas L. McCue Jr. for helping to write this paper. His tireless work in editing this paper, and showing me this problem is what kicked off the entire effort.

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