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Original Article

On a General Integro Differential Equation with Parameter

A.M.A. EL-Sayed[1], M.Sh. Mohamed[2] and A. Basheer[3]

[1,2]*Faculty of Science, Alexandria University, Alexandria, Egypt*
[3]*Faculty of Science and Literature, Northern Border University, Rafha, Saudi Arabia*

Abstract - *In this paper, we study the existence of solutions x for an initial value problem of a general implicit differential equation with parameter in the two classes $x \in C^1[0,T]$ and $x \in AC[0,T]$. The maximal and minimal solution will be proved. The uniqueness of the solution will be proved. The continuous dependence of the unique solution will be studied.*

Keywords - *Implicit differential equation, Continuous solution, Integrable solution, Existence of solutions, Continuous dependence, Maximal and minimal solution.*

Math. Subject Classification: 34A12, 39B12, 47H09.

I. INTRODUCTION

It is well known that the nonlinear initial value problems of implicit differential equations create an important branch of nonlinear analysis and have numerous applications in most fields. For papers studying such kind of equations (see [1]-[2], [4]-[8], [9][15]and references therein).

In this paper, we are concerned with the initial value problem of the implicit differential equation with parameter

$$\frac{dx}{dt} = f_1(t, x(t), \int_0^t f_2(s, \frac{dx}{dt}, \mu)ds), t \in (0,T] \tag{1}$$

with initial data

$$x(0) = x_0. \tag{2}$$

First, we study the existence of at least one solution $x \in C^1[0,T]$. The maximal and minimal solution will be proved. Also, the sufficient conditions for the uniqueness of the solution will be given. The continuous dependence of the unique solution on the parameter μ and the function f_2 will be studied.

Second, we study the existence of at least one solution $x \in AC[0,T]$. The uniqueness of the solution will be studied. The continuous dependence of the unique solution on the parameter μ and the function f_2 will be proved.

II. CONTINUOUS SOLUTIONS

A. Existence of solutions

Consider the following assumptions:

1. $f_1(t, y, v): [0,T] \times R \times R \longrightarrow R$ is measurable in t for all $y \in R$ and satisfies Lipschitz condition

$$|f_1(t, y, v) - f_1(t, y_1, v_1)| \leq k_1(|y - y_1| + |v - v_1|).$$

where k_1 is a positive constant. From this assumption we can deduce

$$|f_1(t, y, v)| - |f_1(t, 0,0)| \leq |f_1(t, y, v) - f_1(t, 0,0)| \leq k_1(|y| + |v|),$$
and
$$|f_1(t, y, v)| \leq k_1(|y| + |v|) + |f_1(t, 0,0)|.$$

2. $f_2: [0,T] \times R \times R \longrightarrow R$ is measurable in $t \in [0,T]$ $\forall$ $x \in R$ and continuous in $x \in R$ for almost all $t \in [0,T]$ and there exist an integrable function $m \in L_1[0,T]$ and a positive constant k_2 such that

$$|f_2(t, y(s), \mu)| \leq |m(t)| + k_2|y(s)| + k_2|\mu|,$$

where

$$\sup_{t\in[0,T]} \int_0^t |m(s)|ds \leq M.$$

3. $kT(1+k) < 1, k = \max\{k_1, k_2\}$

Theorem 1 *Let the assumptions(1)-(3) be satisfied, then problem (1)-(2) has at least one solution* $x \in C^1[0,T]$.

Proof. Let $\frac{dx}{dt} = y \in [0,T]$, then equation (1) will be given by

$$y(t) = f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s, y(s), \mu)ds), \tag{3}$$

where

$$x(t) = x_0 + \int_0^t y(s)ds. \tag{4}$$

Now, define the operator F by

$$Fy(t) = f_1(t, x_0 + \int_0^t y(s), \int_0^t f_2(s, y(s), \mu)ds).$$

Define the set

$$Q_r = \{y \in C[0,T]: \| y \| \leq r\}, r = \frac{k|x_0| + kM + k^2T|\mu| + A}{k(1+kT)}$$

where

$$A = \sup_{t\in[0,T]} |f_1(t, 0,0)|.$$

Now, let $y \in \ Q_r$, then

$$\begin{aligned}
|Fy(t)| &= |f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s, y(s), \mu)ds)| \\
&\leq k_1(|x_0 + \int_0^t y(s)ds| + |\int_0^t f_2(s, y(s), \mu)ds|) + |f_1(t, 0,0)| \\
&\leq k_1|x_0| + k_1 \int_0^t |y(s)|ds + k_1 \int_0^t (|m(s)| + k_2|y(s)| + k_2|\mu|)ds + |f_1(t, 0,0)| \\
&\leq k|x_0| + k\int_0^t |y(s)|ds + k\int_0^t |m(s)|ds + k^2 \int_0^t |y(s)|ds + k^2 \int_0^t |\mu|ds + A \\
&\leq k|x_0| + kTr + kM + k^2Tr + k^2T|\mu| + A = r.
\end{aligned}$$

This proves that $F: Q_r \to Q_r$ and $\{Fy(t)\}, t \in [0,T]$ is uniformly bounded on Q_r.

Now, let $y \in Q_r$. Let $t_1, t_2 \in [0,T]$ be such that $|t_1 - t_2| \leq \delta$, then

$$\begin{aligned}
&|Fy(t_2) - Fy(t_1)| \\
&= |f_1(t_2, x_0 + \int_0^{t_2} y(s)ds, \int_0^{t_2} f_2(s, y(s), \mu)ds) - f_1(t_1, x_0 + \int_0^{t_1} y(s)ds, \int_0^{t_1} f_2(s, y(s), \mu)ds)| \\
&\leq |f_1(t_2, x_0 + \int_0^{t_2} y(s)ds, \int_0^{t_2} f_2(s, y(s), \mu)ds) - f_1(t_2, x_0 + \int_0^{t_1} y(s)ds, \int_0^{t_1} f_2(s, y(s), \mu)ds)| \\
&+ |f_1(t_2, x_0 + \int_0^{t_1} y(s)ds, \int_0^{t_1} f_2(s, y(s), \mu)ds) - f_1(t_1, x_0 + \int_0^{t_1} y(s)ds, \int_0^{t_1} f_2(s, y(s), \mu)ds)| \\
&\leq k\int_{t_1}^{t_2} |y(s)|ds + k\int_{t_1}^{t_2} |f_2(s, y(s), \mu)|ds + \theta(\delta) \\
&\leq k\int_{t_1}^{t_2} |y(s)|ds + k\int_{t_1}^{t_2} (|m(t)| + k|y(s)| + k|\mu|)ds + \theta(\delta) \\
&\leq kr(t_2 - t_1) + k\int_{t_1}^{t_2} |m(t)|ds + k^2(r + |\mu|)(t_2 - t_1) + \theta(\delta).
\end{aligned}$$

This means that the class functions $\{Fy(t)\}, t \in [0,T]$ is equi-continuous on Q_r. Then by Arzela Theorem [6], F is compact operator.

Now, we prove that F is continuous operator. Let $y_n \subset Q_r, y_n \to y$, thus by taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f_1(t, x_0 + \int_0^t y_n(s)ds, \int_0^t f_2(s, y_n(s), \mu)ds)$$

and from Lebesgue dominated convergence theorem [3], we have

$$\lim_{n\to\infty} Fy_n(t) = f_1(t, x_0 + \int_0^t \lim_{n\to\infty} y_n(s)ds, \int_0^t f_2(s, \lim_{n\to\infty} y_n(s), \mu)ds)$$
$$= f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s, y(s), \mu)ds) = Fy(t).$$

This means that $Fy_n(t) \to Fy(t)$. Hence the operator F is continuous.

Now by Schauder fixed point theorem [6], there exists at least one solution $y \in C[0,T]$ of the integral equation (3). Consequently, there exists at least at least one solution $x \in C^1[0,T]$ for the problem (1) and (2) given by (4).

B. Maximal and minimal solution

Lemma 1 *Let the assumptions of Theorem 1 be satisfied and* $y(t)$ *and* $v(t)$ *are two continuous functions on [0,T] satisfying*

$$y(t) \le f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s, y(s), \mu)ds),$$
$$v(t) \ge f_1(t, x_0 + \int_0^t v(s)ds, \int_0^t f_2(s, v(s), \mu)ds)$$

and one of them is strict. If f_1 and f_2 are monotonic nondecreasing, then

$$y(t) < v(t), t > 0. \qquad (5)$$

proof. Let the conclusion (5) be false, then there exists t_1 such that

$$y(t_1) = v(t_2), t_1 > 0$$

and

$$y(t) < v(t), 0 < t < t_1.$$

From the monotonically of f_1 and f_2, we get

$$y(t_1) \le f_1(t_1, x_0 + \int_0^{t_1} y(s)ds, \int_0^{t_1} f_2(s, y(s), \mu)ds)$$
$$< f_1(t_1, x_0 + \int_0^{t_1} v(s)ds, \int_0^{t_1} f_2(s, v(s), \mu)ds) = v(t_1).$$

hence $y_1(t) < v_1(t)$. This contradicts the fact that $y(t_1) = v(t_1)$, then $y(t) < v(t), t \in [0,T]$.

Theorem 2 *Let the assumptions (1)-(3) be satisfied. If* f_1, f_2 *are monotonic nondecreasing, then equations (3) has maximal and minimal solutions.*

proof. Firstly, we prove the existence of maximal solution of (3). Let $\epsilon > 0$, then

$$y_\epsilon(t) = \epsilon + f_1(t, x_0 + \int_0^t y_\epsilon(s)ds, \int_0^t f_2(s, y_\epsilon, \mu)ds). \qquad (6)$$

It's easy to show that equation (6) has at least one solution $y_\epsilon \in [0,T]$.

Now let $\epsilon_1, \epsilon_2 > 0$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$y_{\epsilon_2}(t) = \epsilon_2 + f_1(t, x_0 + \int_0^t y_{\epsilon_2}(s)ds, \int_0^t f_2(s, y_{\epsilon_2}, \mu)ds).$$
$$y_{\epsilon_1}(t) = \epsilon_1 + f_1(t, x_0 + \int_0^t y_{\epsilon_1}(s)ds, \int_0^t f_2(s, y_{\epsilon_1}, \mu)ds)$$
$$> \epsilon_2 + f_1(t, x_0 + \int_0^t y_{\epsilon_1}(s)ds, \int_0^t f_2(s, y_{\epsilon_1}, \mu)ds)$$

and from Lemma 1, we obtain

$$y_{\epsilon_2}(t) < y_{\epsilon_1}(t), t \in [0,T].$$

Now the family $\{y_\epsilon(t)\}$ is uniformly bounded as follows:

$$|y_\epsilon(t)| \le \epsilon + |f_1(t, x_0 + \int_0^t y_\epsilon(s)ds, \int_0^t f_2(s, y_\epsilon, \mu)ds)|$$
$$\le \epsilon + r = r^*$$

Also, the family $\{y_\epsilon(t)\}$ is equi-continuous as follows:

$$|y_\epsilon(t_2) - y_\epsilon(t_1)|$$
$$= |\epsilon + f_1(t_2, x_0 + \int_0^{t_2} y_\epsilon(s)ds, \int_0^{t_2} f_2(s, y_\epsilon, \mu)ds) - \epsilon - f_1(t_1, x_0 - \int_0^{t_1} y_\epsilon(s)ds, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)|$$
$$\le |f_1(t_2, x_0 + \int_0^{t_2} y_\epsilon(s)ds, \int_0^{t_2} f_2(s, y_\epsilon, \mu)ds) f_1(t_2, x_0 + \int_0^{t_1} y_\epsilon(s)ds, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)|$$
$$+|f_1(t_2, x_0 + \int_0^{t_1} y_\epsilon(s)ds, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds) - f_1(t_1, x_0 + \int_0^{t_1} y_\epsilon(s)ds, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)|$$

$$\leq k\int_{t_1}^{t_2}|y_\epsilon(s)|ds + k\int_{t_1}^{t_2}|f_2(s,y(s),\mu)|ds + \theta(\delta)$$
$$\leq k\int_{t_1}^{t_2}|y_\epsilon(s)|ds + k\int_{t_1}^{t_2}(|m(t)| + k|y(s)| + k|\mu|)ds + \theta(\delta)$$
$$\leq kr^*(t_2-t_1)] + k\int_{t_1}^{t_2}|m(t)|ds + k[(kr^* + k|\mu|)(t_2-t_1)] + \theta(\delta).$$

Then$\{y_\epsilon(t)\}$ is equi-continuous and uniformly bounded on [0,T], then $\{y_\epsilon(t)\}$ is relatively compact (Arzela-Ascoli theorem), then there exists decreasing sequence ϵ_n such that $\varepsilon_n \to 0$, $n\to\infty$ and $\lim_{n\to\infty} y_{\epsilon_n}(t)$ exists uniformly on [0,T]. Let

$$\lim_{n\to\infty} y_{\epsilon_n}(t) = q(t).$$

Now form the continuity of the functions f_1 and f_2 and Lebesgue dominated convergence theorem, we have

$$(t, x_0 + \int_0^t y_{\epsilon_n}(s)ds, \int_0^t f_2(s, y_{\epsilon_n}(s),\mu)ds) \to (t, x_0 + \int_0^t q(t)ds, \int_0^t f_2(s,q(t),\mu)ds)$$

and

$$f_1(t, x_0 + \int_0^t y_{\epsilon_n}(s)ds, \int_0^t f_2(s, y_{\epsilon_n}(s),\mu)ds) \to f_1(t, x_0 + \int_0^t q(t)ds, \int_0^t f_2(s,q(t),\mu)ds),$$

then

$$q(t) = \lim_{n\to\infty} y_{\epsilon_n}(t) = f_1(t, x_0 + \int_0^t q(t)ds, \int_0^t f_2(s,q(t),\mu)ds).$$

which implies that $q(t)$ is a solution of equation (3).

Finally, $q(t)$ is the maximal solation of (3). To do this, let $y(t)$ be any solution of (3), then

$$y(t) = f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s,y(s),\mu)ds)$$
$$y_\epsilon(t) = \epsilon + f_1(t, x_0 + \int_0^t y_\epsilon(s)ds, \int_0^t f_2(s,y_\epsilon(s),\mu)ds)$$
$$> f_1(t, x_0 + \int_0^t y_\epsilon(s)ds, \int_0^t f_2(s,y_\epsilon(s),\mu)ds).$$

Applying Lemma 1, we get

$$y(t) < y_\epsilon(t), t\in[0,T].$$

From the uniqueness of the maximal solation, it is clear that $y_\epsilon(t)\to q(t)$ uniformly on [0,T] as $\epsilon\to 0$, thus q is the maximal solation of (3).

By similar wa can prove the existence of the minimal solation.

C. Uniqueness of the solution and continuous dependence

For the uniqueness of the solution of (3) consider the following assumption:

- $f_2: [0,T]\times R\times R\to R$ is measurable in $t\in[0,T]$ and satisfies the Lipschitz condition

$$f_2(t,y,\mu) - f_2(t,v,\mu^*) \leq k_2(|y-v| + |\mu-\mu^*|).$$

Theorem 3 *Let the assumptions (1),(3) and (4) be satisfied, then the solution of the problem (1) and (2) is unique solution.*

Proof. From assumption (4), we have

$$|f_2(t,y,\mu)| \leq k_2(|y| + |\mu|) + |f_2(t,0,0)|, |m(t)| = |f_2(t,0,0)|.$$

Thus assumption (2) is satisfied. Then all assumptions of Theorem 1 are satisfied and the solution of the functional integral equation (3) exists. Let y_1, y_2 be two the solution of (3), then

$$|y_1(t) - y_2(t) \leq k|\int_0^t y_1(s)ds - \int_0^t y_2(s)ds| + k|\int_0^t f_2(s,y_1(s),\mu)ds - \int_0^t f_2(s,y_2(s),\mu)ds|$$
$$\leq k\int_0^t |y_1(s) - y_2(s)|ds + k\int_0^t |f_2(s,y_1(s),\mu) - f_2(s,y_2(s),\mu)|ds$$
$$\leq k\int_0^t |y_1 - y_2|ds + k^2\int_0^t |y_1 - y_2|ds$$
$$\leq kT \parallel y_1 - y_2 \parallel + k^2T \parallel y_1 - y_2 \parallel.$$

Hence

$$\parallel y_1 - y_2 \parallel \leq kT \parallel y_1 - y_2 \parallel + k^2T \parallel y_1 - y_2 \parallel.$$

Then

$$\parallel y_1 - y_2 \parallel (1 - (kT + k^2T)) \leq 0.$$

Since $kT(1+k) < 1$, then $\parallel y_1 - y_2 \parallel = 0$ and this implies that $y_1 = y_2$ and the solution $y\in[0,T]$ of (3) is unique.

Continuous dependence on the parameter μ

Definition 1 *The solution $y \in [0,T]$ of the functional integral (3) depends continuously on the parameter μ, if*

$$\forall \varepsilon > 0, \exists \ \delta(\epsilon) \ \ s.t \ \ |\mu - \mu^*| < \delta \Rightarrow \| y - y^* \| < \epsilon,$$

where y^* is the unique solution of (3) corresponding to μ^*

Definition 2 *The solution $x \in [0,T]$ of the functional integral (4) depends continuously on the parameter μ, if*

$$\forall \varepsilon > 0, \exists \ \delta(\epsilon) \ \ s.t \ \ |\mu - \mu^*| < \delta \Rightarrow \| x - x^* \| < \epsilon$$

where x^* is the solution of (1)-(2) corresponding to μ^*

Theorem 4 *Let the assumptions of Theorem 3 be satisfied, then the solution of the problem (1) and (2) depends continuously on the parameter μ.*

Proof. Let $y \ \ and \ \ y^*$ be the two solutions of equations (3) corresponding to $\mu \ \ and \ \ \mu^*$, then

$$|y(t) - y^*(t) \le k|x_0 + \int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds|$$
$$+k| \int_0^t f_2(s, y(s), \mu)ds - \int_0^t f_2(s, y^*(s), \mu^*)ds|$$
$$\le k \int_0^t |y - y^*|ds + k^2 \int_0^t (|y(s) - y^*(s)| + |\mu - \mu^*|)ds$$
$$\le k \int_0^t \sup_{t\epsilon[0,T]} |y - y^*|ds + k^2 \int_0^t (\sup_{t\epsilon[0,T]} |y(s) - y^*(s)|ds) + k^2T|\mu - \mu^*|$$
$$\le kT \ \| y - y^* \| + k^2T \ \| y - y^* \| + k^2T\delta$$

Hence

$$\| y - y^* \| \le \frac{k^2T\delta}{(1-(kT+k^2T))} = \epsilon.$$

This proves that the solution of Eq (3) depends continuously on the parameter μ. Now

$$|x - x^*| = x_0 + \int_0^t y(s)ds - x_0 + \int_0^t y^*(s)ds$$
$$\le \int_0^t |y - y^*|ds$$
$$\le \| y - y^* \| \ T,$$
$$\| x - x^* \| \le \frac{k^2T^2\delta}{(1-kT-k^2T)} = \epsilon.$$

This proves that the solution of (1)-(2) depends continuously on the parameter μ.

Continuous dependence on the function f_2

Definition 3 *The solution $y \in [0,T]$ of (3) depends continuously on the function f_2, if*

$$\forall \varepsilon > 0, \exists \ \delta(\epsilon) \ \ s.t \ \ |f_2 - f_2^*| < \delta \Rightarrow \| y - y^* \| < \epsilon,$$

where y^* is the unique solution of the functional integral equation (3) corresponding to f_2^*

Theorem 5 *Let the assumptions of theorem (3) be satisfied, then the solution of problem (1) and (2) depends continuously on the function f_2.*

Proof. Let $y \ \ and \ \ y^*$ be the two solutions of (3), then

$$|y(t) - y^*(t) \le k \int_0^t |y(s) - y^*(s)|ds + k \int_0^t |f_2(s, y(s), \mu) - f_2^*(s, y^*(s), \mu)|ds$$
$$\le k \int_0^t |y(s) - y^*(s)|ds + k \int_0^t |f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)$$
$$+f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu)|ds$$
$$\le k \int_0^t |y(s) - y^*(s)|ds + k \int_0^t |f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)|ds$$
$$+k \int_0^t |f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu)|ds$$
$$\le k \int_0^t |y(s) - y^*(s)|ds + k \int_0^t |f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)|$$
$$+k^2 \int_0^t |y(s) - y^*(s)|ds$$
$$\le k \ \| y - y^* \| + kT\delta + k^2T \ \| y - y^* \|.$$

Hence

$$\parallel y - y^* \parallel \leq \frac{kT\delta}{(1-(kT+k^2T))}.$$

Definition 4 *The solution* $x \in [0,T]$ *of (4) depends continuously on the function* f_2, *if*

$$\forall \varepsilon > 0, \exists \ \delta(\epsilon) \ s.t \ |f_2 - f_2^*| < \delta \Rightarrow \parallel x - x^* \parallel < \epsilon,$$

where x^* is the solution of (1)-(2) corresponding to f_2^*.

Theorem 6 *Let the assumptions of Theorem 3 be satisfied, then the solution of the problem (1) and (2) depends continuously on the parameter* f_2.

Now

$$|x - x^*| = x_0 + \int_0^t y(s)ds - x_0 + \int_0^t y^*(s)ds$$
$$\leq \int_0^t |y(s) - y^*|ds$$
$$\leq \parallel y - y^* \parallel T$$
$$\leq \frac{kT^2\delta}{(1-(kT+k^2T))} = \epsilon.$$

This proves that the solution of equation (3) depends continuously on function f_2 consequently the solution of problem (1) and (2) depends continuously on the f_2.

III. INTEGRABLE SOLUTION

A. Existence of solutions

Consider the following initial value problem

$$\frac{dx}{dt} = f_1(t, x(t), \int_0^t f_2(s, \frac{dx}{dt}, \mu)ds), \ a.e. t \in (0,T] \quad (7)$$

with initial data

$$x(0) = x_0. \quad (8)$$

under the following assumptions:

- $f_1: [0,T] \times R \times R \to R$ is measurable in $t \in [0,T]$ for every $y, v \in R$ and continuous in y, v for almost all $t \in [0,T]$ and there exist a function $m_1(t) \in L^1[0,T]$ and constant $N_1 > 0$ such that

$$|f_1(t, y, v)| \leq |m_1(t)| + N_1(|y| + |v|).$$

- $f_1: [0,T] \times R \times R \to R$ is measurable in $t \in [0,T]$ for every $y, \mu \in R$ and continuous in y, μ for almost all $t \in [0,T]$ and there exist a function $m_2(t) \in L^1[0,T]$ and constant $N_2 > 0$ such that

$$|f_2(t, y, \mu)| \leq |m_2(t)| + N_2(|y| + |\mu|).$$

- $2NT < 1, N = \max\{N_1, N_2\}$

Theorem 7 *Let the assumptions* $(i) - (iii)$ *be satisfied, then equation (3) has at least one solution* $y \in L_1[0,T]$, *hence the initial value problem (7)-(8) has at least one solution* $x \in AC[0,T]$.

Proof. Define the operator F by

$$Fy(t) = f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s, y(s), \mu)ds), \qquad t \in [0,T].$$

Define the set $Q_r = \{y \in R: \parallel y \parallel \leq r\}$, where $r = \frac{\|m_1\|_{L_1} + NT|x_0| + NT\|m_2\|_{L_1} + \frac{1}{2}NT^2|\mu|}{1-2NT}$

Let $x \in Q_r$, then

$$|Fy(t)| \leq |m_1(t)| + N|x_0| + N\int_0^t |y(s)|ds + N\int_0^t |f_2(s, y(s), \mu)|ds$$
$$\leq |m_1(t)| + N|x_0| + N\int_0^t |y(s)|ds + N\int_0^t (|m_2(t)| + N|y| + N|\mu|)ds$$
$$\leq |m_1(t)| + N|x_0| + N\int_0^t |y(s)|ds + N\int_0^t |m_2(t)|ds + N\int_0^t |y|ds + N\int_0^t |\mu|)ds.$$

Then

$$\parallel Fy \parallel_{L_1} = \int_0^T |Fy(t)|dt \leq \int_0^T |m_1(t)|dt + N\int_0^T |x_0|dt$$

$$+N\int_0^T\int_0^t |y(s)|dsdt + N\int_0^T\int_0^t |m_2(s)|dsdt$$
$$+N\int_0^T\int_0^t |y(s)|dsdt + N\int_0^T\int_0^t |\mu|dsdt$$
$$\| m_1 \|_{L_1} + NT|zeta_0| + NT \| y \|_{L_1} + NT \| m_2 \|_{L_1} + NT \| y \|_{L_1} + \frac{1}{2}NT^2|\mu| = r.$$

This prove that $F: Q_r \to Q_r$ and $\{Fy\}$ is uniformly bounded in Q_r.

Now let $y \in Q_r$, then

$$\| (Fy)_h - (Fy) \|_{L_1} = \int_0^T |(Fy(s))_h - (Fy(s))|ds$$
$$= \int_0^T \frac{1}{h}|\int_t^{t+h} (Fy(\theta))d\theta - (Fy(s))|ds$$
$$\leq \int_0^T \frac{1}{h}\int_t^{t+h} |(Fy(\theta)) - (Fy(s))|d\theta ds$$
$$\leq \int_0^T \frac{1}{h}\int_t^{t+h} |f_1(\theta, x_0 + \int_0^\theta y(\tau)d\tau, \int_0^\theta f_2(\tau, y(\tau), \mu)d\tau)$$
$$-f_1(s, x_0 + \int_0^s y(\tau)d\tau, \int_0^s f_2(\tau, y(\tau), \mu)d\tau)|d\theta ds.$$

Since $f_1 \in L_1[0,T]$, it follows that

$$\frac{1}{h}\int_t^{t+h} |f_1(\theta, x_0 + \int_0^\theta y(\tau)d\tau, \int_0^\theta f_2(\tau, y(\tau), \mu)d\tau) - f_1(s, x_0 + \int_0^s y(\tau)d\tau, \int_0^s f_2(\tau, y(\tau), \mu)d\tau)|d\theta ds$$

Hence, $Fy(t)_h \to (Fy)$ uniformly in $L_1[0,T]$. Thus the class $\{Fy\}, y \in Q_r$ is relatively compact. Hence F is compact operator.

Now, let $y_n \subset Q_r, y_n \to y,$ then by taking the limit as $n \to \infty,$ then

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f_1(t, x_0 + \int_0^t y_n(s)ds, \int_0^t f_2(s, y_n(s), \mu)ds)$$

since f_1, f_2 are continuous in y, then

$$\lim_{n\to\infty} Fy_n(t) = f_1(t, x_0 + \lim_{n\to\infty}\int_0^t y_n(s)ds, \lim_{n\to\infty}\int_0^t f_2(s, y_n(s), \mu)ds)$$

Now, from assumptions $(i), (ii)$ and Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} Fy_n(t) = f_1(t, x_0 + \int_0^t y(s)ds, \int_0^t f_2(s, y(s), \mu)ds) = Fy(t).$$

Hence the operator F is continuous. Now all the conditions of Schauder fixed point theorem are satisfied. Then the functional integral equation (3) has at least one solution $y \in L_1[0,T]$. Consequently, there exists at least one solution x for the problem (7) and (8) and this solution given by

$$x(t) = x_0 + \int_0^t y(s)ds \qquad t \in [0,T]. \tag{9}$$

B. Unique integrable solution and continuous dependence

Consider following assumption:

- $f_1: [0,T] \times R \times R \longrightarrow R$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition
$$|f_1(t, y_1, v_1) - f_1(t, y_2, v_2)| \leq N_1(|y_1 - y_2| + |v_1 - v_2|).$$

with Lipschitz condition $N_1 > 0$

- $f_2: [0,T] \times R \times R \longrightarrow R$ is measurable in the $t \in [0,T]$ and satisfies the Lipschitz condition
$$|f_2(t, y_1(s), \mu) - f_2(t, y_2(s), \mu^*)| \leq N_2(|y_1 - y_2| + |\mu - \mu^*|)$$

with Lipschitz condition $N_2 > 0$.

- $f_1(t, 0,0),\ f_2(t, 0,0) \in L_1[0,T]$.

- $N_1(1 + N_2)T < 1$.

Theorem 8 *Let the assumptions* $(i^*) - (iv^*)$ *and* (iii) *be satisfied, then (3), has a unique solution* $y \in L_1[0,T]$. *Consequently, the solution* x *of the problem (7) and (8) is unique.*

Proof. From assumption i^*, we obtain

$$|f_1(t,y,v)| \leq N_1(|y|+|v|) + |f_1(t,0,0)| = N_1(|y|+|v|) + m_1(t), m_1(t) = |f_1(t,0,0)|.$$

Similarly,

$$|f_2(t,y,\mu)| \leq N_2(|y|+|\mu|) + m_2(t), m_2(t) = |f_2(t,0,0)|.$$

Then all assumptions of Theorem(7) are satisfied. Then the solution of (3) exists. Now let y, v be two solutions of equation (3), then

$$\begin{aligned}
&|y(t) - v(t)| \leq N_1|(x_0 + \int_0^t y(s) - x_0 - \int_0^t v(s)ds) \\
&+N_1(\int_0^t f_2(s,y(s),\mu)ds - \int_0^t f_2(s,v(s),\mu)ds)| \\
&\leq N_1(\int_0^t |y(s) - v(s)|ds) + N_1(\int_0^t |f_2(s,y(s),\mu) - f_2(s,v(s),\mu)|ds) \\
&\leq (N_1 + N_1N_2)\int_0^t |y(s) - v(s)|ds.
\end{aligned}$$

Thus, we obtain

$$\leq N_1(1+N_2)T \parallel y - v \parallel_{L_1}.$$

Since $N_1(1+N_2)T < 1$, this implies that $y = v$, i.e. the solution of (3) is unique. Consequently, the solution of the problem (7) and (8) is unique.

Continuous dependence on the parameter μ

Definition 5 *The solution* $y \in L_1[0,T]$ *of (3) depends continuously on the parameter* μ, *if*

$$\forall \epsilon > 0, \exists \quad \delta(\epsilon) \quad s.t \quad |\mu - \mu^*| < \delta \Rightarrow \parallel y - y^* \parallel_{L_1} < \epsilon,$$

where $y^* \in L_1[0,T]$ is the unique solution of (3) corresponding to μ^*.

Theorem 9 *Let the assumptions of Theorem 8 be satisfied, then the solution of (3) depends continuously on the parameter* μ.

proof. Let y, y^* be the two solutions of (3) corresponding to μ and μ^* respectively, then

$$\begin{aligned}
&|y(t) - y^*(t)| \leq N_1|x_0 + \int_0^t y(s) - x_0 - \int_0^t y^*(s)ds| \\
&+N_1|\int_0^t f_2(s,y(s),\mu)ds - \int_0^t f_2(s,y^*(s),\mu^*)ds| \\
&\leq N_1\int_0^t |y(s) - y^*(s)|ds + N_1\int_0^t |f_2(s,y(s),\mu) - f_2(s,y^*(s),\mu^*)|ds \\
&\leq N_1\int_0^t |y(s) - y^*(s)|ds + N_1N_2\int_0^t (|y(s) - y^*(s)| + |\mu - \mu^*|)ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&\parallel y - y^* \parallel_{L_1} \leq N_1\int_0^T\int_0^t |y(s) - y^*(s)|dsdt + N_1N_2\int_0^T\int_0^t (|y(s) - y^*(s)| + |\mu - \mu^*|)dsdt \\
&\leq N_1T \parallel y - y^* \parallel_{L_1} + N_1N_2(T \parallel y - y^* \parallel_{L_1} + |\mu - \mu^*|(\frac{T^2}{2})),
\end{aligned}$$

thus

$$\parallel y - y^* \parallel_{L_1} \leq \frac{\frac{1}{2}N_1N_2T^2\delta}{(1 - N_1T(1+N_2))} \leq \epsilon.$$

This proves the continuous dependence of the solution $y \in L_1[0,T]$ of the functional integral equation (3) on the parameter μ.

Corollary 1 *The solution* $x \in AC[0,T]$ *of the problem (7)-(8) depends continuously on the parameter* μ.

Continuous dependence on the parameter f_2

Definition 6 *The solution* $y \in L_1[0,T]$ *of the functional integral equation (3) depends continuously on the function* f_2, *if*

$$\forall \epsilon > 0, \exists \quad \delta(\epsilon), s.t \quad |f_2 - \eta_2^*| < \delta \Rightarrow \parallel y - y^* \parallel_{L_1} < \epsilon,$$

where y^* is the unique solution $y^* \in L_1[0,T]$ of the functional integral equation(3) corresponding to f_2^*

Theorem 10 *Let the assumptions of Theorem 8 be satisfied, then the solution of the integral equation(3) depends continuously on the function* f_2.

Proof. Let y, y^* be the solution of the functional integral equation(3) corresponding to f_2 and f_2^* respectively, then

$$|y(t) - y^*(t)| \leq N_1|x_0 + \int_0^t y(s) - x_0 - \int_0^t y^*(s)ds|$$
$$+N_1|\int_0^t f_2(s, y(s), \mu)ds - \int_0^t f_2^*(s, y^*(s), \mu)ds|$$
$$\leq N_1 \int_0^t |y(s) - y^*(s)|ds + N_1 \int_0^t |f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)$$
$$-f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu|ds$$
$$+N_1 \int_0^t |y(s) - y^*(s)|ds + N_1 \int_0^t |f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu)|ds$$
$$\leq N_1 \int_0^t |y(s) - y^*(s)|ds + N_1 \int_0^t |f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)|ds$$
$$+N_1 N_2 \int_0^t |y(s) - y^*(s)|ds.$$

Hence

$$\| y - y^* \|_{L_1} \leq \frac{\frac{1}{2} N_1 T^2 \delta}{(1 - (N_1(1 + N_2)T))} = \epsilon.$$

This proves the continuous dependence of the solution of the functional integral equation (3) on the function f_2.

Corollary 2 *The solution* x *of the problem (7)-(8) depends continuously on the function* f_2.

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