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Original Article

On Domination Zagreb Polynomials of Graphs and Some Graph Operations

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Abstract — *Graph polynomials are powerful and well-developed tools to express graph parameters. A set $D \subset V(G)$ is a dominating set of G if every vertex not in D is adjacent to at least one vertex in D. The minimal domination set is the dominating set such that $D - v$ is not dominating set. H. Ahmed et al. [4] invented the domination Zagreb Polynomials as follows:*

$$DM_1(G,x) = \sum_{u\epsilon V(G)} x^{d_{d_G}{}^2(u)},$$
$$DM_2(G,x) = \sum_{uv\epsilon E(G)} x^{d_{d_G}(u)d_{d_G}(v)},$$
$$DM_1{}^*(G,x) = \sum_{uv\epsilon E(G)} x^{d_{d_G}(u)+d_{d_G}(v)}.$$

Where $d_d(v)$ is the domination degree of a vertex $v \in V(G)$ which is defined as the number of minimal dominating sets of G contains v. In this paper, we calculate the domination Zagreb polynomials for some important families of graphs and some graph operations like corona product and join of graphs. Some properties of the domination Zagreb polynomials of graphs are also established.

Keywords — *Domination degrees, Domination Zagreb polynomial, Graph operation.*

I. INTRODUCTION

Consider G *to* be a simple, connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of edges that are incident to a vertex is called the degree of that vertex. $\overline{G}$ is a complimentof G, with vertex set $V(G)$ in which two vertices u and v are adjacent in $\overline{G}$ *if* and only if they are not adjacent in G. The dominating set in a graph is a set of vertices $D \subseteq V(G)$such that each vertex is either in D or is adjacent to a vertex in D. The minimal dominating set is the dominating set, such that $D - v$ is not a dominating set. The cardinality of the smallest minimal dominating set is called domination number, represented as $\gamma(G)$ [18], for more information on domination in a graph, readers refer to ([19], [20], [25], [26]). A topological index is a very important and numerical caption of a molecule, based on a certain topological characteristic of the corresponding molecular graph. The indices are exerting to conjecture the phenomenal features allied to the chemical bioactivities reactivities in certain molecules or networks. The topological indices become an enamored fact these days, for more details about topological indices see ([6] - [12], [14], [15], [22], [23], [26]). Domination and topological indices are useful tools in graph theory. Hanan Ahmed, *et el.* [17] introduced a domination degree d_d of vertex v which defined as the cardinality of minimal dominating sets of G which contains v. Let $T_m(G)$ denote the total number of minimal dominating sets of G. The graph G is called $k-$domination regular graph if and only if $d_d(v)=k$ for all $v \in V(G)$, for further information and applications of domination topological indices we refer to ([1]-[3], [5], [21]). The complete graph K_r has a degree $r-1$ of each vertex and each vertex is adjacent. In the bipartite graph $K_{r,s}$, there exists a partition of $V(G)$ into two subsets A and B such that no two vertices in the subset are adjacent. A bipartite graph $\overline{K}_{r,s}$ is said to be a complete bipartite graph if each vertex in A is adjacent to each vertex in B, with $|A|=r$ & $|B|=s$. A Star graph is a complete bipartite graph. Let $K_{r,s}$ is the complement of a complete bipartite graph. A connected acyclic undirected graph is called a tree. A tree containing exactly two non-pendent vertices is called a double star, denoted by $S_{r,s}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be connected simple graph. The Cartesian product $G_1 \times G_2$ [18], is a (V, E) graph where $V = V_1 \times V_2$, such $u = (u_1, u_2)$ and $v = (v_1, v_2)$and two vertices are adjacent if either $u_1 = v_1$ and u_2 is adjacent to $v_2$in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1. Book graph B_r, $r \geq 3$ is a Cartesian product of star S_r and path P_1. The Windmill graph $Wd_r{}^s$is an undirected graph constructed for $r \geq 2$ and $s \geq 2$ by s copies of the complete graph K_r at a shared universal vertex.

Lemma 1.1. [17]
1. If $G \cong K_r$, then $d_d(v) = 1$, for all $v \in V(G)$.
2. If $G \cong S_r$, then $d_d(v) = 1$, for all $v \in V(G)$.
3. If $G \cong K_{r,s}$, then $d_d(v) = d(v)$, for all $v \in V(G)$.
4. If $G \cong S_{r,s}$, then $d_d(v) = 2$, for all $v \in V(G)$.

Definition 1.2. [17] The first, second domination Zagreb and modified first domination Zagreb indices are defined by

$$DM_1(G,x) = \sum_{u \in V(G)} d_{d_G}{}^2(u),$$
$$DM_2(G,x) = \sum_{uv \in E(G)} d_{d_G}(u) d_{d_G}(v),$$
$$DM_1{}^*(G,x) = \sum_{uv \in E(G)} d_{d_G}(u) + d_{d_G}(v).$$

Hanan Ahmed, et al. introduced the domination Zagreb polynomials, which are defined as:

Definition 1.3. [4] The first, second domination Zagreb polynomials and modified first domination Zagreb polynomial are given by:

$$DM_1(G,x) = \sum_{u \in V(G)} x^{d_{d_G}{}^2(u)},$$
$$DM_2(G,x) = \sum_{uv \in E(G)} x^{d_{d_G}(u) d_{d_G}(v)},$$
$$DM_1{}^*(G,x) = \sum_{uv \in E(G)} x^{d_{d_G}(u) + d_{d_G}(v)}.$$

Proposition 1.4. For a simple connected graph G,

$$DM_0(G,x) = \sum_{u \in V(G)} x^{d_{d_G}(u)},$$
$$DM_{c,d}(G,x) = \sum_{uv \in E(G)} x^{(d_{d_G}(u)+c)+(d_{d_G}(v)+d)}.$$

II. DOMINATION ZAGREB POLYNOMIALS OF GRAPHS

Proposition 2.1. 1. For a star graph S_r with $r + 1$ vertices,

$$DM_1(S_r, x) = (r+1)x, \quad DM_2(S_r, x) = rx, \quad DM_1{}^*(S_r, x) = rx^2.$$

2. For the complete graph K_r,

$$DM_1(K_r, x) = rx, \quad DM_2(K_r, x) = \frac{r(r-1)}{2} x, \quad DM_1{}^*(K_r, x) = \frac{r(r-1)}{2} x^2.$$

3. For a double star graph $S_{r,s}$,

$$DM_1(S_{r,s}, x) = (r+s+2)x^4, \quad DM_2(S_{r,s}, x) = DM_1{}^*(S_{r,s}, x) = (r+s+1)x^4.$$

Lemma 2.2. [17] If $G \cong K_{r,s}$, then

$$d_d(v) = \begin{cases} r+1, & \text{if } v \in A; \\ s+1, & \text{if } v \in B. \end{cases}$$

Theorem 2.3. Let G be a complete bipartite graph $K_{r,s}$. Then

$$DM_1(G,x) = x^{(r+1)^2} + sx^{(s+1)^2},$$
$$DM_2(G,x) = rsx^{(r+1)(s+1)},$$
$$DM_1{}^*(G,x) = rsx^{(r+s+2)}.$$

Proof. In a complete bipartite graph $K_{r,s}$, we have two types partitions say $A=\{v_1, v_2, \ldots v_r\}$ & $B=\{u_1, u_2, \ldots u_s\}$ with respect to vertices having domination degree in Lemma 2.2. So we have,

$$\begin{aligned} DM_1(G,x) &= \sum_{u \in V(K_{r,s})} x^{d_{d_G}{}^2(u)} \\ &= \sum_{v \in A} x^{(r+1)^2} + \sum_{v \in B} x^{(s+1)^2} \\ &= x^{(r+1)^2} + sx^{(s+1)^2}. \end{aligned}$$

Since each edge of the set A is connected to each vertex of set B. There are rs edges in $K_{r,s}$. Hence,

$$\begin{aligned} DM_2(G,x) &= \sum_{uv \in E(K_{r,s})} x^{d_{d_G}(u) d_{d_G}(v)}, \\ &= rsx^{(r+1)(s+1)}. \end{aligned}$$

And,

$$DM_1{}^*(G,x) = \sum_{uv \in E(K_{r,s})} x^{d_{d_G}(u) + d_{d_G}(v)}$$

$$= rsx^{(r+s+2)}.$$

Corollary 2.4. In a complete bipartite graph $K_{r,s}$, if r = s then,

$$DM_1 (K_{r,s}, x) = 2rx^{(r+1)^2},$$
$$DM_2 (K_{r,s}, x) = r^2 x^{(r+1)^2},$$
$$DM_1^* (K_{r,s}, x) = r^2\, x^{2(r+1)}.$$

Proposition 2.5. If $\bar{K}_{r,s}$ is complement of $K_{r,s}$, then

$$DM_1 (\bar{K}_{r,s}, x) = rx^{s^2} + sx^{r^2},$$

$$DM_2 (\bar{K}_{r,s}, x) = \frac{r(r-1)}{2} x^{s^2} + \frac{s(s-1)}{2} x^{r^2},$$

$$DM_1^* (\bar{K}_{r,s}, x) = \frac{r(r-1)}{2} x^{2s} + \frac{s(s-1)}{2} x^{2r}.$$

Proof. Note that, $d_{d_{\bar{K}_{r,s}}}(v) = d_{K_{r,s}}(v)$. There are r number of vertices having a domination degree s and s number of vertices having a domination degree r. By using the definition of the first Zagreb polynomial, we get

$$DM_1 (\bar{K}_{r,s}, x) = rx^{s^2} + sx^{r^2}.$$

Next, there are $\frac{r(r-1)}{2}$ number of edges in set A and $\frac{s(s-1)}{2}$ number of edges in set B. Hence we get

$$DM_2 (\bar{K}_{r,s}, x) = \sum_{uv\in E(\bar{K}_{r,s})} x^{d_{d_G}(u) d_{d_G}(v)}$$

$$= \sum_{uv\in A} x^{d_{d_G}(u) d_{d_G}(v)} + \sum_{uv\in B} x^{d_{d_G}(u) d_{d_G}(v)}$$

$$= \frac{r(r-1)}{2} x^{s^2} + \frac{s(s-1)}{2} x^{r^2}.$$

$$DM_1^* (\bar{K}_{r,s}, x) = \sum_{uv\in E(\bar{K}_{r,s})} x^{d_{d_G}(u) + d_{d_G}(v)}$$

$$= \sum_{uv\in A} x^{d_{d_G}(u) + d_{d_G}(v)} + \sum_{uv\in B} x^{d_{d_G}(u) + d_{d_G}(v)}$$

$$= \frac{r(r-1)}{2} x^{2s} + \frac{s(s-1)}{2} x^{2r}.$$

Lemma 2.6. [17] Let G be the Windmill $Wd_r^{\,s}$. Then

$$d_d(v) = \begin{cases} 1, & \text{if } v \text{ is in center;} \\ (r-1)^{s-1}, & \text{otherwise.} \end{cases}$$

Theorem 2.7. Let $G \cong Wd_r^{\,s}$. Then

$$DM_1(G,x) = x + s\,(r-1)x^{(r-1)^{2(s-1)}},$$
$$DM_2(G,x) = s(r-1)x^{(r-1)^{(s-1)}} + \left[\frac{sr(r-1)}{2} - s(r-1)\right] x^{(r-1)^{2(s-1)}},$$
$$DM_1^* (G, x) = s(r-1)x^{(r-1)^{(s-1)}+1} + \left[\frac{sr(r-1)}{2} - s(r-1)\right] x^{2[(r-1)^{(s-1)}]}.$$

Proof. Note that, the center point has domination degree 1 and the remaining $(r-1)$ s vertices have domination degree $(r-1)^{(s-1)}$. By using the definition of the first Zagreb polynomial, we get

$$DM_1(G,x) = x + s\,(r-1)x^{(r-1)^{2(s-1)}}.$$

Next, let E_1 be the set of all edges which are incident with the center vertex and E_2 be the set of all edges of the complete graph. Then,

$$DM_2 (G, x) = \sum_{uv\in E(G)} x^{d_{d_G}(u) d_{d_G}(v)}$$

$$= \sum_{uv\in E_1(G)} x^{d_{d_G}(u) d_{d_G}(v)} + \sum_{uv\in E_2(G)} x^{d_{d_G}(u) d_{d_G}(v)}$$

$$= s(r-1)x^{(r-1)^{(s-1)}} + \left[\frac{sr(r-1)}{2} - s(r-1)\right] x^{(r-1)^{2(s-1)}}.$$

Similarly,

$$DM_1^* (G, x) = \sum_{uv\in E(G)} x^{d_{d_G}(u) + d_{d_G}(v)}$$

$$=\sum_{uv\in E_1(G)} x^{d_{d_G}(u)+d_{d_G}(v)}+\sum_{uv\in E_2(G)} x^{d_{d_G}(u)+d_{d_G}(v)}$$

$$= s(r-1)x^{(r-1)^{(s-1)}+1} + \left[\frac{sr(r-1)}{2} - s(r-1)\right] x^{2[(r-1)^{(s-1)}]}.$$

Lemma 2.8. [17] If $G \cong B_r$, with $T_m(B_r) = 2^r + 3$, then

$$d_d(v) = \begin{cases} 3 & \text{if v is center vertex:} \\ 2^{r-1}+1, & \text{otherwise.}\end{cases}$$

Theorem 2.9. If $G \cong B_r$ with $r \geq 3$, then

$$DM_1(G, x) = 2x^9 + 2rx^{(2^{r-1}+1)^2},$$
$$DM_2(G, x) = x^9 + 2rx^{3(2^{r-1}+1)} + rx^{(2^{r-1}+1)^2},$$
$$DM_1^*(G, x) = x^6 + 2rx^{3+(2^{r-1}+1)} + rx^{2(2^{r-1}+1)}.$$

Proof. Note that, there are two vertices are in the center. Let A be the set of {V (G) −center vertices}. So, by Lemma 2.8, we get

$$DM_1(G, x) = \sum_{u\in V(G)} x^{d_{d_G}{}^2(u)}$$
$$= 2x^9 + \sum_{u\in A} x^{d_{d_G}{}^2(u)}$$
$$= 2x^9 + 2rx^{(2^{r-1}+1)^2}.$$

Next, here three types of edge set in the book graph. Let $E_1 = \{uv \in E(G): d_d(u) = d_d(v) = 3\}$, $E_2 = \{uv \in E(G): d_d(u) = 3, d_d(v) = 2^{r-1}+1\}$, $E_3 = \{uv \in E(G): d_d(u) = d_d(v) = 2^{r-1}+1\}$ and $|E_1| = 1$, $|E_2| = 2r$, $|E_3| = r$. Hence

$$DM_2(G, x) = \sum_{uv\in E(G)} x^{d_{d_G}(u)d_{d_G}(v)}$$

$$=\sum_{uv\in E_1(G)} x^{d_{d_G}(u)d_{d_G}(v)}+\sum_{uv\in E_2(G)} x^{d_{d_G}(u)d_{d_G}(v)}+\sum_{uv\in E_3(G)} x^{d_{d_G}(u)d_{d_G}(v)}$$

$$= x^9 + 2rx^{3(2^{r-1}+1)} + rx^{(2^{r-1}+1)^2}.$$

Similarly,

$$DM_1^*(G, x) = \sum_{uv\in E(G)} x^{d_{d_G}(u)+d_{d_G}(v)}$$

$$=\sum_{uv\in E_1(G)} x^{d_{d_G}(u)+d_{d_G}(v)}+\sum_{uv\in E_2(G)} x^{d_{d_G}(u)+d_{d_G}(v)}+\sum_{uv\in E_3(G)} x^{d_{d_G}(u)+d_{d_G}(v)}$$

$$= x^6 + 2rx^{3+(2^{r-1}+1)} + rx^{2(2^{r-1}+1)}.$$

Proposition 2.10. If G is k−domination regular graph with r vertices and s edges, then

$$DM_1(G, x) = rx^{k^2},$$
$$DM_2(G, x) = sx^{k^2},$$
$$DM_1^*(G, x) = sx^{k^2}.$$

The graph Banana tree $B_{n,k}$ is obtained by connecting one leaf of each of n copies of k−star with a single root vertex that is distinct from all the stars.

Lemma 2.11. Let G be a banana tree $B_{n,k}$, $n \geq 2, k \geq 4$, then

$$d_d(v) = \begin{cases} n+1, & \text{if v is center vertex of } k-\text{star:} \\ 2, & \text{otherwise.}\end{cases}$$

Proof. If G is a Banana tree, there are three types of minimal dominating sets in G. First type: center of n-copies of k-stars and anyone vertices which is connected one leaf of each n-copies of k-stars. Second type: pendent vertices of n-copies of k-stars and root vertex that is distinct from all the stars. Third type: pendent vertices of n-copies of k-stars and one vertex which is connected to one leaf of each n−copies of k-stars. Hence $T_m(B_{n,k})=n + 3$ and for all $v \in V(B_{n,k})$. So, we get,

$$d_d(v) = \begin{cases} n+1, & \text{if v is center vertex of } k-\text{star:} \\ 2, & \text{otherwise.}\end{cases}$$

Theorem 2.12. Let $G \cong B_{n,k}$. Then

$$DM_1(G, x) = x^{(n+1)^2} + (nk+1)x^4,$$
$$DM_2(G, x) = nkx^{2(n+1)} + n\,x^4,$$
$${DM_1}^*(G, x) = nk\,x^{n+3} + nx^4.$$

Proof. If $G \cong B_{n,k}$ one can see that, there are n vertices having domination degree (n+1) and (nk+1) vertices having domination degree 2. Let A= {v: v $\in$ center vertices of n-copies of k-stars} and B={v: v $\in$ V(G) −Center vertices of n-copies of k-stars}. Hence by Lemma 2.11, we have,

$$DM_1(G, x) = \sum_{u\in V(G)} x^{{d_{d_G}}^2(u)}$$
$$= \sum_{u\in A} x^{{d_{d_G}}^2(u)} + \sum_{u\in B} x^{{d_{d_G}}^2(u)}$$
$$= x^{(n+1)^2} + (nk+1)x^4.$$

Next, there are two types of edges in the Banana tree. Let E_1 be the set of nk edges with starting and ending vertices having domination degree 2 and (n + 1) respectively, E_2 be the set of n edges with starting and ending vertices having same domination degree 2. Hence,

$$DM_2(G, x) = \sum_{uv\in E(G)} x^{d_{d_G}(u)d_{d_G}(v)}$$
$$=\sum_{uv\in E_1(G)} x^{d_{d_G}(u)d_{d_G}(v)}+\sum_{uv\in E_2(G)} x^{d_{d_G}(u)d_{d_G}(v)}$$
$$= nkx^{2(n+1)} + nx^4.$$

Similarly,

$${DM_1}^*(G, x) = \sum_{uv\in E(G)} x^{d_{d_G}(u)+d_{d_G}(v)}$$
$$=\sum_{uv\in E_1(G)} x^{d_{d_G}(u)+d_{d_G}(v)}+\sum_{uv\in E_2(G)} x^{d_{d_G}(u)+d_{d_G}(v)}$$
$$= nkx^{n+3} + nx^4.$$

Definition 2.13. [16] Consider L_3 is a tree, embedded in one of its final vertex. For k=2, 3, 4,... assemble embedded tree T_k by analyzing the roots of L_3-trees is the root of T_k.

Definition 2.14. [16] Let $2 \leq d \in Z$. $\alpha_1, \alpha_2, ..., \alpha_d \in \{T_1, T_2, ...\}$. A Kragujevac tree T is a graph having a vertex of degree d, neighboring to the roots $\alpha_1, \alpha_2, ..., \alpha_d$This vertex be the middle vertex of T, where d is the degree of T. The subgraphs $\alpha_1, \alpha_2, ..., \alpha_d$ are the branches of T.

Lemma 2.15. [17] Let T be the Kragujevac tree with V(T) = $1+\sum_{i+1}^{d}(2k_i+1)$ and E(T)=m. Then $d_d(v)$ = 2, for all v∈V (T).

Proposition 2.16. Let T be the Kragujevac tree of order $1+\sum_{i+1}^{d}(2k_i+1)$ and size m. Then

$$DM_1(T, x) = [\,1+\sum_{i+1}^{d}(2k_i+1)]x^4,$$
$$DM_2(T, x) = {DM_1}^*(T, x) = mx^4.$$

Proof. Since Kragujevac tree is 2−domination regular graph with $1+\sum_{i+1}^{d}(2k_i+1)$ vertices and m edges. So we get the result using Proposition 2.10.

Proposition 2.17. If $G \cong K_{r_1,r_2,...,r_k}$, then

$$DM_1(G, x) =\sum_{i=1}^{k} r_i x^{[1+(\sum_{i=1}^{k} r_i)-r_i]^2},$$
$$DM_2(G, x) = \sum_{1\leq l<k}^{k} n_l n_k x^{[1+(\sum_{i=1}^{k} r_i)-r_l][1+(\sum_{i=1}^{k} r_i)-r_k]},$$
$${DM_1}^*(G, x) = \sum_{1\leq l<k}^{k} n_l n_k x^{\left[1+\left(\sum_{i=1}^{k} r_i\right)-r_l\right]+[1+(\sum_{i=1}^{k} r_i)-r_k]}.$$

Proof. In $G \cong K_{r_1,r_2,...,r_k}$, note that for any vertex v∈G we have, d_d (v)=d(v)+1 and |E(G)|= T_m (G)−k. So, using the definition of domination Zagreb polynomials we obtain the results.

Lemma 2.18. Let G be a Healthy spider graph with 2n+1 vertices and 2n edges, then d_d (v)=1, for all v $\in$ V (G) and T_m (G) = 2.

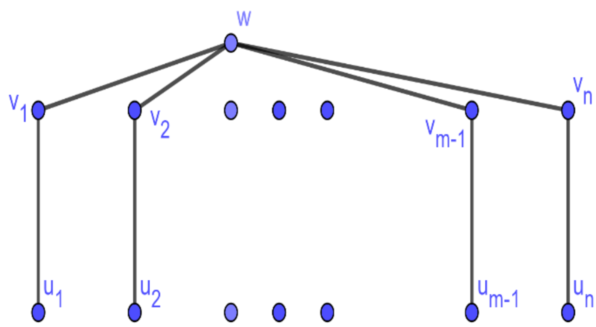

Fig. 1 Healthy spider of 2n+1 vertices

Theorem 2.19. Let G be a Healthy spider graph. Then

$$DM_1(G, x) = (2n+1)\, x,$$
$$DM_2(G, x) = 2n\, x,$$
$$DM_1^*(G,x) = 2n\, x^2.$$

Proof. Since G is 1-domination regular graph. The proof followed by Proposition 2.10.

Lemma 2.20. [17] Let G be a graph with r vertices and s edges. K_{n_2} be a complete graph. If $H \cong GoK_{n_2}$, then $d_d(v)=$ $(n_2+1)^{r-1}$.

Proposition 2.21. If G be a connected graph with r vertices and s edges. Let $H \cong GoK_{n_2}$. Then

$$DM_1(H,x) = (r + rn_2)x^{(n_2+1)^{2(r-1)}},$$
$$DM_2(H,x) = \frac{1}{2}(2s + rn_2{}^2 + rn_2)x^{(n_2+1)^{2(r-1)}},$$
$$DM_1^*(H,x) = \frac{1}{2}(2s + rn_2{}^2 + rn_2)x^{2[(n_2+1)^{(r-1)}]}.$$

Proof. Since H has order $r(1+n_2)$ so, by Proposition 2.10, we get

$$DM_1(H, x) = \sum_{u\in V(H)} x^{d_{d_G}{}^2(u)},$$
$$= (r + rn_1)x^{(n_2+1)^{2(r-1)}}.$$

Note that, in K_{n_2} there are $\frac{n_2(n_2-1)}{2}$ edges so, there are $\frac{1}{2}(2s + rn_2{}^2 + rn_2)$ edges in the graph H. Hence

$$DM_2(H, x) = \sum_{uv\in E(H)} x^{d_{d_G}(u) d_{d_G}(v)}$$

$$= \frac{1}{2}(2s + rn_2{}^2 + rn_2)x^{(n_2+1)^{2(r-1)}}.$$

$$DM_1^*(H,x) = \sum_{uv\in E(G)} x^{d_{d_G}(u)+d_{d_G}(v)}$$

$$= \frac{1}{2}(2s + rn_2{}^2 + rn_2)x^{2[(n_2+1)^{(r-1)}]}.$$

Lemma 2.22. [17] Let $H \cong Go\,\overline{K}_{n_2}$. Then $d_d(v)=T_m(H)-2^{r-1}$, for all $v \in H$.

Proposition 2.23. If $H \cong Go\,\overline{K}_{n_2}$, then

$$DM_1(H,x) = (r + rn_2)x^{(n_2+1)^{2(r-1)}},$$
$$DM_2(H,x) = \frac{1}{2}(2s + rn_2{}^2 + rn_2)x^{(n_2+1)^{2(r-1)}},$$
$$DM_1^*(H,x) = \frac{1}{2}(2s + rn_2{}^2 + rn_2)x^{2[(n_2+1)^{(r-1)}]}.$$

Proof. By Definition 1.3, Proposition 2.10 and Lemma 2.22, we get the results.

Proposition 2.24. If $G_1, G_2, G_3 \ldots G_n$ are any graphs, and $G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n$. Then

$$DM_1(G,x) = \sum_{i=1}^{n} DM_1(G_i,x),$$
$$DM_2(G,x) = \sum_{i=1}^{n} DM_2(G_i,x),$$
$$DM_1^*(G,x) = \sum_{i=1}^{n} DM_1^*(G_i,x).$$

Proof. The proof follows from the definition of $G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n$ and Definition 1.3.

Definition 2.25. [18] Let $G_1 = (r_1, s_1)$, $G_2 = (r_2, s_2)$be two graphs. Then the join $G_1 + G_2$ is a graph with $V(G) = r_1 + r_2$ and $E(G) = |E(G_1)| + |E(G_2)| \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Lemma 2.26. [17] Let G_1 and G_2 be any non complete graphs with r_1, r_2 vertices respectively.Then T_m $(G_1 + G_2)$ = T_m (G_1) + $T_m(G_2)$ + r_1r_2, and

$$\mathbf{d_{d_{G_1+G_2}}(v)}=\begin{cases} d_d(v) + r_2, & \text{if } v \in V(G_1); \\ d_d(v) + r_1, & \text{if } v \in V(G_2). \end{cases}$$

Therem 2.27. If $G_1 = (r_1, s_1)$, $G_2 = (r_2, s_2)$ be any two non complete graphs and $G \cong G_1 + G_2$, then

$$DM_1(G,x) = x^{(r_2)^2} DM_1(G_1,x) DM_0(G_1,x^{2r_2}) + x^{(r_1)^2} DM_1(G_2,x) DM_0(G_2,x^{2r_1}),$$
$$DM_2G, x) = DM_{r_2,r_2}(G_1,x) + DM_{r_1,r_2}(G_2,x) + x^{r_1+r_2} DM_0(G_1,x)\, DM_0(G_2,x),$$
$$DM_1^*(G,x) = x^{2r_2} DM_1^*(G_1,x) + x^{2r_1} DM_1^*(G_2,x) + x^{r_1+r_2} DM_0(G_1,x)\, DM_0(G_2,x).$$

Proof. Since the graph $G \cong G_1 + G_2$ has an order $|V(G)| = |r_1| + |r_2|$ and size $E(G) = |E(G_1)| + |E(G_2)| \cup \{uv: u \in V(G_1), v \in V(G_2)\}$.

$$DM_1(G, x) = \sum_{v\in V(G_1+G_2)} x^{{d_d}^2(v)}$$
$$= \sum_{v\in V(G_1)} x^{{d_d}^2(v)} + \sum_{v\in V(G_2)} x^{{d_d}^2(v)}$$
$$= \sum_{v\in V(G_1)} x^{(d_d(v)+r_2)^2} + \sum_{v\in V(G_2)} x^{(d_d(v)+r_1)^2}$$
$$= \sum_{v\in V(G_1)} x^{{d_d}^2(v)} x^{(r_2)^2} x^{2r_2 d_d(v)} + \sum_{v\in V(G_2)} x^{{d_d}^2(v)} x^{(r_1)^2} x^{2r_1 d_d(v)}$$
$$= x^{(r_2)^2} DM_1(G_1,x)\, DM_0(G_1,x^{2r_2}) + x^{(r_1)^2} DM_1(G_2,x)\, DM_0(G_2,x^{2r_1}).$$

Next, by Definition 1.3, and Proposition 1.4, we get

$$DM_2(G, x) = \sum_{uv\in E(G)} x^{d_d(u)d_d(v)}$$
$$= \sum_{uv\in E(G_1)} x^{d_d(u)d_d(v)} + \sum_{uv\in E(G_2)} x^{d_d(u)d_d(v)} + \sum_{u\in E(G_1), v\in E(G_2)} x^{d_d(u)d_d(v)}$$
$$= \sum_{uv\in E(G_1)} x^{(d_d(u)+r_2)(d_d(u)+r_1)} + \sum_{uv\in E(G_2)} x^{(d_d(u)+r_2)(d_d(u)+r_1)} + \sum_{u\in E(G_1), v\in E(G_2)} x^{(d_d(u)+r_2)(d_d(u)+r_1)}$$
$$= DM_{r_2,r_2}(G_1,x) + DM_{r_1,r_2}(G_2,x) + x^{r_1+r_2} DM_0(G_1,x)\, DM_0(G_2,x).$$

Similarly,

$$DM_1^*(G,x) = \sum_{uv\in E(G)} x^{d_d(u)+d_d(v)}$$
$$= \sum_{uv\in E(G_1)} x^{d_d(u)+d_d(v)} + \sum_{uv\in E(G_2)} x^{d_d(u)+d_d(v)} + \sum_{u\in E(G_1), v\in E(G_2)} x^{d_d(u)+d_d(v)}$$
$$= x^{2r_2} DM_1^*(G_1,x) + x^{2r_1} DM_1^*(G_2,x) + x^{r_1+r_2} DM_0(G_1,x)\, DM_0(G_2,x).$$

Definition 2.28. A polynomial Q(x) is called graphical domination Zagreb polynomial if there exists at least one simple connected graph G such that,

1. For first domination Zagreb polynomial
   $DM_1(G, x) = Q(x)$,
2. For second domination Zagreb polynomial:
   $DM_2(G, x) = Q(x)$,
3. For modified first domination Zagreb polynomial:
   $DM_1^*(G, x) = Q(x)$.

Definition 2.29. Two graphs G_1 and G_2 are called equal domination Zagreb polynomial if andonly if

1. For first domination Zagreb polynomial:
   $DM_1(G_1, x) = DM_1(G_2, x)$,

2. For second domination Zagreb polynomial:
   $DM_2(G_1, x) = DM_2(G_2, x)$,

3. For modified first domination Zagreb polynomial:
   $DM_1^*(G_1, x) = DM_1^*(G_2, x)$.

The following theorem gives us some basic properties of domination Zagreb polynomials:

Theorem 2.30. 1. If the graph G is a k−domination regular graph then the Zagreb polynomials are monomial.
2. The domination Zagreb polynomials does not have a constant term.
3. The domination Zagreb polynomials are increasing in $[0, \infty)$.
4. The domination Zagreb polynomials have at least one real root.
5. Let G be a graph, then the coefficient of first domination Zagreb polynomial is the order of the graph.
6. Let G be a graph, then the sum of coefficient of second and also modified first domination Zagreb polynomials is the sizeof the graph.

Proposition 2.31. If $G_1 \cong G_2$, then the polynomials of G_1 and G_2 are the same. But the converse need not be true.

Example 2.32. Let $G_1 \cong K_3$ and $G_2 \cong S_{3+1}$, note that
$DM_2(G_1, x) = DM_2(G_2, x) = 3x$,
$DM_1^*(G, x) = DM_1^*(G, x) = 3x^2$.
But, $G_1 \ncong G_2$.

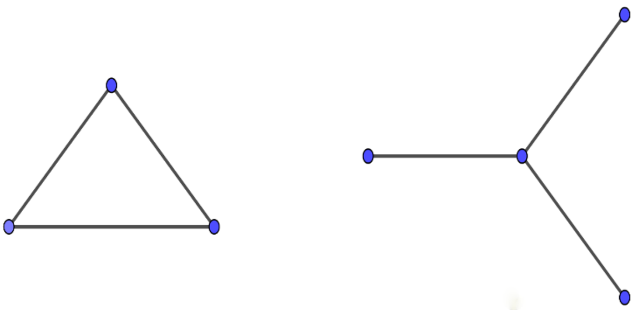

Fig. 2

III. CONCLUSION

In this paper, domination Zagreb polynomials of graphs are studied. We have calculated the domination Zagreb polynomials for some important families of graph. The exact formula of domination Zagreb polynomials of some graph operations like corona product and the join of graphs was calculated. Some properties of the domination Zagreb polynomials of graphs are also established.

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