International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 / https://doi.org/10.14445/22315373/IJMTT-V68I2P515

Volume 68 Issue 2, 95-103, February 2022

Original Article

Existence and Uniqueness Solutions of Differential Equations in Banach Spaces by Using Matlab

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Abstract **-** *Banach space is a finished standard space, which implies that each Cauchy succession of the components of this space closes inside the actual space, and this is the thing that makes it a shut space It also has many applications in functional analysis. The aim of this paper is design and develop highly efficient algorithms that provide the existence of unique solutions to the differential equation in Banach spaces using MATLAB. The quality algorithm was used and developed to solve the differential equation in Banach spaces. For accurate results. The proposed model contributed to providing an integrated computer solution for all stages of the solution starting from the stage of solving differential equations in Banach space and the stage of displaying and representing the results graphically in the MATLAB program. This was done by solving three different types in type and rank to reach accurate readings that provide an ideal model applicable in many of the techniques that contribute to solving our daily problems. This study was distorted in the accuracy of the results with their comparisons from their platforms in the stages and type of solution, as the model achieved high accuracy in the presence of unique solutions to the differential equation in Banach spaces in a computerized form.*

Keywords — *Banach Spaces, Existence and Uniqueness, Differential Equations and MATLAB.*

I. INTRODUCTION

Computer software is one of the most widespread developments in mathematical solutions, which enables us to provide a more accurate and graphically representative solution very quickly. Traditional solutions lead to inaccurate results, accompanied by many errors, and may be difficult to display and represent graphically. In recent years the studies of functional analysis is very important tool in the investigation of all kind of problems in pure mathematics, physics, biology, economics, etc.[1]. Many differential equations may be formulated in terms of a suitable linear operator acting on a Banach space[2]. In mathematics, more specifically in functional analysis, a Banach space (pronounced ˈbanax) is a complete normed vector space. Thus, a Banach space is a vector space with a metric that allows the computation of vector length and distance between vectors and is complete in the sense that a Cauchy sequence of vectors always converges to defined limit that is within the space[3]. Banach spaces are named after the Polish mathematician Stefan Banach, who introduced this concept and studied it systematically in 1920–1922 along with Hans Hahn and Eduard Helly. Banach spaces originally grew out of the study of function spaces by Hilbert, Fréchet, and Riesz earlier in the century. Banach spaces play a central role in functional analysis. In other areas of analysis, the spaces under study are often Banach spaces[4-6]. This study solved differential equation in a Banach space by converting to a computing system for existence and uniqueness solution differential equation in a Banach space as electronic solution for some differential equation in a Banach space as sample.

II. EXISTENCE AND UNIQUENESS OF SOLUTIONS

A differential equation of any type, in conjunction with any other information such as an initial condition, is said to describe a well-posed problem if it satisfies three conditions, known as Hadamard's conditions for well-pawedness:

- A solution of the problem exists.
- A solution of the problem is unique.
- The unique solution depends continuously on the problem data, which may include initial values or coefficients of the differential equation. That is, a small change in the data corresponds to a small change in the solution[7].

A. Delay Differential Equation on A Closed Subset of a Banach Spaces.

In an earlier work [8], sufficient conditions for the existence of solutions in a closed subset F of a Banach space E for the Cauchy problem

$$X'(t)f(t,x_t), Xt_0 = \phi_0 \qquad (1)$$

Where $f: R^+ \times C \to E, C[-\tau,0], \emptyset_0 C_f = \{\emptyset_0 C \emptyset(0) F\}$ are obtained by requiring f to satisfy (i) a compactness-type condition in terms of the Kuratowski measure of non-compactness and (ii) a boundary condition, namely,

$$\lim_{h\to 0} inf \frac{1}{h}\, d(\emptyset(0) + hf(t,\emptyset), f) = 0 \qquad (2)$$

Lemma(A.1)[9].Suppose that the hypotheses (A1) and (A2) are satisfied.

If $\{\varepsilon_n\} \subset (0,1)$ is a non-increasing sequence with $\lim_{n\to\infty} \varepsilon_n = 0$, then n new n there exists a sequence $\{x_n(t)\}$ of en-approximate solutions for the Cauchy problem (1), that is, for every n , there exists a function $x_n(t) : [t_0 - \tau, t_0 + \gamma] \to E$ with the following properties:

(i) there exists a sequence $\{t_i^n\}_{i=1}^{\infty}\ in\ [t_0 - \tau, t_0 + \gamma]$ such that $t_n = t_0$,

$t_{i+1}^n = t_i^n + \delta_i^n, \delta_i^n > 0\ if\ t_i^n \le t_0 + \gamma, \lim_{i\to\infty} t_i^n t_0 + \gamma$

(ii) $x_n(t) = \emptyset_0(t - t_0)\ \ fort\ \epsilon\ [t_0 - \tau, t_0, ||\, x_n(t) - x_n(s)\, || \le M|\, t - s\, |\ for\ \ t, s\epsilon[t_0, t_0 + \gamma]$;

(iii) for each $i \ge 0$, $(t_i^n, x_n, t_i^n)\ \epsilon\ C_0(b)$ and $x_n(t)$ is linear on each of the intervals $[t_i^n, t_{i+1}^n]$;

(iv) if $t\ \epsilon\ (t_i^n, t_{i+1}^n)\ and\ t_n < t + \gamma$, then $||\, x_n'(t) - f(t_i^n, x_n, t_n)|| < \varepsilon_n$;

(v) δ_i^n can be chosen less than min $\left\{\varepsilon_n, \delta_0^{\emptyset}\left(\frac{x_n}{2}\right), \frac{n_n}{2M}\right\}$ where $\delta_0^{\emptyset}(n_n)$ is the number associated with n_n by the uniform continuity of $\emptyset_0\ on\ [-\tau, 0]$and n_n is such that

$$|t - t_i^n| \le n_n, ||\, \phi - xx_n, t_i^n||0 \le n_n \text{ imply that } ||\, f(t,\phi) - f\,(t_i^n, x_n, t_i^n)\,|| \le \varepsilon_n;$$

Remark (A.2)[10]. Lemma (A.1) remains valid when in (A1) and (A2), the set C_F is replaced by $C_F^{\wedge}$.

Lemma (A.3) [7]. Let the assumptions of Lemma (A.1). hold. If the sequence $\{x_n(t)\}$ of an-approximate solutions of the Cauchy problem (1) which exist by virtue of Lemma (A.1).is such that $\{x_n(t)\}$ converges to x(t) uniformly on $[t_0 - \tau, t_0 + \gamma]$, then x(t) is a solution of (1) such that $x(t)\ \epsilon\ F\ for\ t\ \epsilon[t_0, t_0 + \gamma]$.

We give some known convergence and comparison lemmas[11]. Let us consider the following assumptions:

Suppose that

(S1) $r(t) = r(t, t_0, u_0)$ is the maximal solution of the scalar differential equation

$$u' = g(t,u), u(t_0) = u_0 \geq 0, s_0^t \epsilon R^+ \qquad (3)$$

on $[t_0, \infty)$, where $g \epsilon C[R^+ \times R^+, R^+]$;

(S2) r(t)= $r(t, t_0, u_t)$ is the maximal solution of the scalar functional differential equation

$$u' = g(t, u, u_t), u_0^t = \emptyset_0, t \epsilon R^+ \qquad (4)$$

on $[t_0, \infty)$, where $g \epsilon C[R^+ \times R^+ \times C^+, R^+]$, $C^+ = C[[-t, 0], R^+]$and $g(t, u, \phi)$is non decreasing in ϕ for each (t, u), that is, $\phi_1(\theta) < \phi_2$ for every, $\emptyset_0$ $[-t, 0]$ implies $g(t, u, \phi_1) \leq g(t, u, \phi_2)$for each (t, u).

Lemma (A.4) [7]. Let (S1) ((S2) respectively) hold.‘ Then, for any interval $[t_0, t_1] \subset [t_0, \infty)$, there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the maximal solution $r(t, \varepsilon)$= $r(t, t_0, u_0, \varepsilon)$of

$$u' = g(t, u) + \varepsilon, u(t_0) = u_n + \varepsilon \qquad (5)$$

(the maximal solution $r(t, \varepsilon) = r(t, t_0, u_0, \varepsilon)$ of

$$u' = g(t, u, u_t) + \varepsilon, \ u_0^t = \phi_0 + \varepsilon \qquad (6)$$

respectively) exists on $[t_0, t_1]$ *and* $\lim_{\varepsilon \to 0} \gamma(t, \varepsilon)$ uniformly on $[t_0, t_1]$.

Lemma (A.5) [7]. Assume that (S1) holds. Let $m \ \grave{o} \ C[t_0 - \tau, \infty), R^+]$ and for every $t^{\wedge} \in [t_0, \infty)$ where S is a countable subset of $[t_0, \infty)$, the differential inequality

$$D_m(t^{\wedge}) \leq g(t_m^{\wedge}(t^{\wedge}))$$

(with Dbeing any one of the Dini derivativesD, D^-, D^+) be satisfied provided E is such that $m_t^{\wedge}(\theta) \leq m(t^{\wedge}), -\tau \leq \theta \leq 0$. Then *m(t)* $\leq r(t, t_0, u_0)$.

t ≥ t0 whenever $m_{t0}(\theta) \leq u_0, -\tau \leq \theta \leq 0$.

M. The proof of this lemma is quite known [ ] for $D = D^-$(and therefore for $D = D^+$). we sketch the proof for the case $D = D^+$.

In view of Lemma (A.4)., it is enough to show that for every $\varepsilon > 0$, sufficiently small. $m(t) \leq u(t, \varepsilon)$, $t \epsilon [t_0, \infty) \backslash S$, where $u(t, \varepsilon) \equiv u(t, t_0, u_0, \varepsilon)$ is any solution of (5). Suppose that$Z = \{t \epsilon [t_0, \infty) \backslash S: m(t) > u(t, \varepsilon)\}$ is nonempty and let $t_1 = \inf Z$.

Since $m(t_0) \leq u_0 \leq u_0 + \varepsilon = u(t_0, \varepsilon)$.

we have $t_1 > t_0$, $m(t_1) = u(t_1, \varepsilon)$ and $m(t) \leq u(t, \varepsilon)$, $t \epsilon [t_0, t_1]$.

As $u(t, \varepsilon)$ is an increasing function and mt0 m_0^t $D(\varepsilon) < u_0$, $-\tau \leq \theta \leq 0$,

$$m(t) \leq u(t_1, \varepsilon) = m(t_1)$$

for every $t \epsilon [t_0 - \tau, t_1]$. Hence by hypothesis on m,

$$D^+ m(t_1) - g(t_1, m(t_1)) \leq 0 \qquad (7)$$

Let $v(t) = m(t) - u(t, \varepsilon)$. Since Zis nonempty, there exists sequence [12], $h_n > 0$, $h_n \to 0$ as $n \to m$, such that

$$v(t_1 + h_n) = m(t_1 + h_n) - u(t_1 + h_n, \varepsilon)$$

Since $D + v(t_1) \geq \lim_{n \to \infty} sup \frac{v(t_1 + h_n) - (t1)}{h_n}$

$$= \lim_{n \to \infty} sup \frac{v(t_1 + h_n) - (t1)}{h_n}$$

We have $D^+ m(t_1) - g\big(t_1, m(t_1)\big) - \varepsilon = D^+ v(t_1) > 0$ which contradicts (7). The lemma is therefore proved.
The following known comparison result involving a functional differential inequality [13] is also used in the sequel.

Solution :

```
% Existence and Uniqueness of Solutions of Delay Differential  Equation on A Closed Subset of a Banach  Spaces
clear all
clc
syms t0 tl e e0 u un pi0 ut t u gr u0 g(t,u) d(t,u,ut)
% the maximal solution r(t,e) = r(t,e) = r(t,t0,u0,e)
f = g(t,u) + e
u(t0)= un + e
t0=0
t1=0.5
e=0
% the maximal solution r(t,e) = r(t,e) = r(t,t0,u0,e)
h=d(t,u,ut) + e
s = pi0 + e
r = sin(t)/t;
limit(r,t,0)
sol = dde23(@ddex1de,[1, 0.2],@ddex1hist,[0, 0.5]);
tint = linspace(t0,t1);
yint = deval(sol,tint);

plot(tint,yint);
```

Represent the Solution Graphically:

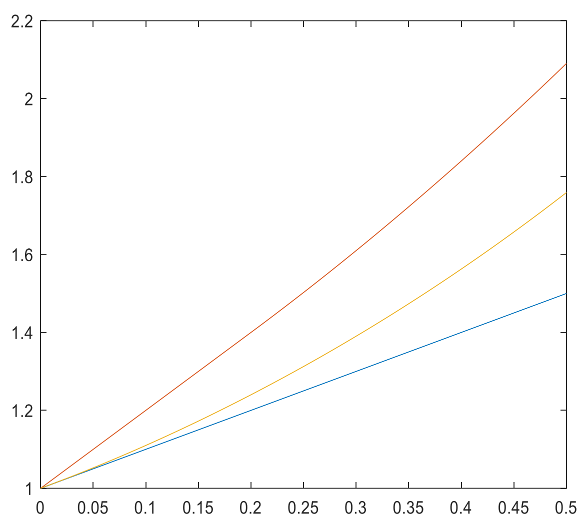

Fig. 1 Existence and Uniqueness of Solutions of Delay Differential Equation on A Closed Subset of a Banach Spaces

B. Ordinary Differential Equation in Banach Space

Let E be an arbitrary Banach space and A a K-positive definite operator defined in a densedomain $D(A) \subseteq E$. Let B be a linear unbounded operator such that $D(B) \supseteq D(A)$. We prove that the equation

$$L_u = f, \qquad (8)$$

where $L = A + B$, has a unique solution and construct an iterative scheme that converges to the unique solution of this equation. Let

$$L_u = (A + B)u = f. \qquad (9)$$

Multiplying both sides of (5) *by* $A - 1$, we have

$$u + Tu = g, \qquad (10)$$

where $T = A^{-1}B, g = A^{-1}f$. Since A is continuously invertible, the operator $T = A^{-1}B$ is completely continuous. Hence T is locally lipschitzian and accretive. It follows that (10) has a unique solution (see [14]).

If $A = B$, then $L = A + B = 2A$. In this case $\langle L_u, K_u \rangle = 2\langle A_u, K_u \rangle \geq 2\alpha \left\| K_u \right\|^2 = \beta \left\| K_u \right\|^2$.Thus L is K-positive definite and so the equation L_u= f has a unique solution (see [10]). Examples of such A are all positive operators when $K = l_2$ and are all invertible operators when $K = A$. If $A \neq B$, then let $E = l2$, for instance, and define $A : l_2 \to l_2$ by $Ax = (ax_1, ax_2, ax_3, \dots)$ for $x = (x_1, x_2, x_3, \dots) \in l_2$ *and* $a > 0$. Let $K = I$, the identity operator, then $\langle Ax, x \rangle = a \sum_{i=1}^{\infty} x_i^2 = a \|x\|^2 > (\frac{1}{2})\|x\|^2$. Thus A is K-positive definite. Let B be any

linear operator; in particular, let $B : l_2 \to l_2$ be defined by $Bx = (0, x_1, x_2, x_3, \dots)$. Then by (9) and (10), the equation $L_u = f$, where $L = A + B$, has a unique solution. Next we derive the solution to (9) from the inverse function theorem and construct an iterative scheme which converges to the unique solution of this equation.

Theorem (B.1) [7] the inverse function theorem_. Suppose that E, Y are Banach spaces and $L : E \to Y$ is such that L has uniformly continuous Fréchet derivatives in a neighborhood of some point u_0 of E. Then if $L'(u_0)$ is a linear homeomorphism of E onto Y, then L is a local homeomorphism of neighborhood $U(u_0)$ of u_0 to a neighborhood $L(u_0)$.

Proof. For a sketch of proof of this theorem, see .

By mimicking the proof of Theorem(B.1) of[15], we get that, if $\| g - Lu_0) \|$ is sufficiently small, $Lu_0 g$ has a unique solution $u = u_0 + \rho^*$, where $\rho^*$is the limit of the sequence ρ0= 0, $\rho_{n+1} = Q\rho_n$, where Q is a contraction mapping of a sphere $S(0, \varepsilon)$ in E into itself, for some ε sufficiently small. It follows that the sequence $u_n = u_0 + \rho_n$converges to $u_0 + \rho^*$, the unique solution of $L_u = g$ in $U(u_0)$. Now

$$u_n = u_0 + \rho_n = u_0 + Q\rho_{n-1}$$

$$= u_0[L'(u_0)] - 1\,[g - L(u_0) - R(u_0, \rho_{n-1})]$$

from Taylors theorem

$$= u_0 + [L'(u_0)] - 1\, g\_L'(u_0)\rho_n - 1 - L(u_0 + , \rho_n - 1)] \qquad (11)$$

$$= u_0 + \rho_n - 1 + [L'(u_0)] - 1\,[g - L(u_{n-1})]$$

$$= u_{n-1} + [L'(u_0)] - 1\,[g - Lu_{n-1}].$$

Hence

$$u_{n-1} = u_n + [L'(u_0)] - 1\,[g - Lu_n]. \tag{12}$$

Special Cases

If $B = I$, then (12) becomes

$$u_{n-1} = u_n + [A'(u_0)] - 1\,[g - Au_n + u_n]. \tag{13}$$

If $B = 0$, then we have Corollary of [12].

For the case $B = 0$, we prove the following theorem for an asymptotically K-positive definite operator.

Recall see [12] the definition of an asymptotically $K-$pd operator. For simplicity and ease of reference, we repeat the definition.

Solution :

```
clear all
clc
syms x y A B L Lu u Tu g L K I Au Ku X Y n f Ax a
Lu = f
Lu = (A + B)*u
u + Tu == g
B = A
L = A + B
n=5
a=20
xi= 1:n
Ax =a*xi
s=a*(sum(xi))^2
x=4
t=a*(abs(x))^2
z=0.5*(abs(x))^2
plot(Ax,xi);
```

Represent the solution graphically

.

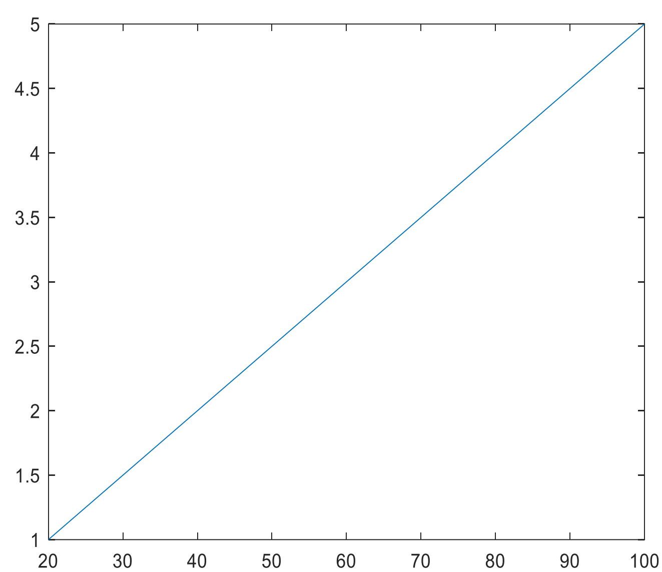

Fig. 2 Uniqueness Theorem for Ordinary Differential Equation in Banach Space

C. Peano's Theorem in Locally Convex Spaces

It is worth noticing, in this connection, that can also be applied to discontinue functions. This is shown by the following easy example: $F(x,y) = \frac{1}{y_0 - y}$ for $y \neq y_0$ and $f(x,y_0) = 0$.

Let f = fα (0 <α< 2) given by $$f\alpha(x,y) = \begin{cases} |y|^{\alpha}\sin\left(\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0 \end{cases} \tag{14}$$

The initial value problem

$$\begin{cases} y = f_{\alpha}(x,y) \\ y(0) = 0 \end{cases} \tag{15}$$

is satisfied only by the zero solution. This can be proved by a simple application of classical results on scalar differential inequalities [7], since

$$y = \frac{1}{k\pi}, \qquad k = \pm l, \pm 2, \pm 3, \ldots$$

are under functions with respect to (15) if $k > 0$ and over functions if $k < 0$.

$$u = t|u|^{\alpha} sin(l/u) \; for \; u \neq 0 \tag{16}$$

and $u = 0$ otherwise.

If $\alpha = 1/2$, one can verify that

$$u = \left(\frac{\pi}{2} + 2n\pi\right) - 1, t = \left(\frac{\pi}{2} + 2n\pi\right) - \frac{1}{2}, \qquad n \in N \tag{17}$$

are solutions of (16). If $\alpha = 1$, becomes

$$sgn(u) = tsin(w),$$

which cannot be satisfied if $t < 1$. when f is neither Lipschitzian nor decreasing in y.

Solution :

```
clear all
clc
syms x y y0 u Tu g Lv ds Au Tu X Y n f t a F fa k
n=5
a=1
u=1
F(x,y) = 1/(y0-y)
f(x, y0) = 0

if (y ~= 0)
   fa(x,y)= (abs(y)^a)*(sin(1/y))
else
      (y == 0)
      fa(x,y)= 0
end
for k=a:n

   y(k) = 1/(k*pi)
```

```
end
   k=[1:5]
   plot(y,k)
   u = (t*(abs(u))^a) * (sin(1/u))

%     If ? = 1/2 , one can verify that

u= ((pi/2) + (2*n*pi))^-1
t= ((pi/2) + 2*n*pi)^(-1/2)
a=1
u = (t*(abs(u))^a) * (sin(1/u))
```

Represent the solution graphically

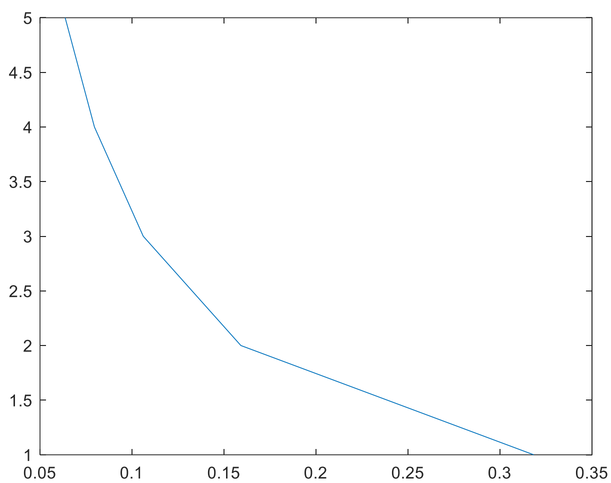

Fig. 3 A uniqueness Theorem and Peano's Theorem in Locally Convex Spaces

III. RESULTS AND DISCUSSION

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A. Results

After implementing all the steps described in the previous sections, the model is evaluated, and the final results are discussed. Results obtained from the proposed model automatically during implementation, and testing process to get accurate results by used MATLAB by applying This is to repeat the experiment three times using the values mentioned in the previous section for different example of differential equation in Banach spaces such as existence and uniqueness of solutions of delay differential equation on a closed subset of a Banach spaces, uniqueness Theorem for ordinary differential equation in Banach space and a uniqueness theorem and Peano's theorem in locally convex spaces , as each investigation resulted in a curve of graphics. The mean was calculated for accuracy to obtain excellent accuracy due to repetition, training, and verification in each example.

B. Discussion

The performance of the computer solution model for existence and uniqueness solutions of differential equation in Banach spaces was evaluated in this study through the graphics presented in the previous sections to verify its validity. This study used a new method to solve of existence and uniqueness solutions of differential equation in Banach spaces by make powerful algorithm for results and represent automatically in graphics unlike other studies that focused on a specific application, such as the linear system of mixed Volterra-Fredholm. Fixed point solutions for variation inequalities in image restoration overq-uniformly smooth etc. This study achieved great success in solving many types of differential equations in Banach spaces. But it failed to direct these solutions into the application in the problems of our daily lives, as it was satisfied only with the solution and represented it graphically. This problem was solved by increasing the solution of more than one computer example so that each applicator finds the solution that fits the problem in our daily life (such as the problems of industrial, agricultural, animal and economic production) and thus the model achieved an accuracy of more than 97.5% and the presence of unique solutions to the differential equation in Banach spaces.

IV. CONCLUSION

This paper presented an existence and uniqueness solutions of differential equation in Banach spaces to solve and presented several differential equation examples by using Matlab programming language. examples are based on the collection of three kinds of differential equations from different sources with applied in all stages: solution stage results stage, and presented graphically.The number of examples and lemma for each category of differential equation in the proposed model was applied power full algorithms to reach high accuracy of results.When entering a solve existence and uniqueness solutions of differential equation in Banach spaces, it is tested by a comparing with the previous study to obtain it automatically the accuracy of its test is compared to the study’s results, the model results showed excellent efficiency with 97.5% accuracy in existence and uniqueness solutions of differential equation in Banach spaces.

ACKNOWLEDGMENT

First of all, I would like to family of Omdurman Islamic University for providing the mathematical information, I would also like to thank my supervisor, Dr. Abdel Radi Abdel Rahman Abdel Gadir for his immense support and guidance. And I would like to thank my father, mother, and my little family for their moral support.

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