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Contraction Theorem in E-Fuzzy Metric Space

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Abstract -*In this paper we discussed some of the properties of generalized E-fuzzy metric space and proved contraction theorem in E-fuzzy metric space.*
Key Words - *Fuzzy Metric Space, G-Metric Space, E-fuzzy metric space*

1 Introduction

In Mathematics,the concept of Fuzzy set was introduced by L A Zadeh[2].It is a new way to repreesent vagueness in our daily life.In 1975 Kramosil and Michalek[7] introduced the concept of fuzzy metric spaces which opened a new way for further development of analysis in such spaces.George and Veeramani[1] modified the concept of fuzzy metric space.After that several fixed point theorems have been proved in fuzzy metric spaces. In 2006, Mustafa.Z and B.Sims[4] presented a definition of G-metric space.After that several fixed point results have been proved in G-metric spaces.

We have defined generalized E-fuzzy metric space[18] and proved common fixed point theorem for mappings in generalized E-fuzzy metric space [19]. In this paper we discuss some of the properties of E-fuzzy metric space and prove contraction theorem in E-fuzzy metric space.

2 Preliminary Notes

Definition 1. [2] A fuzzy set A in X is a function with domain X and values in [0,1]

Definition 2. [5] A binary operation * :[0,1]× [0,1]→[0,1] is a continuous t-norm if * satifies the following conditions

- *is commutative and associative

- * is continuous
- $a * 1 = a$ for all $a \in$ [0,1]
- $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$

Definition 3. [1] A 3-tuple(X,M,*) is said to be a fuzzy metric space if X is an arbitrary set,* is a continuous t-norm and M is a fuzzy set on $X^2\times$[0,∞) satisfying the following conditions ,for all x,y,z $\in$ X and $s, t > 0$

- $M(x, y, t) > 0$
- $M(x, y, t) = 1$, if and only if $x = y$
- $M(x, y, t) = M(y, x, t)$
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- $M(x, y, .) : (0, \infty) \rightarrow (0, 1]$ is continuous

M(x,y,t) denotes the degree of nearness between x and y with respect to t.

Definition 4. [4] Let X be a nonempty set and let G:X $\times$X $\times$ X $\rightarrow$ [0,∞) be a function satisfying the following

- $G(x, y, z) = 0$ if $x = y = z$
- $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$
- $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$
- $G(x, y, z) = G(p\{x, y, z\})$ (symmetry) where p is a permutation function
- $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (Rectangle inequality)

Then the function is called a generalized metric , or, more specifically a G-metric on X and the pair (X,G) is a G-metric space.

Definition 5. [18] A 3-tuple $(X, E, *)$ is called an E- fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and E is a fuzzy set on $X^3\times(0, \infty)$ satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

1. $E(x, y, z, t) > 0$ and $E(x, x, y, t) \geq E(x, y, z, t)$ for all $x, y, z \in X$ with $z \neq y$
2. $E(x, y, z, t) = 1$, for all $t > 0$ if and only if $x = y = z$
3. $E(x, y, z, t) = E(p(x, y, z), t)$ (symmetry), where p is a permutation function
4. $E(x, y, z, t + s) \geq E(x, a, z, t) * E(a, y, z, s)$
5. $E(x, y, z, .) : (0, \infty) \rightarrow [0, 1]$ is continuous

This is a generalization of fuzzy metric space called E-fuzzy metric space.

Example 6. Let $X = R$ and G is a G- metric on X. The t- norm is $a * b = ab$ for all $a, b \in [0, 1]$. For each $t > 0$ and

$E(x, y, z, t) = [exp(\frac{G(x,y,z)}{t})]^{-1}$

Then $(X, E, *)$ is an E-fuzzy metric space.

Lemma 7. [18] If $(X, E, *)$ be a E-fuzzy metric space , then E(x,y,z,t) is non-decreasing with respect to t for all $x, y, z \in X$

Lemma 8. [18] Let $(X, E, *)$ be a generalized E-Fuzzy Metric space. If there exists, $k \in (0, 1)$ such that $E(x, y, z, kt) \geq E(x, y, z, t)$ for all $x, y, z \in X$ and $t > 0$ then $x = y = z$

3 Properties of E-Fuzzy Metric Space

Definition 9. $(X, E, *)$ be a E-fuzzy metric space. An open ball with centre x_0 and radius r is given by
$B_E(x_0, r, t) = \{x \in X; E(x_0, x, x, t) > 1 - r\}$

Definition 10. $(X, E, *)$ be a E-fuzzy metric space. A sequence (x_n) in X converges to a point x $\in$ X if and only if E(x_n,x, x, t) $\rightarrow 1$ as n $\rightarrow \infty$.

Lemma 11. If (x_n) is a sequence in $(X, E, *)$ converges to $x \in X$, then

1. E(x_n,x_n, x, t) $\rightarrow 1$ as n $\rightarrow \infty$
2. E(x_n,x_m, x, t) $\rightarrow 1$ as $m, n \rightarrow \infty$
3. $E(x_n, x_m, x, t) > 1 - \epsilon$, for $\epsilon > 0$ and $m, n \geq n_0$

Definition 12. A sequence (x_n) in $(X, E, *)$ is said to be a Cauchy sequence if for each $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that E(x_m,x_n,x_l, t) $> 1 - \epsilon$ for each $l, m, n \geq n_0$.

Definition 13. An E- fuzzy metric space in which every Cauchy sequence is convergent is said to be a complete E- fuzzy metric space.

Definition 14. $(X_1, E_1, *)$ and $(X_2, E_2, *)$ be two generalized E- fuzzy metric spaces. A function $f : X_1 \rightarrow X_2$ is said to be continuous at a point $a \in X_1$ if $\forall$ $\epsilon > 0$ there exists $\delta > 0$ such that $E_2(f(x), f(a), f(a), s) > 1 - \epsilon$ whenever $E_1(x, a, a, t) > 1 - \delta$, $t > 0$.

Definition 15. $(X_1, E_1, *)$ and $(X_2, E_2, *)$ be two generalized E- fuzzy metric spaces. A function $f : X_1 \to X_2$ is said to be uniformly continuous if $\forall\ \epsilon > 0$ there exists $\delta > 0$ such that $E_2(f(x), f(y), f(z), s) > 1 - \epsilon$ whenever $E_1(x, y, z, t) > 1 - \delta$.

Definition 16. Let $(X, E, *)$ be a E-fuzzy metric space. A mapping $f : X \to X$ is said to be a contraction on X if there exists some α with $0 < \alpha < 1$ such that

$$\frac{1}{E(fx,fy,fz,t)} - 1 \leq \alpha\left(\frac{1}{E(x,y,z,t)} - 1\right)$$

Lemma 17. A contraction map f on $(X, E, *)$ is both continuous and uniformly continuous.

Proof. Let f is a contraction on $(X, E, *)$.
Then for some α with $0 < \alpha < 1$ we have

$$\begin{aligned}
\frac{1}{E(fx,fy,fz,t)} - 1 &\leq \alpha\left(\frac{1}{E(x,y,z,t)} - 1\right) \\
\frac{1}{E(fx,fy,fz,t)} - 1 &\leq \alpha\left(\frac{1-E(x,y,z,t)}{E(x,y,z,t)}\right) \\
\frac{1}{E(fx,fy,fz,t)} - 1 &\leq \alpha\left(\frac{1-\delta}{\delta}\right) \\
\frac{1}{E(fx,fy,fz,t)} - 1 &\leq \alpha\epsilon_0 \\
\frac{1}{E(fx,fy,fz,t)} &\leq 1 + \alpha\epsilon_0 \\
E(fx, fy, fz, t) &\geq \frac{1}{1+\alpha\epsilon_0} \\
E(fx, fy, fz, t) &\geq 1 - \frac{\alpha\epsilon_0}{1+\alpha\epsilon_0} \\
E(fx, fy, fz, t) &\geq 1 - \epsilon
\end{aligned}$$

Hence f is uniformly continuous. □

4 Banach Contraction Theorem

Theorem 18. *Let $(X, E, *)$ be a complete generalized E-fuzzy metric space. Let f be a contraction on X.*
ie $\frac{1}{E(fx,fy,fz,t)} - 1 \leq \alpha\left(\frac{1}{E(x,y,z,t)} - 1\right)$ *for some α with $0 < \alpha < 1$*

Then f has a unique fixed point.

Proof. Let $x_0 \in X$, consider a sequence in X such as

$$\begin{array}{llll}
x_1 & = & fx_0 & \\
x_2 & = & fx_1 & = & f^2x_0 \\
x_3 & = & fx_2 & = & f^3x_0 \\
 & --- & --- & --- & \\
 & --- & --- & --- & \\
x_n & = & fx_{n-1} & = & f^nx_0
\end{array}$$

First we assert that (x_n) is cauchy.
By the definition of contraction, we have

$$\begin{aligned}
(\tfrac{1}{E(x_n,x_m,x_l,t)} - 1) &= (\tfrac{1}{E(fx_{n-1},fx_{m-1},fx_{l-1},t)} - 1) \\
&\leq \alpha(\tfrac{1}{E(x_{n-1},x_{m-1},x_{l-1},t)} - 1) \\
&\leq \alpha(\tfrac{1}{E(fx_{n-2},fx_{m-2},fx_{l-2},t)} - 1) \\
&\leq \alpha^2(\tfrac{1}{E(x_{n-2},x_{m-2},x_{l-2},t)} - 1) \\
& \quad ---- \\
&\leq \alpha^n(\tfrac{1}{E(x_0,x_{m-n},x_{l-n},t)} - 1) \\
&\rightarrow 0, \text{ as } n \rightarrow \infty \\
&\implies \tfrac{1}{E(x_n,x_m,x_l,t)} - 1 \rightarrow 0 \\
&\implies E(x_n,x_m,x_l,t) \rightarrow 1
\end{aligned}$$

Hence (x_n) is cauchy. Also X is complete
Hence there exists an x such that $(x_n) \rightarrow x$
Now we will prove that $f(x_n) \rightarrow f(x)$

$$\begin{aligned}
(\tfrac{1}{E(fx,fx_n,fx_m,t)} - 1) &\leq \alpha(\tfrac{1}{E(x,x_n,x_m,t)} - 1) \\
&\rightarrow 0, \text{ as } n \rightarrow \infty \\
&\implies \tfrac{1}{E(fx,fx_n,fx_m,t)} - 1 \rightarrow 0 \\
&\implies E(fx,fx_n,fx_m,t) \rightarrow 1 \\
&\implies f(x_n) \rightarrow f(x)
\end{aligned}$$

Now we establish that $f(x) = x$

$$
\begin{aligned}
(\frac{1}{E(x_n,fx_n,fx_m,t)}-1) &= (\frac{1}{E(fx_{n-1},fx_n,fx_m,t)}-1) \\
&\leq \alpha(\frac{1}{E(x_{n-1},x_n,x_m,t)}-1) \\
&\leq \alpha^2(\frac{1}{E(x_{n-2},x_{n-1},x_{m-1},t)}-1) \\
& \quad ----\,----- \\
&\leq \alpha^{n-1}(\frac{1}{E(x_0,x,x_{m-n},t)}-1)
\end{aligned}
$$

Taking limit $n \to \infty$, we have

$$
\begin{aligned}
&\Longrightarrow \quad \frac{1}{E(x,fx,fx,t)}-1 \to 0 \\
&\Longrightarrow \quad E(x,fx,fx,t)=1 \\
&\Longrightarrow \quad f(x)=x
\end{aligned}
$$

Hence x is a fixed point.
To prove uniqueness, let $x^{'}$ be another fixed point.
Then $fx^{'}=x^{'}$

$$
\begin{aligned}
\frac{1}{E(x,x^{'},x,t)}-1 &= \frac{1}{E(fx,fx^{'},fx,t)}-1 \\
&\leq \alpha(\frac{1}{E(x,x^{'},x,t)}-1) \\
ie \quad & \quad \frac{1}{E(x,x^{'},x,t)}-1=0 \\
ie \quad & \quad E(x,x^{'},x,t)=1 \\
ie \quad & \quad x=x^{'}
\end{aligned}
$$

□

Corollary 19. *Let f be a mapping of a complete E-fuzzy metric space X to itself. If f is a contraction on a closed ball $\overline{B_E(x_0,r,t)}$, then there exists a unique fixed point of f in $\overline{B_E(x_0,r,t)}$.*

Proof. Consider a sequence (x_n) such that

$$
\begin{aligned}
x_1 &= f(x_0) \\
x_2 &= f(x_1)=f^2(x_0) \\
--- &- -------- \\
--- &- -------- \\
x_n &= f(x_{n-1})=f^n(x_0)
\end{aligned}
$$

Put $n=0$ and $l=m$ in the inequality of above theorem, then

$$\begin{aligned}
\left(\frac{1}{E(x_0,x_m,x_m,t)}-1\right) &\leq \alpha\left(\frac{1}{E(x_0,x_m,x_m,t)}-1\right)\\
\frac{1}{E(x_0,x_m,x_m,t)}(1-\alpha) &\leq (1-\alpha)\\
E(x_0,x_m,x_m,t) &\geq 1 > 1-r, 0 < r < 1
\end{aligned}$$

Hence $x_m \in B_E(x_0,r,t)$
ie (x_n) is a sequence in $B_E(x_0,r,t)$
Now using above theorem we can prove that $x_n \to x$.
since $\overline{B_E(x_0,r,t)}$ is closed, $x \in B_E(x_0,r,t)$.
Hence this result follows from Banach Contraction Theorem.
Then by Banach Contraction Theorem f has a unique fixed point in $\overline{B_E(x_0,r,t)}$.
□

5 Conclusion

Fixed point theory has many applications in several branches of science such as game theory, nonlinear programming,economics,theory of differential equations, etc. In this paper we discussed some of the properties of generalized E-fuzzy metric space. Also we proved contraction theorem in generalized E-fuzzy metric space. Our results presented in this paper generalize and improve some known results in fuzzy metric space.

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