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Disjoint Fair Domination in the Join and Corona of Two Graphs

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Abstract *- Let $G = (V(G), E(G))$ be a connected simple graph. A subset S of $V(G)$ is a dominating set of G if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $u, v \in E(G)$. A dominating set S is called a fair dominating set if for each distinct vertices $u, v \in V(G) \setminus S$, $|N_G(u) \cap S| = |N_G(v) \cap S|$. Further, if D is a minimum fair dominating set of G, then a fair dominating set $S \subseteq V(G) \setminus D$ is called an inverse fair dominating set of G with respect to D. A disjoint fair dominating set of G is the set $C = D \cup S \subseteq V(G)$. In this paper, we give the characterizations in the join and corona of two graphs.*

Keywords - *Fair dominating set, Inverse fair dominating set, Disjoint fair dominating set, Join, Corona.*

I. INTRODUCTION

Suppose that $G = (V(G), E(G)$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$. In simple graph, we mean, finite and undirected graph with neither loops nor multiple edges. For the general graph theoretic terminology, the readers may refer to [1].

A vertex v is said to dominate a vertex u if uv is an edge of G or $v = u$. A set of vertices $S \subseteq V(G)$ is called a dominating set of G if every vertex not in S is dominated by at least one member of S. The size of a set of least cardinality among all dominating sets for G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called γ-set of G. Domination in a graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [2]. Domination in graphs has been studied in [3 - 15].

A dominating set S is called a fair dominating set [16] of G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N_G(u) \cap S| = |N_G(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S, N_G(v) \cap S = k$. The fair domination number of G, is the minimum cardinality of a fair dominating set of G and is denoted by $\gamma_{fd}(G)$. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set of G. Fair domination in graphs has been studied in [17 - 22].

A fair dominating set S is called an inverse fair dominating set of G if each $S \subseteq V(G) \setminus D$ is a γ_{fd}-set of G. The inverse fair domination number of G, is the minimum cardinality of an inverse fair dominating set of G and is denoted by $\gamma_{fd}^{-1}(G)$. An inverse fair dominating set of cardinality $\gamma_{fd}^{-1}(G)$ is called γ_{fd}^{-1}-set of G. The inverse domination has been studied in [23 - 31]. A disjoint dominating set of G is the set $C = D \cup S \subseteq V(G)$. The disjoint domination number of G, is the minimum cardinality of a disjoint dominating set of G and is denoted by $\gamma\gamma(G)$. A disjoint dominating set of cardinality $\gamma\gamma(G)$ is called $\gamma\gamma$-$set\ of\ G$. The idea of disjoint domination in graphs has been studied in [32-34].

Let D be a minimum fair dominating set and S be an inverse fair dominating set of G with respect to D. A disjoint fair dominating set of G is the set $C = D \cup S \subseteq V(G)$. The disjoint fair domination number of G, is the minimum cardinality of a disjoint fair dominating set of G and is denoted by $\gamma\gamma_{fd}(G)$. A disjoint fair dominating set of cardinality $\gamma\gamma_{fd}(G)$ is called $\gamma\gamma_{fd}$-set of G [35]. In this paper, we extend the concept of disjoint fair dominating set by introducing some of its binary operations such as the join and corona of two graphs.

II. RESULTS

Let a graph $G = C_5$ as shown in Figure 1. This illustrates a graph that is not a disjoint fair dominating set.

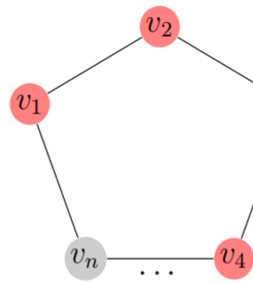

Fig. 1 A Graph G with $\gamma_{fd}(G) = 3$

The set $D = \{v_1, v_2, v_4\}$ is a minimum fair dominating set of the graph G and $S \subseteq V(G) \setminus D = \{v_3, v_5\}$ is not a fair dominating set of G. Hence, $C = D \cup S$ is not a disjoint fair dominating set of G.

Since $\gamma\gamma_{fd}(G)$ does not always exist in a connected nontrivial graph G, we denote $\mathcal{DF}(G)$, a family of all graphs with disjoint fair dominating set of G. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered, belong to the family $\mathcal{DF}(G)$. From the definitions, the following remarks follow.

Remark 2.1 Let G be a nontrivial connected graph and $S \subset V(G)$. If $N_G(v) \cap S = S$ for every $v \in V(G) \setminus S$, then S is an $|S|$-fair dominating set of G.

Remark 2.2 Let G and H be connected graphs of order m and n respectively. Then $V(G)$ is an m-fair dominating set of a graph $G + H$ and $V(H)$ is an n-fair dominating set of $G + H$.

The following lemmas are needed for the characterization of the disjoint fair dominating set in the join of two graphs.

Lemma 2.3 *Let G and H be nontrivial connected graphs. If S is an $|S|$-fair dominating set of G, then S is a fair dominating set of $G + H$.*

Proof: Suppose that S is an $|S|$-fair dominating set of G. Let $u \in V(G) \setminus S$. Then $N_G(u) \cap S = S$. This implies that $uv \in E(G)$ for all $v \in S$. Since for every $w \in V(H), N_{G+H}w \cap S = S$, it follows that for all $u \in V(G + H) \setminus S, N_{G+H}u \cap S = S$. Hence, S is an $|S|$-fair dominating set of $G + H$, that is S is a fair dominating set of $G + H$. ∎

Lemma 2.4 *Let G and H be nontrivial connected graphs. If S is an inverse $|S|$-fair dominating set of H, then S is a fair dominating set of $G + H$.*

Proof: The proof is similar to the proofs of Lemma 2.3 ∎

Lemma 2.5 *Let G and H be nontrivial connected graphs. If $S = S_G \cup S_H$ where S_G is an inverse $|S_G|$-fair dominating set of G and S_H is an $|S_H|$-fair dominating set of H, then S is an inverse fair dominating set of $G + H$.*

Proof: Since S_G is an inverse $|S_G|$-fair dominating set of G there exists $D \subseteq V(G) \setminus S$ such that D is a minimum $|D|$-fair dominating set of $G + H$ by using the proof of Lemma 2.3. Further, $N_G(u) \cap S_G = S_G$ for all $u \in V(G) \setminus S_G$. Since S_H is an $|S_H|$-fair dominating set of H, $N_H(v) \cap S_H = S_H$ for all $v \in V(H) \setminus S_H$. Let $u, v \in V(G + H) \setminus S$. If $u, v \in V(G) \setminus S_G$, then $|N_G(u) \cap S_G| = |N_G(v) \cap S_G$ because S_G is a fair dominating set of G. Thus

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |[(N_G(u) \cap S_G) \cup V(H)] \cap S| \\ &= |[(N_G(v) \cap S_G) \cup V(H)] \cap S| \\ &= |N_{G+H}(v) \cap S|. \end{aligned}$$

If $u, v \in V(H) \setminus S_H$, then $|N_H(u) \cap S_H| = |N_H(v) \cap S_H|$ because S_H is a fair dominating set of H. Thus

$$\begin{aligned}|N_{G+H}(u)\cap S| &= |[(N_H(u)\cap S_H)\cup V(G)]\cap S|\\ &= |[(N_H(v)\cap S_H)\cup V(G)]\cap S|\\ &= |N_{G+H}(v)\cap S|.\end{aligned}$$

If $u\in V(G)\setminus S_G$ and $v\in V(H)\setminus S_H$, then $N_G(u)\cap S_G=S_G$ and $N_H(v)\cap S_H=S_H$. Thus,

$$\begin{aligned}|N_{G+H}(u)\cap S| &= |[(N_G(u)\cap S_G)\cup V(H)]\cap S|\\ &= |(S_G\cup V(H))\cap S|\\ &= |(S_G\cap S)\cup(V(H)\cap S)]\\ &= |S_G\cup S_H|, \text{since } S_G\cap S=S_G \text{ and } V(H)\cap S=S_H\\ &= |(V(G)\cap S)\cup(S_H\cap S)|\\ &= |(V(G)\cup S_H)\cap S|\\ &= |(S_H\cup V(G))\cap S|\\ &= |[(N_H(v)\cap S_H)\cup V(G)]\cap S|\\ &= |N_{G+H}(v)\cap S|\end{aligned}$$

Therefore, for any $u,v\in V(G+H)\setminus S$, $|N_{G+H}(u)\cap S|=|N_{G+H}(v)\cap S|$, that is, S is a fair dominating set of $G+H$. Since $S\subseteq V(G+H)\setminus D$ where D is a minimum fair dominating set of $G+H$, it follows that S is an inverse fair dominating set of $G+H$. ■

Lemma 2.6 *Let G and H be nontrivial connected graphs. If $S=S_G\cup S_H$ where S_H is an inverse $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G, then S is an inverse fair dominating set of $G+H$.*

Proof: The proof is similar to the proofs of Lemma 2.5. ■

The following result shows the characterization of disjoint fair dominating set in the join of two graphs.

Theorem 2.7 *Let G and H be nontrivial connected graphs. Then a subset C is a disjoint fair dominating set of $G+H$ if and only if $C=D\cup S$ and one of the following statements is satisfied.*

(i) $a)$ *S is an $|S|$-fair dominating set of G and D is a minimum $|D|$-fair dominating set of H where $|D|\leq|S|$ for all S; or*
$b)$ *S and D are $|S|$-fair and $|D|$-fair dominating sets of G where $D\subseteq V(G)\setminus S$ and $|D|\leq|S|$.*
(ii) $a)$ *S is an $|S|$-fair dominating set of H and D is a minimum $|D|$-fair dominating set of G where $|D|\leq|S|$ for all S; or*
$b)$ *S and D are $|S|$-fair and $|D|$-fair dominating sets of H where $D\subseteq V(H)\setminus S$ and $|D|\leq|S|$.*
(iii) *$S=S_G\cup S_H$ where*
$a)$ *S_G is an inverse $|S_G|$-fair dominating set of G and S_H is an $|S_H|$-fair dominating set of H; or*
$b)$ *S_H is an inverse $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G.*

Proof: Suppose that C is a disjoint fair dominating set of $G+H$. Then there exist a γ_{fd}-set D of $G+H$ and an inverse fair dominating set S of $G+H$ such that $C=D\cup S$. Consider the following cases:

Case 1. If $S\cap V(H)=\emptyset$, then $S\subseteq V(G)$. Suppose that S is not an $|S|$-fair dominating set of G. Then there exists $u\in V(G)\setminus S$ such that $N_G(u)\cap S\neq S$ by Remark 2.1. Thus, there exists $u\in V(G+H)\setminus S$ such that

$$\begin{aligned}|N_{G+H}(u)\cap S| &= |(N_G(u)\cup V(H))\cap S|\\ &= |(N_G(u)\cap S)\cup(V(H)\cap S)|\\ &\neq |S\cup\emptyset| \text{ since } N_G(u)\cap S\neq S \text{ and } V(H)\cap S=\emptyset\\ &= |S|=|N_{G+H}(v)\cap S| \text{for some } v\in V(H).\end{aligned}$$

This contradict to our assumption that S is an inverse fair dominating set of $G+H$. Hence, S must be an $|S|$-fair dominating set of G. Since D is a γ_{fd}-set of $G+H$, if $D\subset V(H)$, by using the same argument, D must be a minimum $|D|$-fair dominating set of H. This means that $|D|\leq|S|$ for all S. This proves statement $(i)a)$. Similarly, if $D\subset V(G)$, then $|D|$-fair dominating sets of G where $D\subseteq V(G)\setminus S$ and $|D|\leq|S|$. This proves statement $(i)b)$.

Case 2. If $S \cap V(H) \neq \emptyset$, then consider the following. If $S \cap V(G) = \emptyset$, then $S \subseteq V(H)$. Suppose that S is not an $|S|$-fair dominating set of H. Then there exists $u \in V(H) \setminus S$ such that $N_H(u) \cap S \neq S$ by Remark 2.1. Thus, there exists $u \in V(G+H) \setminus S$ such that

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |(N_H(u) \cup V(G)) \cap S| \\ &= |(N_H(u) \cap S) \cup (V(G) \cap S)| \\ &\neq |S \cup \emptyset| \text{ since } N_H(u) \cap S \neq S \text{ and } V(G) \cap S = \emptyset \\ &= |S| = |N_{G+H}(v) \cap S| \text{for some } v \in V(G). \end{aligned}$$

This contradicts the assumption that S is an inverse fair dominating set of $G+H$. Hence, S must be an $|S|$-fair dominating set of H. Since D is a γ_{fd}-set of $G+H, if\ D \subset V(G)$, by using the same argument, D must be a minimum $|D|$-fair dominating set of G. This means that $|D| \leq |S|$ for all S. This proves statement $(ii)a$). Similarly, if $D \subset V(H)$, then $|D|$-fair dominating sets of H where $D \subseteq V(H) \setminus S$ and $|D| \leq |S|$. This proves statement $(ii)b$). If $S \cap V(G) \neq \emptyset$, then let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Now,

$$\begin{aligned} S &= S \cap V(G+H) \\ &= S \cap (V(G) \cup V(H)) \\ &= [S \cap V(G)] \cup [S \cap V(H)] \\ &= S_G \cup S_H. \end{aligned}$$

Consider first that S_G is an inverse $|S_G|$-fair dominating set of G. Suppose S_H is not a fair dominating set of H. Then there exists $u \in V(H) \setminus S_H$ such that $|N_H(u) \cap S_H| \neq |N_H(v) \cap S_H|$ for some $S \in V(H) \setminus S_H$. Thus, there exist $u \in V(G+H) \setminus S$ such that $|N_{G+H}(u) \cap S| \neq |N_{G+H}(v) \cap S|$ for some $v \in V(G+H) \setminus S$. This is contrary to our assumption that S is a fair dominating set of $G+H$. Thus, S_H must be a fair dominating set of H. Suppose S_H is not an $|S_H|$-fair dominating set of H. Then there exists $u \in V(H) \setminus S_H$ such that $N_H(u) \cap S_H \neq S_H$. Thus, there exists $u \in V(G+H) \setminus S$ such that

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\ &= |(\emptyset) \cup (N_H(u) \cap S_H)| \\ &= |N_H(u) \cap S_H| \\ &\neq |S_H| \\ &= |N_{G+H}(v) \cap S_H| \text{ for all } v \in V(G) \\ &\subset |N_{G+H}(v) \cap S| \text{ for all } v \in V(G) \end{aligned}$$

Thus, there exists $u \in V(G+H) \setminus S$ such that $|N_{G+H}(u) \cap S| \neq |N_{G+H}(v) \cap S|$ for all $v \in V(G)$. This contradict to our assumption that S is a fair dominating set of $G+H$. Hence, S_H must be an $|S_H|$-fair dominating set of H. This proves statement $(iii)a$). Similarly, if S_H is an inverse $|S_H|$-fair dominating set of H, then S_G must be an $|S_G|$-fair dominating set of G. This proves statement $(iii)b$).

For the converse, suppose that statement $(i)a$) is satisfied. Since S is an $|S|$-fair dominating set of G, S is also a fair dominating set of $G+H$ by Lemma 2.3. Similarly, D is an $|D|$-fair dominating set of H implies that D is also a fair dominating set of $G+H$ by Lemma 2.4. Since D is a minimum fair dominating set of H and $|D| \leq |S|$ for all S, it follows that D is a minimum fair dominating set of $G+H$. Since $S \subseteq V(G+H) \setminus D$, it follows that S is an inverse fair dominating set of $G+H$ with respect to D. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G+H$.

Suppose that statement $(i)b$) is satisfied. Since S is an $|S|$-fair dominating set of G and D is an $|D|$-fair dominating set of G, it follows that S and D are fair dominating set of $G+H$ by Lemma 2.3. Since $D \subseteq V(G) \setminus S$ and $|D| \leq |S|$ for all fair dominating set S and D, it follows that D is a minimum fair dominating set of $G+H$. Since $S \subseteq V(G) \setminus D$, S is an inverse fair dominating set of $G+H$ with respect to D. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G+H$.

Suppose that statement $(ii)a$) is satisfied. Since S is an $|S|$-fair dominating set of H, S is also a fair dominating set of $G+H$ by Lemma 2.4. Similarly, D is an $|D|$-fair dominating set of G implies that D is also a fair dominating set of $G+H$ by Lemma 2.3. Since D is a minimum fair dominating set of G and $|D| \leq |S|$ for all S, it follows that D is a minimum fair dominating set of $G+H$. Since $S \subseteq V(G+H) \setminus D$, it follows that S is an inverse fair dominating set of $G+H$ with respect to D. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G+H$.

Suppose that statement $(ii)b)$ is satisfied. Since S is an $|S|$-fair dominating set of H and D is an $|D|$-fair dominating set of H, it follows that S and D are fair dominating sets of $G+H$ by Lemma 2.4. Since $D \subseteq V(H) \setminus S$ and $|D| \leq |S|$ for all fair dominating set S and D, it follows that D is a minimum fair dominating set of $G+H$. Since $S \subseteq V(G) \setminus D$, S is an inverse fair dominating set of $G+H$ with respect to D. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G+H$.

Suppose that statement $(iii)a)$ is satisfied, that is, $S = S_G \cup S_H$ where S_G is an inverse $|S_G|$-fair dominating set of G and S_H is an $|S_H|$-fair dominating set of H. Then S is an inverse fair dominating set of $G+H$ by Lemma 2.5. Thus, there exists a minimum fair dominating set D of $G+H$ such that $C = D \cup S$ is a disjoint fair dominating set of $G+H$.

Suppose that statement $(iii)b)$ is satisfied, that is, $S = S_G \cup S_H$ where S_H is an inverse $|S_H|$-fair dominating set of H and S_G is an $|S_G|$-fair dominating set of G. By similar arguments used in $(iii(a)$, and by Lemma 2.6, S is an inverse fair dominating set of $G+H$. Thus, there exists a fair dominating set D of $G+H$ such that $C = D \cup S$ is a disjoint fair dominating set of $G+H$. ■

The following next result is an immediate consequences of Theorem 2.7.

Corollary 2.8 *Let G and H be nontrivial connected graphs. Then $\gamma\gamma_{fd}(G+H) = 2$ if and only if one of the following holds.*

(i) $\gamma^{-1}(G) = 1$.
(ii) $\gamma^{-1}(H) = 1$.
(iii) $\gamma(G) = 1$ *and* $\gamma(H) = 1$.

Proof: Suppose that $\gamma\gamma_{fd}(G+H) = 2$. Let $D = \{v\}$ be a fair dominating set of $G+H$ and $S = \{x\}$ be an inverse fair dominating set of $G+H$. Then $C = D \cup S$ is a disjoint fair dominating set of $G+H$. If S and D are $|S|$-fair and $|D|$-fair dominating sets of G where $D \subseteq V(G) \setminus S$ and $|D| \leq |S|$ (by Theorem 2.7 $(i)b$), then $\gamma^{-1}(G) = 1$, proving statement (i). If S and D are $|S|$-fair and $|D|$-fair dominating sets of H where $D \subseteq V(H) \setminus S$ and $|D| \leq |S|$ (by Theorem 2.7 $(ii)b$), then $\gamma^{-1}(H) = 1$ proving statement (ii). If S is an $|S|$-fair dominating set of G and D is a minimum $|D|$-fair dominating set of H where $|D| \leq |S|$ for all S (by Theorem 2.7 $(i)a)$), then $\gamma(G) = 1$ and $\gamma(H) = 1$, proving statement (iii).

For the converse, suppose that statement (i) is satisfied. Then there exist a minimum fair dominating set $D = \{v\}$ and an inverse fair dominating set $S = \{x\}$ of G such that $C = \{v, x\}$ is a disjoint fair dominating set of G and hence a minimum disjoint fair dominating set of $G+H$. Thus, $\gamma\gamma_{fd}(G+H) = |C| = 2$.

Suppose that statement (ii) is satisfied. Then there exist a minimum fair dominating set $D = \{v\}$ and an inverse fair dominating set $S = \{x\}$ of H such that $C = \{v, x\}$ is a disjoint fair dominating set of H and hence a minimum disjoint fair dominating set of $G+H$. Thus, $\gamma\gamma_{fd}(G+H) = |C| = 2$.

Suppose that statement (iii) is satisfied. Let $D = \{v\}$ be a dominating set of G and $S = \{x\}$ be a dominating set of H. Clearly, D is a fair dominating set of G and hence a fair dominating set of $G+H$. Similarly, S is a fair dominating set of $G+H$. Since $S \subseteq V(G+H) \setminus D$, it follows that S is a minimum inverse fair dominating set of $G+H$ with respect to D. Thus, $C = D \cup S$ is a minimum disjoint fair dominating set of $G+H$. Accordingly, $\gamma\gamma_{fd}(G+H) = |C| = 2$. ■

Remark 2.9 *For any connected graph G and graph H, $V(G)$ is a minimum fair dominating set in $G \circ H$.*

The following results are needed for the characterization of disjoint fair dominating set in the corona of two graphs.

Lemma 2.10 *Let G and H be nontrivial connected graphs. If $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$ where S_v is a fair dominating set of H^v and $D_v = \{x\}$ is a dominating set of H^v with $S_v \cap D_v = \emptyset$ for all $v \in V(G)$, then $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$.*

Proof: Since $D_v = \{x\}$ is a dominating set of H^v, $D = \bigcup_{v \in V(G)} D_v$ is a dominating set of $G \circ H$. Since H is a nontrivial connected graph, $V(H^v) \setminus D_v \neq \emptyset$ for each $v \in V(G)$. Let $y \in V(H^v) \setminus D_v$. Then

$$|N_{G \circ H}(v) \cap D| = |N_{v+H^v}(v) \cap D_v| = |N_{v+H^v}(y) \cap D_v| = |N_{G \circ H}(y) \cap D|$$

for all $v, y \in V(G \circ H) \setminus D$. Hence, D is a fair dominating set of $G \circ H$. Since $V(G)$ is a minimum fair dominating set of $G \circ H$ and $|D| = \left|\bigcup_{v \in V(G)} D_v\right| = \sum_{v \in V(G)} |D_v| = |V(G)||D_v| = |V(G)| \cdot 1 = |V(G)|$, it follows that D is a minimum fair dominating set of $G \circ H$. Now, S_v is a fair dominating set of H^v implies that $V(v + \langle S_v \rangle)$ is a fair dominating set of $v + H^v$ for all $v \in V(G)$. Thus, $S = V(G) \cup (\bigcup_{v \in V(G)} S_v) = \bigcup_{v \in V(G)} V(v + \langle S_v \rangle)$ is a fair dominating set of $G \circ H$. Since $S_v \cap D_v = \emptyset$ for all $v \in V(G)$,

$$\begin{aligned} S \cap D &= \left(V(G) \cup \left(\bigcup_{v \in V(G)} S_v\right)\right) \cap \left(\bigcup_{v \in V(G)} D_v\right) \\ &= \left(V(G) \cap \left(\bigcup_{v \in V(G)} D_v\right)\right) \cup \left(\left(\bigcup_{v \in V(G)} S_v\right) \cap \left(\bigcup_{v \in V(G)} D_v\right)\right) \\ &= \left(\bigcup_{v \in V(G)} (V(G) \cap D_v)\right) \cup \left(\bigcup_{v \in V(G)} (S_v \cap D_v)\right) \\ &= \left(\bigcup_{v \in V(G)} \emptyset\right) \cup \left(\bigcup_{v \in V(G)} \emptyset\right), \textit{since } V(G) \cap D_v = \emptyset. \end{aligned}$$

Thus, $S \cap D = \emptyset$. Let $S \subset V(G \circ H) \setminus D$. Since D is a minimum fair dominating set and S is a fair dominating set of $G \circ H$, it follows that S is an inverse fair dominating set of $G \circ H$. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$. ■

Lemma 2.11 *Let G and H be nontrivial connected graphs. If $S = \bigcup_{v \in V(G)} S_v$ where S_v is an $|S_v|$-fair dominating set of $v + H^v$ and $D_v = \{x\}$ is a dominating set of H^v with $S_v \cap D_v = \emptyset$ for all $v \in V(G)$, then $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$.*

Proof: Suppose that $D_v = \{x\}$ is a dominating set of H^v. Let $D = \bigcup_{v \in V(G)} D_v$. By similar reasoning that is used in the proof of Lemma 2.10, D is a minimum fair dominating set of $G \circ H$. Let $x, v \in V(v + H^v) \setminus S_v$ for each $v \in V(G)$. Since S_v is an $|S_v|$-
fair dominating set of $v + H^v$ for each $v \in V(G)$, $|N_{G \circ H}(x) \cap S| = |N_{v + H^v}(x) \cap S_v| = |N_{v + H^v}(v) \cap S_v| = |N_{G \circ H}(v) \cap S|$ for all $x, v \in V(G \circ H) \setminus S$. Hence, S is a fair dominating set of $G \circ H$. Since $S_v \cap D_v = \emptyset$ for all $v \in V(G)$,

$$S \cap D = \left(\bigcup_{v \in V(G)} S_v\right) \cap \left(\bigcup_{v \in V(G)} D_v\right) = \bigcup_{v \in V(G)} (S_v \cap D_v) = \bigcup_{v \in V(G)} (\emptyset).$$

Thus, $S \cap D = \emptyset$. Let $S \subset V(G \circ H) \setminus D$. Since D is a minimum fair dominating set and S is a fair dominating set of $G \circ H$, it follows that S is an inverse fair dominating set of $G \circ H$. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$. ■

Lemma 2.12 *Let G and H be nontrivial connected graphs. If $S = \bigcup_{v \in V(G)} V(H^v)$ and $D = V(G)$, then $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$.*

Proof: Since $D = V(G)$, D is a minimum fair dominating set of $G \circ H$ by Remark 2.9. If $S = \bigcup_{v \in V(G)} V(H^v)$, then $S = V(G \circ H) \setminus D$ is a dominating set of $G \circ H$ by Theorem 2.11. Let $x, y \in V(G \circ H) \setminus S = V(G)$. Then $|N_{G \circ H}(x) \cap S| = |V(H)| = |N_{G \circ H}(y) \cap S|$. Hence, S is a fair dominating set of $G \circ H$. Since D is a minimum fair dominating set of $G \circ H$ and $S = V(G \circ H) \setminus D$, it follows that S is an inverse fair dominating set of $G \circ H$. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$. ■

Lemma 2.13 *Let G and H be nontrivial connected graphs. If $S = \bigcup_{v \in V(G)} S_v$ where S_v is an $|S_v|$-fair dominating set of $v + H^v$ for each $v \in V(G)$ and $D = V(G)$, then S is an inverse fair dominating set of $G \circ H$.*

Proof: Since $D = V(G)$, D is a minimum fair dominating set of $G \circ H$. Let $x, v \in V(v + H^v) \setminus S_v$ for each $v \in V(G)$. Since S_v is an $|S_v|$-fair dominating set of $v + H^v$ for each $v \in V(G)$,

$$|N_{G \circ H}(x) \cap S| = |N_{v + H^v}(x) \cap S_v| = |S_v| = |N_{v + H^v}(v) \cap S_v v| = |N_{G \circ H}(v) \cap S|$$

for all $x, v \in V(G \circ H) \setminus S$. Hence, S is a fair dominating set of $G \circ H$. Since D is a minimum fair dominating set of $G \circ H$ and $S \subseteq V(G \circ H) \setminus D$, it follows that S is an inverse fair dominating set of $G \circ H$. Accordingly, $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$. ■

The following result, shows the characterization of disjoint fair dominating set in the corona of two graphs

Theorem 2.14 *Let G and H be nontrivial connected graphs. A nonempty subset $C = S \cup D$ of $V(G \circ H)$ is a disjoint fair dominating set of $G \circ H$ if and only if for each $v \in V(G)$, one of the following is satisfied.*

(i) $S = V(G)$ and $\gamma(H) = 1$.
(ii) $D_v = \{x\}$is a dominating set of H^v for all $v \in V(G)$ and
a) $S = V(G) \cup (\bigcup_{v\in V(G)} S_v)$ where S_v is a fair dominating set of H^v and $S_v \cap D_v = \emptyset$; or
b) $S = \bigcup_{v\in V(G)} S_v$ where S_v is an $|S_v|$-fair dominating set of $v + H^v$ and $S_v \cap D_v = \emptyset$.
(iii) $D = V(G)$ and $S = \bigcup_{v\in V(G)} S_v$ where
a) $S_v = V(H^v)$ for each $v \in V(G)$; or
b) S_v is an $|S_v|$-fair dominating set of $v + H^v$ for each $v \in V(G)$.

Proof: Suppose that a nonempty subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint fair dominating set of $G \circ H$. Let D be a γ_{fd}-set of $G \circ H$ such that $S \cap D = \emptyset$. Consider the following cases:
Case 1. Suppose that $D \subseteq V(G \circ H) \setminus V(G)$. If $D = V(G \circ H) \setminus V(G)$, then $D = \bigcup_{v\in V(G)} V(H^v)$. Since H is nontrivial, $|V(H)| \geq 2$ and $|D| = \left|\bigcup_{v\in V(G)} V(H^v)\right| = \sum_{v\in V(G)} |V(H^v)| = |V(G)||V(H)| \geq |V(G)| \cdot 2 > |V(G)|$, that is, $|D| > |V(G)|$. By Remark 2.9, $V(G)$ is a minimum fair dominating set of $G \circ H$ contradict to our assumption that D is γ_{fd}-set of $G \circ H$. This implies that $D \neq V(G \circ H) \setminus V(G)$. Thus, $D \subset V(G \circ H) \setminus V(G)$. Let $D = \bigcup_{v\in V(G)} V(D_v)$ where $D_v \subset V(H^v)$ for all $v \in V(G)$. Since D and $V(G)$ are minimum fair dominating sets of $G \circ H, |D| = |V(G)|$. First, Consider that $S = V(G)$. Then $S = V(G) \subseteq V(G \circ H) \setminus D$. Moreover, $|V(G)| = |D| = |\bigcup_{v\in V(G)} D_v| = \sum_{v\in V(G} |D_v| = |V(G)||D_v|$, where $D_v \subset V(H^v)$ for all $v \in V(G)$. Thus, $|D_v| = 1$. Since D is a dominating set of $G \circ H$, D_v must be a dominating set of H^v for all $v \in V(G)$. Hence $\gamma(H) = |D_v| = 1$ for all $v \in V(G)$. This proves statement (i).

Next, consider that $S \neq V(G)$. If $V(G) \subset S$, then for each $v \in V(G)$, it follows that $S = V(G) \cup (\bigcup_{v\in V(G)} S_v)$, where $S_v \subset V(H^v)$. Since S is a fair dominating set of $V(G \circ H), S_v$ must be a fair dominating set of H^v for each $v \in V(G)$. Similarly, because D and $V(G)$ are minimum fair dominating sets of $G \circ H$, $|D| = |V(G)|$ and so, $|D_v| = 1$ for all $v \in V(G)$. Let $D_v = \{x\}$. Since $S \cap D = \emptyset$ and $V(G) \cap D_v = \emptyset$ where $D_v \subset V(H^v)$ for all $v \in V(G)$,

$$
\begin{aligned}
S \cap D &= \left((V(G) \cup (\bigcup_{v\in V(G)} S_v)\right) \cap \left(\bigcup_{v\in V(G)} D_v \right) \\
&= \left((V(G) \cap (\bigcup_{v\in V(G)} D_v)\right) \cup \left((\bigcup_{v\in V(G)} S_v) \cap (\bigcup_{v\in V(G)} D_v)\right) \\
&= \left(\bigcup_{v\in V(G)} (V(G) \cap D_v)\right) \cup \left(\bigcup_{v\in V(G)} (S_v \cap D_v)\right) \\
&= \left(\bigcup_{v\in V(G)} \emptyset\right) \cup (\bigcup_{v\in V(G)} (S_v \cap D_v)) \\
&= \bigcup_{v\in V(G)} (S_v \cap D_v) = \emptyset
\end{aligned}
$$

This implies that $S_v \cap D_v = \emptyset$ for all $v \in V(G)$. This proves statement (iia).

Now, the fact that $V(G)$ is a minimum fair dominating set of $G \circ H$, $S \not\subset V(G)$. If $V(G) \not\subset S$, then let $S = \bigcup_{v\in V(G)} S_v$, where $S_v \subset V(v + H^v) \setminus D_v$ and $D_v = \{x\}$ is a dominating set of H^v for all $v \in V(G)$. Since S is a fair dominating set of $V(G \circ H), S_v$ must be a fair dominating set of $v + H^v$ for each $v \in V(G)$. This means that $|N_{v+H^v}(v) \cap S_v| = |N_{v+H^v}(x) \cap S_v|$ for each $v, x \in V(v + H^v) \setminus S_v$. Since $|N_{v+H^v}(v) \cap S_v| = |S_v|$, it follows that S_v is an $|S_v|$-fair dominating set of $v + H^v$ for each $v \in V(G)$. Similarly, $S \cap D = \emptyset$ implies that $S_v \cap D_v = \emptyset$ This proves statement (iib).

Case 2. Suppose that $D \not\subseteq V(G \circ H) \setminus V(G)$. Then $D \subseteq V(G)$. If $D \neq V(G)$, then $D \subset V(G)$ contradict to the fact that D and $V(G)$ are both minimum fair dominating sets of $G \circ H$. This implies that $D = V(G)$. Since $S \cap D = \emptyset$, let $S = \bigcup_{v\in V(G)} S_v$, where $S_v \subseteq V(H^v)$ and $S_v \neq \emptyset$. If $S_v = V(H^v)$ for each $v \in V(G)$, then statement $(iiia)$ is satisfied. Suppose that $S_v \neq V(H^v)$. Let $x \in V(H^v) \setminus S_v$ for each $v \in V(G)$. Since S is a fair dominating set of $V(G \circ H), S_v$ must be a fair dominating set of $v + H^v$ for each $v \in V(G)$. This means that $|N_{v+H^v}(v) \cap S_v| = |N_{v+H^v}(x) \cap S_v|$ for each $v \in V(G)$ and for each $v, x \in V(v + H^v) \setminus S_v$. Since $|N_{v+H^v}(v) \cap S_v| = |S_v|$, it follows that S_v is an $|S_v|$-fair dominating set of $v + H^v$ for each $v \in V(G)$. This proves statement $(iiib)$.

For the converse, suppose that statement (i) is satisfied. In view of Remark 2.10, $S = V(G)$ is a fair dominating set of $G \circ H$. For each $v \in V(G)$, let $D_v = \{x\}$ be a dominating set of H^v. Then for each

$$u, v \in V(v + H^v) \setminus D_v, |N_{v+H^v}(u) \cap D_v| = 1 = |N_{v+H^v}(v) \cap D_v|.$$

Hence D_v is a fair dominating set of $v + H^v v$. Let $D = \bigcup_{v \in V(G)} D_v$ and let $u, v \in V(v + H^v) \setminus D_v$. Then $|N_{G \circ H}(u) \cap D| = |N_{v+H^v}(u) \cap D_v| = 1 = |N_{v+H^v}(v) \cap D_v| = |N_{G \circ H}(v) \cap D|$ for all $u, v \in V(G \circ H) \setminus D$. Thus, D is a fair dominating set of $G \circ H$. Since $|D| = \left|\bigcup_{v \in V(G)} D_v\right| = \left|\sum_{v \in V(G)} D_v\right| = |V(G)||D_v| = |V(G)| \cdot 1 = |V(G)|$, it follows that D is also a minimum fair dominating set of $G \circ H$. Since

$$S \cap D = V(G) \cap (\bigcup_{v \in V(G)} D_v) = \bigcup_{v \in V(G}(V(G) \cap D_v) = \bigcup_{v \in V(G)}(\emptyset), S \cap D = \emptyset.$$

Let $S \subseteq V(G \circ H) \setminus D$. Since D is a minimum fair dominating set of $G \circ H$ and S is a fair dominating set of $G \circ H$, it follows that S is an inverse fair dominating set of $G \circ H$ with respect to D. Thus, $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$. Next, if statement (iia) is satisfied, then by Lemma 2.11, $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$. Similarly, if statement (iib) is satisfied, then $C = D \cup S$ is a disjoint fair dominating set of $G \circ H$ by Lemma 2.11. Finally, if statement $(iiia)$ (or statement $(iiib)$) is satisfied, then $C = D \cup S$ is an inverse fair dominating set of $G \circ H$ by Lemma 2.12 (or Lemma 2.13). This completes the proofs. ■

The following result is an immediate consequence of Theorem 2.14.

Corollary 2.15 *Let G and H be nontrivial connected graphs with $|V(G)| = m$ and $|V(H)| = n$, and $k = |S_v|$ where S_v is a γ_{fd}-set of $v + H^v$ for all $v \in V(G)$. Then*

$$\gamma\gamma_{fd}(G \circ H) = \begin{cases} 2m, & if \quad \gamma(H) = 1 \\ (k+1)m, & if \quad \gamma(H) \geq 2 \end{cases}$$

Proof: Suppose that a nonempty subset $C = D \cup S$ of $V(G \circ H)$ is a disjoint fair dominating set of $G \circ H$. Then $\gamma\gamma_{fd}(G \circ H) \leq |C|$. Consider the following cases.

Case 1. Suppose that $\gamma(H) = 1$. Then $S = V(G)$, by Theorem 2.14i). This implies that $\gamma_{fd}^{-1}(G \circ H) \leq |S| = |V(G)|$. Further, for each $v \in V(G)$, let S_v be a γ-set of H^v. Then $|S_v| = \gamma(H^v) = 1$. Let $D = \bigcup_{v \in V(G)} S_v$. Then

$$|D| = \left|\bigcup_{\{v \in V(G)\}} S_v\right| = \sum_{v \in V(G)} |S_v| = |V(G)| \cdot 1 = |V(G)|.$$

That is, D is also a γ_{fd}-set of $G \circ H$ by Remark 2.10. Thus,

$$m = |V(G)| = |D| = \gamma_{fd}(G \circ H) \leq \gamma_{fd}^{-1}(G \circ H) \leq |S| = |V(G)| = m.$$

This implies that $\gamma\gamma_{fd}(G \circ H) \leq |C| = |D \cup S| = |D| + |S| = m + m = 2m$. Since, $\gamma_{fd}(G \circ H) = \gamma_{fd}^{-1}(G \circ H) = m$, it follows that $2m = \gamma_{fd}(G \circ H) + \gamma_{fd}^{-1}(G \circ H) \leq \gamma\gamma_{fd}(G \circ H) \leq |C| = 2m$. Hence, $\gamma\gamma_{fd}(G \circ H) = 2m$.

Case 2. Suppose that $\gamma(H) \neq 1$. Then $\gamma(H) \geq 2$. Let S_v be a minimum k-fair dominating set of H^v for all $v \in V(G)$ where $k = |S_v|$. Then $k = |S_v| \geq 2$ (since $\gamma(H) \geq 2$). Let $D = V(G)$ and $S = \bigcup_{v \in V(G)} S_v$ (by Theorem 2.14iii)). Then,

$$|S| = \left|\bigcup_{v \in V(G)} S_v\right| = \sum_{v \in V(G)} |S_v| = |V(G)||S_v| = mk > m = |V(G)| = |D|.$$

Thus, $mk = |S| > |D| = m$. Hence, $\gamma\gamma_{fd}(G \circ H) \leq |C| = |D \cup S| = |D| + |S| = m + mk$.
Since $k = |S_v|$ where S_v is a minimum k-fair dominating set of H^v for all $v \in V(G)$, it follows that, S is a γ_{fd}^{-1}-set of $G \circ H$. Thus, $|S| = \gamma_{fd}^{-1}(G \circ H)$and $|D| = |V(G)| = \gamma_{fd}(G \circ H)$, that is ,

$$m + mk = \gamma_{fd}(G \circ H) + \gamma_{fd}^{-1}(G \circ H) \leq \gamma\gamma_{fd}(G \circ H) \leq |C| = m + mk. \text{ Hence, } \gamma\gamma_{fd}(G \circ H) = (k+1)m. \blacksquare$$

III. CONCLUSION

In this paper, we extend the concept of disjoint fair domination in graphs by characterizing the join and corona of two connected nontrivial graphs. We further give the disjoint fair domination number of the join and corona of two graphs. It is interesting to note that some related problems on disjoint fair domination in graphs are still open for research. We can extend further the study to the following:

1. Characterize the disjoint fair dominating sets of the Cartesian product and lexicographic product of two graphs.
2. Find the disjoint fair domination number of the Cartesian product and lexicographic product of two graphs.

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