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Original Article

Numerical Solutions for Nonlinear Volterra Integral Equations of the Second Kind with a Domain Decomposition and Modified Decomposition Methods

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Abstract - *In this work, a domain decomposition method (ADM) and Modified a domain decomposition method (MADM) is used to solve the nonlinear Volterra integral equations, also some comparison between the two methods is done for the same class of nonlinear equations. We outline some differences between the two methods and show that the Modified decomposition method is more effective than the standard decomposition method . Numerical examples and their solutions are obtained using MATLAB.*

Keywords - *Volterra integral equations, A domain decomposition method, Modified decomposition method.*

I. INTRODUCTION

The Volterra integral equation arises from various fields of science as physics, chemisty , medical phenomena, biological models and engineering field, which clearly appear as the population dynamics, spread of epidemics, and in engineering such as vibration and wave equation also heat transfer and diffusion problems.

Due to the increased interest in non-linear integral equations, a broad class of analytical and numerical solution methods have been used to handle these problems [1–6]. However,these analytical solution methods are not easy to use and require difficult work and for this reason we will presented numerical methods for solution.

Numerical solution of nonlinear Volterra integral equation of the second kind have been developed by many researches. Such as Linz et al. [1], Kanwal et al. [2], Atkinson et al. [3])and WAZWAZ [7-15] contain many different methods for solving this equations numerically, The use of the a domian decomposition method for solving nonlinear Volterra equation is mentioned in [16]- [25].

However, how to solve any kind of the volterra integral equations is a great challenge for researches.As we see the main objective of this work is to use some powerful numerical methods to solve nonlinear Volterra integral equations.

This paper is arranged as follows, the next section present a domian decomposition method; Section III, presents modified a domian decomposition method, A comparison between the ADM and MADM methods is demonstrated in Section Iv , Section v concern to numerical implementation of the ADM and MADM in MATLAB , two examples are presented to illustrate the accuracy of the MADM, and the conclusion is drawn in Section VI.

In this work we shall deal with nonlinear Volterra equation of the second kind in which characterized by at least one variable limit of integration. This equation can be represented by the form

$$\phi(x) = f(x) + \lambda \int_0^x k(x,t)\, f(\phi(t))\, dt \qquad (1,1)$$

Where k(x,t) and f(x) are real-valued function of $f(\phi(x))$ is a nonlinear function of $\phi(x)$ such as $\phi^2(x)$,sin($\phi(x)$) and $e^{\phi(x)}$, $\phi^3(x), \cos(\phi(x))$.

Where λ is a constant parameter. K(x,t) is called kernel (nucleus), if f(x) is zero the equation is called homogenous.
First we will state the existence and uniqueness and converegence of the theorems of nonhomogeneous nonlinear Volterra integral equations **.**

II. EXISTENCE AND UNIQUENESS OF THE SOLUTION VOLTERRA EQUATIONS

In this section we will present an existence theorem for the solution of nonlinear Volterra integral equations Consider the nonlinear Volterra integral equation .of the form

$$\phi(x) = f(x) + \lambda \int_0^x G(x,t,\phi(t))dt \qquad (2,1)$$

In order to guarantee the existence of a unique solution of
(2.1) We assume the following:
(i) The functions f(x)and G(x,t,φ(t)) is integrable and bounded in in $a \le x, t \le b$.

(ii) The functions f(x) and G(x,t,φ(t))must satisfy the Lipschitz condition in the interval
i.e.
$|f(x) - f(y)| < k|x-y|$ and $|G(x,t,z) - G(x,t,z')| < M|z-z'|$

Theorem

Consider the nonlinear Volterra integral equation (2.1) if the functions f(x) and G(x,t,φ(t)) satisfying conditions (i),and (ii), then the nonlinear Volterra integral equation (2.1) have a unique solution $\phi(x)$ which converges absolutely and almost uniformly.

The complete proof of this theorem can be found in [1–6]. The aim of this work will be on solving the nonlinear Volterra integral equations rather than proving this theorems

III. THE A DOMAIN DECOMPOSITION METHOD

In the 1980's, George Adomian[16-18] introduced a new method for solving linear and nonlinear Volterra equations Since then, this method has been known as the Adomian decomposition method (ADM) [3,4] also a theoretical foundation of Adomian method was developed in Gabet [5], Venkatarangan and Rajalakshmi [6].
The Adomain decomposition methods consists of decomposing the unknown function $\phi(x)$ of any equation into sum of infinite series components that will be determined, given by such as

$$\phi(x) = \phi_0(x) + \phi_1(x) + \phi_2(x) + \ldots = \sum_{n=0}^{\infty} \phi_n(x) \qquad (3)$$

Where $\phi_n(x), n \ge 0$ will be determine by recursive manner. And the nonlinear terms should be expressed by Adomain polynomials An, $n \ge 0$ so we will develop an algorithm to represent the adomain polynomials An which is given by the form

$$An = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{i=0}^{n} \lambda^i \phi_i\right)\right]_{\lambda=0} , n = 0,1,2,\ldots \qquad (4)$$

The first four Adomain polynomials are represented as follows

$$A0 = f(\phi_0)$$
$$A_1 = \phi_1 f'(\phi_0)$$
$$A_2 = \phi_2 f'(\phi_0) + \frac{1}{2!}\phi_1^2 f''(\phi_0)$$
$$A_3 = \phi_3 f'(\phi_0) + \phi_1\phi_2 f''(\phi_0) + \frac{1}{3!}\phi_1^3 f'''(\phi_0)$$

In fact the first componentA0 must depend only on ϕ_0 , and A1 only on ϕ_0 and ϕ_1 ,and so on. To establish the recurrence relation, we substitute (2) into Volterra integral equation (1) to obtain

$$\sum_{n=0}^{\infty} \phi_n(x) = f(x) + \lambda \int_0^x k(x,t) \sum_{n=0}^{\infty} \phi_n(x) dt \qquad (5)$$

$$\phi_0(x) + \phi_1(x) + \phi_2(x) + \ldots = f(x) + \lambda \int_0^x k(x,t)\phi_0(t)dt + \lambda \int_0^x k(x,t)\phi_1(t)dt \qquad (6)$$

The zeroth component $\phi_0(x)$ is identified by all terms that are not include the integral sign. So

$$\phi_0(x) = f(x)$$

And

$$\phi_i(x) = \lambda \int_0^x k(x,t)\phi_{i-1}(t)dt \quad , i \geq 1 \qquad (7)$$

i.e $\phi_i(x)$ is obtained by computation of the integrals so the final solution $\phi(x)$ is determined by this components, note that few terms of is obtained series for approximation solution

IV. THE MODIFIED AND DECOMPOSITION METHOD

A reliable modification of the a Adomian decomposition method was developed by Wazwaz and presented in [8,15]. The modified decomposition method will simplify the computational difficulties process and further accelerate the convergence of the origin Adomain method.

The main idea of this method depend to splitting the function f(x) into two parts. So it can be Appling to all integral equation wherever it is appropriate and it is clear that if f(x) is consists of one term it will be not suitable to use modification method . to give a clear description of this process let

$$f(x) = f_1(x) + f_2(x) \qquad (8)$$

where $f_1(x)$ consists of one term only or more terms in rare case . and $f_2(x)$ contains the remaining terms of $f(x)$. hence based on this assumption and as a result, the modified decomposition method introduces the recursive relation

$$\phi_0(x) = f_1(x) \qquad (9)$$

$$\phi_1(x) = f_2(x) + \lambda \int_0^x k(x,t)\phi_0(t)dt \qquad (10)$$

$$\phi_n(x) = \lambda \int_0^x k(x,t)\phi_{n-1}(t)dt \quad n \geq 2 \qquad (11)$$

Equation 5-7 show that the difference between the a domain decomposition method and the modified ADM. Rests only in formulation of the first two terms ϕ_0, ϕ_1 . the reaming terms with no change. Accordingly, 2 becomes

$$\phi_0(x) + \phi_1(x) + \phi_2(x) + \ldots = f_1(x) + f_2(x) + \lambda \int_0^x k(x,t)\phi_0(t)dt + \lambda \int_0^x k(x,t)\phi_1(t)dt$$

V. NUMERICAL RESULTS

In this section, we present some examples to illustrate the above methods for solving nonlinear Volterra integral equation of the second kind, also comparison between the ADM , MADM and the exact solution will be presented in implementation we used matlab .The validity of the modified technique was verified through illustrative examples. numerical results show how the modified decomposition method is more effective converges faster than the standard one.

Example 1

Consider the nonlinear Volterra equation

$$\phi(x) = x - \sin x + \int_0^x \cos\phi(t)dt$$

First we use the adomain decomposition method

Let

$$\phi(x) = f(x) + \int_0^x k(x,t)F(u(t))dt$$

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x) \qquad n \geq 0$$

$$\phi_0(x) = x - \sin x$$

$$\phi_1 = \int_0^x \phi_0(t)dt = \int_0^x (t - \sin t)dt = \frac{x^2}{2} + \cos x - 1$$

$$\phi_2 = \int_0^x 2\phi_0(t)\phi_1(t)dt = \int (\mathrm{t - sint})\left(\frac{\mathrm{t}^2}{2} + \cos t - 1\right)\mathrm{dt}$$

$$\phi_3 = \int_0^x (2\phi_0(t)\phi_2(t) + \phi_1^2)dt$$

And the solution is

$$\phi(x) = \phi_0 + \phi_1 + \phi_2 + \ldots$$

Secondly we will use the modified Adomain method

Let

$$\phi_0 = x$$

$$\phi_1 = -\sin x + \int_0^x \cos(\phi_0(t))dt = 0$$

The exact solution is $\phi(x) = x$

Table 1. Show the exact solution compare with the (ADM)and (MADM) solution

x	Exact_solution	ADM_solution	MADM_solution
0	0	0	0
0.1000	0.1000	0.0100	0.1000
0.2000	0.2000	0.0399	0.2000
0.3000	0.3000	0.0897	0.3000
0.4000	0.4000	0.1590	0.4000
0.5000	0.5000	0.2476	0.5000
0.6000	0.6000	0.3553	0.6000
0.7000	0.7000	0.4817	0.7000
0.8000	0.8000	0.6267	0.8000
0.9000	0.9000	0.7903	0.9000
1.0000	1.0000	0.9724	1.0000

Example 2 (this example appear in [20]
Consider the following nonlinear Volterra integral equation

$$\phi(x) = \sec x + \tan x - \int_0^x \phi^2(t)dt$$

$$\phi_0(x) = \sec x + \tan x$$

$$\phi_1 = -\int_0^x A_0(t)dt = -\int_0^x (\sec t + \tan t)^2 dt$$

Using the adomain decomposition methods $\quad = -2\tan x - 2\sec x + x + 2$

$$\phi_2 = -\int_0^x A_1(t)dt = -\int_0^x 2\phi_1(t)\phi_0(t)dt$$

$$= -\int_0^x 2(-2\tan x - 2\sec x + x + 2)(\sec t + \tan t)dt$$

which is difficult to calculate the approximate solution will be in table 2.
Now we use the modified decomposition method
Let

$$\phi_0 = \sec x$$

$$\phi_1 = \tan x - \int_0^x \sec^2(t)dt = 0$$

Hence $\phi(x) = \phi_0(x) = \sec x$

Table 2. Show the exact solution , the (ADM)and (MADM) solution

x	Exact solution	ADM solution	MADM solution
0	1.0000	1.0000	1.0000
0.1000	1.0050	0.9946	1.0050
0.2000	1.0203	0.9770	1.0203
0.3000	1.0468	0.9439	1.0468
0.4000	1.0857	0.8915	1.0857
0.5000	1.1395	0.8142	1.1395
0.6000	1.2116	0.7042	1.2115
0.7000	1.3075	0.5503	1.3070
0.8000	1.4353	0.3350	1.4333
0.9000	1.6087	0.0311	1.6015
1.0000	1.8508	-0.4082	1.8274

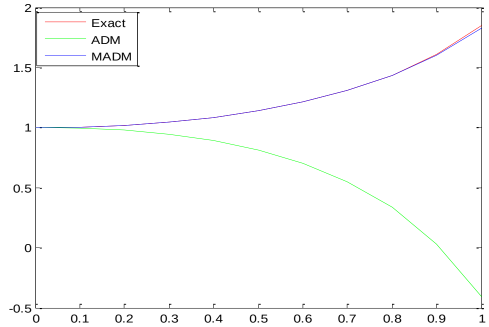

Fig. 1 Plot of the solutions of Volterra integral equation for example 2 using two different methods and exact solution

VI. CONCLUSION

The purpose of this study is use Adomian and Modified decomposition method to solve nonlinear Volterra equations. The results obtained by using Adomian technique, are compared to those obtained by using Modified Adomian decomposition one. The numerical results, demonstrate that MADM technique, gives the approximate solution with faster convergence rate and higher accuracy than using the standard ADM

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REFERENCES

[1] P. Linz, Analytical and Numerical Methods for Volterra equations. Society for Industrial and Applied Mathematics, (1985).
[2] R.P. Kanwal, Linear Integral Equations Theory and Technique. Boston, (1996).
[3] K.E.Atkinson, The Numerical Solution of Integral Equation of the Second Kind. Cambridge, (1997).
[4] A.M. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications. Springer ,(2011).
[5] L. Gabet, The Decomposition Method and Distributions. Computers & Mathematics with Applications, 27(3) (1994) 41-49.
[6] S. N. Venkatarangan, & K. Rajalakshmi, A Modification of a Domain's Solution for Nonlinear Oscillatory Systems. Computers & Mathematics with Applications, 29(6) (1995) 67-73.
[7] A.M. Wazwaz, A Reliable Modification of Decomposition Method, Appl. Math. Comput. 102 (1999) 77–86.
[8] A.M. Wazwaz, A Reliable Treatment for Mixed Volterra–Fredholm Integral Equations, Appl. Math. Comput. 127 (2002) 405–414.
[9] A.M. Wazwaz, The Numerical Solution of Special Fourth-Order Boundary Value Problems by the Modified Decomposition Method, Int.J. Comput. Math. 79(3) (2002) 345–356.
[10] A.M. Wazwaz, A First Course in Integral Equations, World Scientific, (1997).
[11] A.M. Wazwaz, Partial Differential Equations: Methods and Applications, Balkema, Netherlands, (2002).
[12] A.M. Wazwaz, A Reliable Technique for Solving the Weakly Singular Second-Kind Volterra-Type Integral Equations, Appl. Math. Comput. 80 (1996) 287–299.
[13] A.M. Wazwaz, Analytical Approximations and Pade' approximants for Volterra's Population Model, Appl. Math. Comput. 100 (1999) 13–25.
[14] A.M. Wazwaz, the Modified Decomposition Method and Pade' Approximants for Solving Thomas–Fermi equation, Appl. Math.Comput. 105 (1999) 11–19.
[15] A.M. Wazwaz, A new algorithm for calculating A Domain Polynomials for Nonlinear Operators, Appl. Math. Comput. 111 (2000) 53–69.
[16] G. A domain, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, 1994.
[17] G. A domain, A review of the Decomposition Method and Some Recent Results for Nonlinear Equation, Math. Comput. Modell. 13(7) (1992) 17–43.
[18] G. A domian, The Decomposition Method for Nonlinear Dynamical Systems. Journal of mathematical analysis and applications, 120(1) (1986) 370-383.
[19] K. Maleknejad, K. Nouri, &R. Mollapourasl, Existence of solutions for some nonlinear integral equations. Communications in Nonlinear Science and Numerical Simulation, 14(6) (2009) 2559-2564.
[20] L. J. Xie, A New Modification of a Domain Decomposition Method for Volterra Integral Equations of the Second Kind. Journal of Applied Mathematics, (2013).
[21] H. H., Ali, &F. Abdelwahid, Modified A Domain Techniques Applied to Non-Linear Volterra Integral Equations. (2013).
[22] A. Jerri, Introduction to Integral Equations with Applications, Wiley, New York, (1999).
[23] R.K. Miller, Nonlinear Volterra Integral Equations, W.A. Benjamin, Menlo Park, CA, (1967).
[24] H. Brunner, On the Numerical Solution of Nonlinear Volterral–Fredholm Integral Equation bBy Collocation Methods, SIAM J. Numer. Anal. 27 (4) (1990) 987–1000.
[25] L. Hacia, On Approximate Solution for Integral Equations of Mixed Type, ZAMM. Z. Angew. Math. Mech. 76 (1996) 415–416.