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Original Article

Oscillatory Behavior of Forth Order Mixed Neutral Delay Difference Equations

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Abstract *- This paper is concerned with the forth Order mixed neutral delay difference equation of the form*

$$\Delta\left(a_\xi \Delta^2\left(d_\xi \Delta\left(y_\xi + b_\xi y_{\xi-\mu_1} + c_\xi y_{\xi+\mu_2}\right)\right)\right) + q_\xi y^{\varsigma}_{\xi+1-\varphi_1} + p_\xi y^{\eta}_{\xi+1+\varphi_2} = 0,$$

we obtain some new oscillation criteria by using riccati transformation technique. Examples are given to illustrate the results.

Keywords *- Difference equation, Oscillation, Nonoscillation, Mixed type neutral delay difference equation.*

I. INTRODUCTION

Consider the oscillation for certain forth Order neutral delay difference equation

$$\Delta\left(a_\xi \Delta^2\left(d_\xi \Delta\left(y_\xi + b_\xi y_{\xi-\mu_1} + c_\xi y_{\xi+\mu_2}\right)\right)\right) + q_\xi y^{\varsigma}_{\xi+1-\varphi_1} + p_\xi y^{\eta}_{\xi+1+\varphi_2} = 0, \tag{1.1}$$

where $\xi_0 \in N = \{\xi_0, \xi_0 + 1, \ldots\}$ ξ_0 - is nonnegative integer. Here $\varphi_1, \varphi_2, \mu_1$ and μ_2 are nonnegative integers and Δ is forward difference operator. $\Delta y_\xi = y_{\xi+1} - y_\xi$. Throughout this paper the following conditions are assumed to hold:

[H1] $\{a_\xi\}$ and $\{d_\xi\}$ are positive nondecreasing sequences and $\sum_{\xi=\xi_0}^{\infty}\frac{1}{a_\xi} = \sum_{\xi=\xi_0}^{\infty}\frac{1}{d_\xi} = \infty.$

[H2] $\{b_\xi\}$ and $\{c_\xi\}$ are positive real sequences such as $0 \le b_\xi \le b$ and $0 \le c_\xi \le c$ with $b + c < 1$.

[H3] $\{p_\xi\}$ and $\{q_\xi\}$ are real positive sequences.

[H4] ς, η are positive integers. μ_1, μ_2, φ_1 and φ_2 are nonnegative integers. For the basic theory of difference equations one can refer the monographs by Agarwal, Bohner and Grace [1]. The oscillation solution for third Order and higher Order difference equations [2, 3, 4, 5, 6, 8, 9, 10, 11, 12,13] has recaused more attention in the last few years. Let $\sigma = \max\{\mu_1, \varphi_1\}$. A solution of equation (1.1) we mean a real sequence $\{y_\xi\}$ which is defined for all $\xi \ge \xi_0 - \sigma$ and satisfying equation (1.1) for all $\xi \in N$. A solution $\{y_\xi\}$ is said to be oscillatory. If it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. Recently Kaleeswari [7] deals with oscillation for third Order difference equation of the form

$$\Delta\left(a_n\Delta^2\left(x_n+b_nx_{n-\tau_1}+c_nx_{n+\tau_2}\right)\right)+q_nx^{\beta}_{n+1-\sigma_1}+p_nx^{\beta}_{n+1+\sigma_2}=0$$

and discussed some oscillatory properties by assuming $\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty.$ Our aim in this paper is to discuss the oscillatory behavior of fourth Order difference equation when $\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty$ and $\sum_{n=n_0}^{\infty}\frac{1}{d_n}=\infty.$

So, the author is concerned fourth Order mixed neutral delay difference equation of the form

$$\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta\left(y_\xi+b_\xi y_{\xi-\mu_1}+c_\xi y_{\xi+\mu_2}\right)\right)\right)+q_\xi y^{\varsigma}_{\xi+1-\varphi_1}+p_\xi y^{\eta}_{\xi+1+\varphi_2}=0,$$

where $\sum_{\xi=\xi_0}^{\infty}\frac{1}{a_\xi}=\sum_{\xi=\xi_0}^{\infty}\frac{1}{d_\xi}=\infty.$

II. OSCILLATION RESULTS

In this section, we present some new oscillation criteria for equation (1.1) will be established. For simplicity, we use the following notations:

$$z_\xi=y_\xi+b_\xi y_{\xi-\mu_1}+c_\xi y_{\xi+\mu_2},\ P_\xi+Q_\xi=R_\xi,\ Q_\xi=\min\left\{q_\xi,q_{\xi-\mu_1},q_{\xi+\mu_2}\right\},\ P_\xi=\min\left\{p_\xi,p_{\xi-\mu_1},p_{\xi+\mu_2}\right\}.$$

We need the following lemma to prove the main results.

Lemma 2.1.

Assume $A\geq 0$ and $B\geq 0,\ \beta\geq 1.$ Then

$$\left(A+B\right)^{\beta}\leq 2^{\beta-1}\left(A^{\beta}+B^{\beta}\right).$$

The proof of lemma is simple and it is omitted.

Lemma 2.2. Let $\{y_\xi\}$ be a positive solution of equation (1.1). Then there are two cases for $\xi\geq\xi_1\in N$ sufficiently large n.

$$1)\ z_\xi>0,\Delta z_\xi>0,\Delta\left(d_\xi\Delta z_\xi\right)>0,\Delta^2\left(d_\xi\Delta z_\xi\right)>0,\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)\leq 0.$$

$$2)\ z_\xi>0,\Delta z_\xi<0,\Delta\left(d_\xi\Delta z_\xi\right)>0,\Delta^2\left(d_\xi\Delta z_\xi\right)>0,\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)\leq 0.$$

Proof.

Let $\{y_\xi\}$ be a positive solution of (1.1). Then there is an integer $\xi_1\geq\xi_0$ such that $y_\xi>0, y_{\xi-\mu_1}>0,$ $x_{\xi+\mu_2}>0, y_{\xi-\varphi_1}>0$ and $y_{\xi+\varphi_2}>0$ for all $\xi\geq\xi_1.$ Then $z_\xi>0$ for all $\xi\geq\xi_1.$ It follows from equation (1.1) that

$$\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)=-q_\xi y^{\varsigma}_{\xi+1-\varphi_1}-p_\xi y^{\eta}_{\xi+1+\varphi_2}<0;\qquad \xi\geq\xi_1 \tag{2.1}$$

Therefore $a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)$ is strictly decreasing for all $\xi\geq\xi_1$. We can proved that $\Delta^2\left(d_\xi\Delta z_\xi\right)>0$ for all $\xi\geq\xi_1$. If not, then there is an integer $\xi_2\geq\xi_1$ and $G<0$ such that

$$a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\leq a_{\xi_2}\Delta^2\left(d_{\xi_2}\Delta z_{\xi_2}\right)\leq G,\ \xi\geq\xi_2.$$

Summing the last inequality from ξ_2 to $\xi-1,$ we get

$$\sum_{s=\xi_2}^{\xi-1}\Delta^2\left(d_s\Delta z_s\right)\leq\sum_{s=\xi_2}^{\xi-1}\frac{1}{a_s}a_{\xi_2}\Delta^2\left(d_{\xi_2}\Delta z_{\xi_2}\right)$$

$$\Delta\left(d_\xi \Delta z_\xi\right) \leq \Delta\left(d_{\xi_2} \Delta z_{\xi_2}\right) G \sum_{s=\xi_2}^{\xi-1} \frac{1}{a_s}.$$

Letting $\xi \to \infty,$ then $\Delta\left(d_\xi \Delta z_\xi\right) \to -\infty.$ Then there exist an integer $\xi_3 \geq \xi_2$ and $L < 0$ such that

$$d_\xi \Delta z_\xi \leq d_{\xi_3} \Delta z_{\xi_3} \leq L; \qquad \xi \geq \xi_3.$$

Summing the last inequality from ξ_3 to $\xi - 1,$ we have

$$\sum_{s=\xi_3}^{\xi-1} \Delta z_s \leq \sum_{s=\xi_3}^{\xi-1} \frac{1}{d_s} d_{\xi_3} \Delta z_{\xi_3},$$

$$z_\xi \leq z_{\xi_3} + L \sum_{s=\xi_3}^{\xi-1} \frac{1}{d_s}.$$

Letting $n \to \infty,$ then $z_\xi \to -\infty,$ which is contradiction. Hence $\Delta^2\left(d_\xi \Delta z_\xi\right) > 0$ for $\xi \geq \xi_1.$

Lemma 2.3.

Let $z_\xi > 0,$ $\Delta z_\xi > 0,$ $\Delta^2 z_\xi > 0,$ $\Delta^3 z_\xi > 0$ and $\Delta^4 z_\xi \leq 0$ for all $n \geq m \in N.$ Then for any $k \in (0,1)$ and for some integer $m_1.$

$$\frac{z_{\xi+1}}{\Delta z_\xi} \geq \left(\frac{n-m}{2}\right) \geq \frac{k\xi}{2}. \tag{2.2}$$

Proof.

Since $\Delta z_\xi = \Delta z_m + \sum_{s=m}^{n-1} \Delta^2 z_s,$ we have $\Delta z_\xi \geq (\xi - m)\Delta^2 z_\xi.$

Summing the last inequality

$$\sum_{s=m}^{\xi-1} \Delta z_s \geq \sum_{s=m}^{n-1} (n-m)\Delta^2 z_s$$

$$z_\xi \geq z_m + (\xi - m)\Delta z_\xi - z_\xi + z_m$$

$$\text{(or)} \quad 2z_\xi \geq 2z_m \Delta z_\xi$$

$$z_{\xi+1} \geq \left(\frac{\xi - m}{2}\right)\Delta z_\xi \geq \frac{k\xi}{2}\Delta z_\xi; \qquad \xi \geq m_1 \geq m.$$

The proof is now complete.

Theorem 2.4.

Assume that there exist a positive real sequence $\{\rho_\xi\}$ and $\varphi_1 \geq \mu_1,\ \varsigma \leq \eta$ and $\varsigma, \eta \geq 1$ holds. If

$$\sum_{s=N}^{\xi-1}\left(\frac{\rho_\xi R_\xi h_\xi^{(\varsigma+\eta)-1}k(\xi-\varphi_1)}{2^{\varsigma+\eta}}+\frac{\left(1+b^\eta+\frac{c^\eta}{2^{\eta-1}}\right)\left(\Delta\rho_\xi^2\right)a_{\xi-\varphi_1}}{4\rho_\xi}\right)=\infty, \tag{2.3}$$

$$\sum_{s=\xi_1}^{\infty}\frac{1}{a_s}\sum_{t=s}^{\infty}(p_t+q_t)=\infty, \tag{2.4}$$

holds, then every solution $\{y_\xi\}$ of equation (1.1) oscillates or $\lim_{\xi\to\infty} y_\xi = 0.$

Proof.

Let $\{y_\xi\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists an integer $N \geq \xi_0$ such that $y_\xi > 0,\ y_{\xi-\varphi_1} > 0,\ y_{\xi+\varphi_2} > 0,\ y_{\xi-\mu_1} > 0$ and $y_{\xi+\mu_2} > 0$ for all $\xi \geq N.$ Then we have $z_\xi > 0$ and (2.1) for all $\xi \geq N.$ From (1.1) for all $\xi \geq N$ we have

$$\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)+q_\xi y_{\xi+1-\varphi_1}^{\varsigma}+p_\xi y_{\xi+1+\varphi_2}^{\eta}+b^\eta\left(\Delta a_{\xi-\mu_1}\Delta^2\left(d_{\xi-\mu_1}\Delta z_{\xi-\mu_1}\right)\right)+b^\eta\left(q_{\xi-\mu_1}y_{\xi+1-\mu_1-\varphi_1}^{\varsigma}\right)$$

$$+b^\eta p_{\xi-\mu_1}y_{\xi-\mu_1+\varphi_2+1}^{\eta}+\frac{c^\eta}{2^{\eta-1}}\left(\Delta a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta z_{\xi+\mu_2}\right)\right)+\frac{c^\eta}{2^{\eta-1}}q_{\xi+\mu_2}y_{\xi+1+\mu_2-\varphi_1}^{\varsigma}+\frac{c^\eta}{2^{\eta-1}}p_{\xi+\mu_2}y_{\xi+1+\mu_2+\varphi_2}^{\eta}=0 \tag{2.5}$$

Using lemma 2.1 in (2.5), we have

$$\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)+b^\eta\Delta\left(a_{\xi-\mu_1}\Delta^2\left(d_{\xi-\mu_1}\Delta z_{\xi-\mu_1}\right)\right)+\frac{c^\eta}{2^{\eta-1}}\Delta\left(a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta\left(z_{\xi+\mu_2}\right)\right)\right)$$

$$+\frac{Q_\xi}{4^{\varsigma-1}}z_{\xi+1-\varphi_1}^{\varsigma}+\frac{P_\xi}{4^{\eta-1}}z_{\xi+1+\varphi_2}^{\eta}\leq 0. \tag{2.6}$$

By lemma 2.2, there are two cases for $z_\xi.$ First assume that case 1 holds for all $\xi \geq N_1 \geq N.$ It follows from $\Delta z_\xi > 0$ that $z_{\xi+1} - z_\xi > 0$ then $z_{\xi+\varphi_2} \geq z_{\xi-\varphi_1}.$ Thus, by (2.6) we obtain

$$\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)+b^\eta\Delta\left(a_{\xi-\mu_1}\Delta^2\left(d_{\xi-\mu_1}\Delta z_{\xi-\mu_1}\right)\right)+\frac{c^\eta}{2^{\eta-1}}\Delta\left(a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta z_{\xi+\mu_2}\right)\right)+\frac{R_\xi}{4^{\varsigma+\eta-2}}z_{\xi+1-\varphi_1}^{\varsigma+\eta}\leq 0. \tag{2.7}$$

Define

$$w_1(\xi)=\rho_\xi\frac{a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)}{\Delta\left(d_\xi\Delta\left(z_\xi-\varphi_1\right)\right)}. \tag{2.8}$$

Then $w_1(\xi) > 0$ for $\xi \geq N_1.$ Then from (2.8) we obtain

$$\Delta w_1(\xi)=\frac{\Delta\rho_\xi}{\rho_{\xi+1}}w_1(\xi+1)+\rho_\xi\frac{\Delta\left(a_\xi\Delta^2\left(d_\xi\Delta z_\xi\right)\right)}{\Delta\left(d_\xi\Delta z_{\xi-\varphi_1}\right)}-w_1(\xi+1)\frac{\Delta^2\left(d_\xi\Delta z_{\xi-\varphi_1}\right)}{\Delta\left(d_\xi\Delta z_{\xi-\varphi_1}\right)}.$$

By equation (2.1), we have $a_{\xi-\varphi_1}\Delta^2\left(d_{\xi-\varphi_1}\Delta z_{\xi-\varphi_1}\right)\geq a_{\xi+1}\Delta^2\left(d_{\xi+1}\Delta z_{\xi+1}\right).$ Thus from (2.8), we obtain

$$\Delta w_1(\xi) \le \frac{\Delta\rho_\xi}{\rho_{\xi+1}} w_1(\xi+1) + \rho_\xi \frac{\Delta\left(a_\xi \Delta^2\left(d_\xi \Delta z_\xi\right)\right)}{\Delta\left(d_\xi \Delta z_\xi\right)} - \frac{w_1(\xi+1)^2 \rho_\xi}{a_{\xi-\varphi_1}\left(\rho_{\xi+1}\right)^2}. \tag{2.9}$$

Next we define

$$w_2(\xi) = \rho_\xi \frac{a_{\xi-\mu_1}\Delta^2\left(d_{\xi-\mu_1}\Delta z_{\xi-\mu_1}\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)} \tag{2.10}$$

Then $w_2(\xi) > 0$ for $\xi \ge N_1$. Then from (2.10), we obtain

$$\Delta w_2(\xi) = \frac{w_2(\xi+1)}{\rho_{\xi+1}} \Delta\rho_\xi + \rho_\xi \frac{\Delta\left(a_{\xi-\mu_1}\Delta^2\left(d_{\xi-\mu_1}\Delta z_{\xi-\mu_1}\right)\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)} - w_2(\xi+1)\frac{\Delta^2\left(d_\xi \Delta z_{\xi-\varphi_1}\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)}.$$

By equation (2.1) and $\varphi_1 \ge \mu_1$ we have

$$a_{\xi-\varphi_1}\Delta^2\left(d_{\xi-\varphi_1}\Delta z_{\xi-\varphi_1}\right) \ge a_{\xi+1-\mu_1}\Delta^2\left(d_{\xi+1-\mu_1}\left(\Delta z_{\xi+1-\mu_1}\right)\right).$$

Thus from (2.10), we get

$$\Delta w_2(\xi) \le \frac{\Delta\rho_\xi}{\rho_{\xi+1}} w_2(\xi+1) + \rho_\xi \frac{\Delta\left(a_{\xi-\mu_1}\Delta^2\left(d_{\xi-\mu_1}\Delta z_{\xi-\mu_1}\right)\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)} - \frac{w_2^2(\xi+1)\rho_\xi}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}. \tag{2.11}$$

In the following we define

$$w_3(\xi) = \rho_\xi \frac{a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta z_{\xi+\mu_2}\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)}. \tag{2.12}$$

Then $w_3(\xi) > 0$ for $\xi \ge N_1$. From (2.12) we obtain

$$\Delta w_3(\xi) = \frac{w_3(\xi+1)}{\rho_{\xi+1}} \Delta\rho_\xi + \frac{\rho_\xi \Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)\Delta\left(a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta z_{\xi+\mu_2}\right)\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)\Delta\left(d_{\xi+1}\Delta z_{\xi+1-\varphi_1}\right)}$$
$$- \frac{\rho_\xi a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta z_{\xi+\mu_2}\right)\Delta^2\left(d_\xi \Delta z_{\xi-\varphi_1}\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)\Delta\left(d_{\xi+1}\Delta z_{\xi+1-\varphi_1}\right)}.$$

By equation (2.1), we obtain $a_{\xi-\varphi_1}\Delta^2\left(d_{\xi-\varphi_1}\Delta z_{\xi-\varphi_1}\right) \ge a_{\xi+1+\mu_2}\Delta^2\left(d_{\xi+1+\mu_2}\Delta z_{\xi+1+\mu_2}\right)$. Hence by (2.12) we obtain

$$\Delta w_3(\xi) \le \frac{\Delta\rho_\xi}{\rho_{\xi+1}} w_3(\xi+1) + \rho_\xi \frac{\Delta\left(a_{\xi+\mu_2}\Delta^2\left(d_{\xi+\mu_2}\Delta z_{\xi+\mu_2}\right)\right)}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)} - \frac{w_3^2(\xi+1)\rho_\xi}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}$$

(2.13)

Therefore (2.9), (2.11) and (2.13), we obtain

$$\Delta w_1(\xi) + b^\eta \Delta w_2(\xi) + \frac{c^\eta}{2^{\eta-1}}\Delta w_3(\xi) \le \frac{-\rho_\xi R_\xi}{4^{\zeta+\eta-2}} \frac{z_{\xi+1-\varphi_1}^{\zeta+\eta}}{\Delta\left(d_\xi \Delta z_{\xi-\varphi_1}\right)} + \frac{\Delta\rho_\xi}{\rho_{\xi+1}} w_1(\xi+1) - \frac{w_1^2(\xi+1)\rho_\xi}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}$$
$$+ b^\eta\left(\frac{\Delta\rho_\xi}{\rho_{\xi+1}} w_2(\xi+1) - \frac{w_2^2(\xi+1)\rho_\xi}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}\right)$$

$$+\frac{c^{\eta}}{2^{\eta-1}}\left(\frac{\Delta\rho_{\xi}}{\rho_{\xi+1}}w_3(\xi+1)-\frac{w_3^2(\xi+1)\rho_{\xi}}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}\right). \tag{2.14}$$

On the other hand $\{a_{\xi}\}$ and $\{d_{\xi}\}$ nondecreasing $\Delta^3 z_{\xi}>0$ for $\xi\geq m_1$ we have $\Delta^4 z_{\xi}\leq 0$ for $\xi\geq m_1$. Then by lemma 2.3 for any $k\in(0,1)$ and ξ is sufficiently large

$$\frac{z_{\xi+1-\varphi_1}}{\Delta z_{\xi-\varphi_1}}\geq\frac{k(\xi-\varphi_1)}{2}. \tag{2.15}$$

Due to (2.2) . Since $z_{\xi}>0,\ \Delta z_{\xi}>0,\ \Delta^2 z_{\xi}>0$ and $\Delta^3 z_{\xi}>0$ for $\xi\geq m_1$ we have

$$z_{\xi}=z_{m_1}+\sum_{s=m_1}^{\xi-1}\Delta z_s\geq(\xi-m_1)\Delta z_{m_1}\geq\frac{h\xi}{2} \tag{2.16}$$

for some $h>0$ and ξ is sufficiently large. From (2.15) and (2.16) and $\zeta,\eta\geq 1$ we have

$$\frac{z_{\xi+1-\varphi_1}^{\zeta-\eta}}{\Delta z_{\xi-\varphi_1}}\geq\frac{h_{\xi}^{(\zeta+\eta)-1}k(\xi-\varphi_1)}{2^{\zeta+\eta}}.$$

(2.14) becomes

$$\Delta w_1(\xi)+b^{\eta}\Delta w_2(\xi)+\frac{c^{\eta}}{2^{\eta-1}}\Delta w_3(\xi)\leq-\frac{\rho_{\xi}R_{\xi}h_{\xi}^{(\zeta+\eta)-1}k(\xi-\varphi_1)}{2^{\zeta+\eta}}+\frac{\Delta\rho_{\xi}}{\rho_{\xi+1}}w_1(\xi+1)-\frac{w_1^2(\xi+1)\rho_{\xi}}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}$$
$$+b^{\eta}\left(\frac{\Delta\rho_{\xi}}{\rho_{\xi+1}}w_2(\xi+1)-\frac{w_2^2(\xi+1)\rho_{\xi}}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}\right)$$
$$+\frac{c^{\eta}}{2^{\eta-1}}\left(\frac{\Delta\rho_{\xi}}{\rho_{\xi+1}}w_3(\xi+1)-\frac{w_3^2(\xi+1)\rho_{\xi}}{\rho_{\xi+1}^2 a_{\xi-\varphi_1}}\right).$$

By using completing the square in the right hand side of the above inequality, we get

$$\Delta w_1(\xi)+b^{\eta}\Delta w_2(\xi)+\frac{c^{\eta}}{2^{\eta-1}}\Delta w_3(\xi)\leq-\frac{\rho_{\xi}R_{\xi}h_{\xi}^{(\zeta+\eta)-1}k(\xi-\varphi_1)}{2^{\zeta+\eta}}+\frac{\left(1+b^{\eta}+\frac{c^{\eta}}{2^{\eta-1}}\right)(\Delta\rho_{\xi})^2 a_{\xi-\varphi_1}}{4\rho_{\xi}}$$

Summing the last inequality from $N_2\geq N_1$ to $\xi-1$, we obtain

$$\sum_{s=N_2}^{\xi-1}\left(\frac{\rho_{\xi}R_{\xi}h_{\xi}^{(\zeta+\eta)-1}k(\xi-\varphi_1)}{2^{\zeta+\eta}}+\frac{\left(1+b^{\eta}+\frac{c^{\eta}}{2^{\eta-1}}\right)(\Delta\rho_{\xi})^2 a_{\xi-\varphi_1}}{4\rho_{\xi}}\right)\leq w_1(N_2)+b^{\eta}w_2(N_2)+\frac{c^{\eta}}{2^{\eta-1}}w_3(N_2).$$

Taking limsup in the last inequality, we get a contradiction to (2.3). Assume that lemma 2.2(2) holds. Let $\{y_{\xi}\}$ be a positive solution of equation (1.1). Since $z_{\xi}>0$ and $\Delta z_{\xi}<0$, then $\lim_{\xi\to\infty}z_{\xi}=l>0$ exists. We shall prove that

$l = 0.$ Assume $l > 0$ then for any $\varepsilon > 0,$ we have $l + \varepsilon > z_\xi$ eventually. Choose $0 < \varepsilon < \frac{l(l-b-c)}{b+c}.$ It is easy to verify that

$$y_\xi > l - (b+c)l + \varepsilon > kz_{\xi.}$$

Where $k = \frac{l-(b+c)l+\varepsilon}{l+\varepsilon} > 0.$ Using the above inequality, we obtain from (2.1)

$$\Delta\left(a_\xi \Delta^2\left(d_\xi \Delta z_\xi\right)\right) \le -k^{\zeta+\eta}\left(q_\xi + p_\xi\right) z^{\zeta+\eta}_{\xi+1-\mu_1}.$$

Summing the last inequality from ξ to ∞ and using $z_\xi > l,$ we obtain

$$\sum_{s=\xi}^{\infty} \Delta\left(a_\xi \Delta^2\left(d_\xi \Delta z_\xi\right)\right) \le -\sum_{s=\xi}^{\infty} k^{\zeta+\eta}\left(q_\xi + p_\xi\right) z^{\zeta+\eta}_{\xi+1-\mu_1}$$

$$\Delta^2\left(d_\xi \Delta z_\xi\right) \ge (kl)^{\zeta+\eta} \frac{1}{a_\xi} \sum_{s=\xi}^{\infty}\left(q_\xi + p_\xi\right).$$

Summing again ξ_1 to $\infty,$ $\xi \ge \xi_1$ we obtain

$$\sum_{s=\xi_1}^{\infty} \Delta^2\left(d_s \Delta z_s\right) \ge \sum_{s=\xi_1}^{\infty} (kl)^{\zeta+\eta} \frac{1}{a_s} \sum_{t=s}^{\infty}\left(p_t + q_t\right)$$

$$-d_{\xi_1} \Delta z_{\xi_1} \le -(kl)^{\zeta+\eta} \sum_{s=\xi_1}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty}\left(p_t + q_t\right)$$

$$-z_{\xi_1+1} \le -\frac{(kl)^{\zeta+\eta}}{d_{\xi_1}} \sum_{s=\xi_1}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty}\left(p_t + q_t\right)$$

$$\sum_{s=\xi_1}^{\infty} \frac{1}{a_s} \sum_{t=s}^{\infty}\left(p_t + q_t\right) \le z_{\xi_1+1}.$$

This contradicts to (2.4). So the proof is complete.

III. APPLICATIONS

Example 3.1.

Consider the forth Order mixed neutral type difference equation of the form

$$\Delta\left(\xi^2 \Delta^3\left(y_\xi + \frac{1}{4} y_{\xi-1} + \frac{1}{2} y_{\xi+2}\right)\right) + 2\xi^3 y_{\xi-1} + (2\xi+2) y_{\xi+1} = 0. \tag{3.1}$$

Let $a_\xi = \xi^2, b_\xi = \frac{1}{4}, c_\xi = \frac{1}{2}, d_\xi = \mu_1 = 1, \mu_2 = 2, p_\xi = 2\xi + 2, q_\xi = 2\xi^3, \zeta = \eta = 1, \varphi_1 = 2, \varphi_2 = 1.$ Take $\rho_\xi = 1.$ Then condition (2.3) holds. On the other hand, condition (2.4) also holds. We can easily see that the conditions of Theorem 2.4 are satisfied. Hence all the solutions of equation (3.1) are oscillatory. In fact $\{y_\xi\} = (-1)^\xi$ is one such a solution of equation (3.1).

Example 3.2.

Consider the forth Order mixed neutral type difference equation of the form

$$\Delta\left(\frac{\xi}{2}\Delta^2\left(2\xi\Delta\left(y_\xi+\frac{2}{3}y_{\xi-2}+\frac{1}{4}y_{\xi+3}\right)\right)\right)+\left(\xi^3+2\right)y_\xi+2\xi^2y_{\xi+1}^2=0. \qquad (3.2)$$

Let $a_\xi=\frac{\xi}{2}, b_\xi=\frac{2}{3}, c_\xi=\frac{1}{4}, d_\xi=2\xi, \mu_1=2, \mu_2=3, p_\xi=2\xi^2, q_\xi=\xi^3+2, \zeta=1, \eta=2, \varphi_1=1.$ Take condition (2.3) holds. On the other hand condition (2.4) also holds. We can easily see that the conditions of Theorem 2.4 are satisfied. Hence all the solutions of equation (3.2) are oscillatory.

In fact $\{y_\xi\}=(-1)^\xi$ is one such a solution of equation (3.2).

CONCLUSION

In this paper, by using Ricatti type transformation and the summing averaging technique, the oscillatory behavior of every solution of the equation (1.1) are discussed in Theorem 2.4. Here some sufficient conditions are proved. These sufficient conditions which are new, extend and complement some of the known results in the literature. Also the example reveals the illustration of the proved results.

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