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Original Article

Comparisons of 2^3 Factorial Designs by Frequentist and Bayesian Approach

R. Vijayaragunathan[1], S. Kanaga Vel[2], M. R. Srinivasan[3]

[1]*Department of Statistics, Indira Gandhi College of Arts and Science, Puducherry, India.*
[2]*Department of Mathematics, Perunthalaivar Kamarajar Arts College, Puducherry, India.*
[3]*School of Mathematical Sciences, University of Hyderabad, Hyderabad, India.*

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Abstract - *In this study, we discussed the effect of factors in full factorial and fractional factorial designs, also we considered reduced factorial design which consists of significant factors alone. Sometimes, the experimenter wants to know/get additional information than the fractional factorial design if there is no restriction in the experimental run. The Bayes factors are used and found to identify and quantify the original weightage of the main/interaction effects in these three designs through the simulation datasets.*

Keywords - *Bayes Factor, Jeffreys-Zellner-Siow Prior, 2^3 full factorial design, Fractional and reduced factorial designs.*

I. INTRODUCTION

In a 2^p factorial design, as the number of factors increases then the number of trials required for a full replicate of the design rapidly increases in the experiments. In such cases, we cannot perform a full replicate of the design and in that situation, a fractional factorial design has to be run. Suppose certain interactions involving a large number of factors are negligible, information on the lower order effects can be obtained by running a suitable fraction of the 2^p full factorial design. Two-level fractional factorial designs are broadly divided into regular and non-regular fractional factorial designs are discussed (Tang and Deng, 1999). Statisticians have designated fractional factorial experiments to reduce the number of runs or trials, only selected treatment combinations are tried instead of all combinations. A fractional factorial design employs a systematic approach to reduce the number of experimental conditions to allow meaningful study. To run-size economy and be cost-effective, we use fractional factorial designs, which are widely applied in various fields such as engineering, industrial and scientific researches. The higher-order interactions are confounded, or aliased, with lower-order effects such that negligible in size in the fractional factorial designs. The experimenters have found that higher-order interactions of three or more factors tend to be small and can be ignored often. Thus we decide to omit three-factor interaction from the analysis

a) 2^3 *Full factorial design*

We consider three factors A, B, and C each at two levels, so that there are eight treatment combinations. The standard order of treatment combinations is 1, A, B, AB, C, AC, BC, and ABC. In this design, we have three main effects, three first-order interaction effects, and one second-order effect. These seven effects are mutually orthogonal contrasts of the treatment means. Thus, the 2^3 full factorial design with *n* replications model is

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\beta\gamma)_{jk} + (\alpha\gamma)_{ik} + (\alpha\beta\gamma)_{ijk} + \rho_l + \epsilon_{ijkl} \quad (1)$$

To estimate the average of the main effect of factor *A, B and C are* in a 2^3 factorial design replicated *n* times, we use the following formula:

$$A = \frac{1}{4n}[abc + ab + ac + a - bc - b - c - 1]$$
$$B = \frac{1}{4n}[abc + ab + bc + b - ac - a - c - 1]$$
$$C = \frac{1}{4n}[abc + ab + bc + c - ab - a - b - 1]$$

The interaction effect of *AB, AC* and *BC are*

$$AB = \frac{1}{4n}[abc + ab + c + 1 - bc - ac - a - b]$$
$$AC = \frac{1}{4n}[abc + ac + b + 1 - ab - bc - c - a]$$
$$BC = \frac{1}{4n}[abc + bc + a + 1 - ab - ac - b - c]$$

and the average difference between the interaction AB with the two different levels of C is the effect of interaction ABC

$$ABC = \frac{1}{4n}[abc + a + b + c - ab - bc - ac - 1]$$

Finally, based on the ANOVA table for 2^3 full factorial designs, we can identify the significance of the main and interaction effects.

b) $\mathbf{2^{3-1}}$*Fractional factorial design*

Suppose, the factor A and BC, B and AC, and C and AB have identical signs to each other. The calculation of effect also would be the same as the effect is calculated as the difference between average response at a high level and low level of the factor. Thus, we will be unable to distinguish between the effect of A and BC being confounded, A and BC are called aliases. Based on the extent of confounding, experiments are denoted by their resolution codes. The resolution of a design indicates its power and ability to separately estimate the effects of the factors and interactions. In general, the resolution of a design is one more than the smallest order interaction that some main effects is confounded with. If some main effects are confounded with two-factor interactions, the resolution would be 3. The resolution 2 designs do not exist as it will imply confounding of main effects. It is customary to write these codes in Roman letters. Resolutions III, IV and V designs are popular (Montgomery 2017). Therefore, the 2^{3-1} fractional factorial design is designated as 2_{III}^{3-1} and the model becomes

$$y_{ijkl} = \alpha_i + \beta_j + \gamma_k + (\alpha\beta\gamma)_{ijk} + \rho_l + \varepsilon_{ijkl} \tag{2}$$

The average effects of these factors in this half-fraction factorial design are determined by

$$A = \tfrac{1}{2}[abc - b - c + a]\ ,\ \mathrm{B} = \tfrac{1}{2}[\text{abc-b-c} + \mathrm{b}]\ \text{ and } \mathrm{C} = \tfrac{1}{2}[\text{abc-a-b} + \mathrm{c}]$$

In terms of confounding, $2_{\mathrm{III}}^{3\text{-}1}$ fractional replicate designs can be estimated main effects, but they are confounded with two-factor interactions then based on the ANOVA table for 2_{III}^{3-1} factorial design can identify the significance of the main and interaction effects. The minimizing aberration in a design of resolution III ensures that the design has the minimum number of main effects aliased with the first-order interaction effects. Firstly, we compare the ANOVA output of full, reduced and fractional factorial designs in a classical approach. Further, we performed the same comparisons in a Bayesian approach to identify the main and interaction effects in the full, reduced and fractional factorial designs. To avoid the undisputed conclusions, we used Jeffreys-Zellner-Siow prior to find the Bayes factor values. Finally, we used simulation datasets to find the Bayes factors to generalize an instructive conclusion.

c) 2^3 Reduced factorial design

Suppose, the number of significant factors in the full factorial design is more than the factors in the half-fraction factorial design, our choice may be a reduced factorial design. Our idea is to build a reduced factorial design with all significant factors alone. We cannot predetermine before we do the full factorial design. Suppose, the experimenter decides never to lose any kind of information from all the factors in the experiment in the future study this reduced factorial design will be useful and informative. This screening design is preferable if there is no constraint or deliberately wants to the experimenter, for taking into account of adding all the main and interaction factors except the non-significant factor(s). After identifying the significant factors in the full factorial design, then demonstrate this reduced factorial design, as usual, did in the full and fractional factorial design. Also, we used this as a tool to the decision of whether choosing the fractional factorial design is effective or not. If the reduced factorial design gave better results than fractional factorial design then the experimenter may think of this reduced factorial design.

II. PRIOR AND BAYES FACTOR

In this study, we used Jeffreys-Zellner-Siow prior to find the Bayes factors for full, reduced and fractional factorial designs. This prior is considered in the comparison of hierarchical two-way ANOVA models (Vijayaragunathan and Srinivasan 2021).

A. Jeffreys-Zellner-Siow Prior

Jeffreys-Zellner-Siow (JZS) prior is a mixture of priors we estimate g from the data, (Liang et al. 2008). The Bayes Factor for the full model to the null model is

$$BF = \frac{(n/2)^{1/2}}{\Gamma(1/2)} \int_0^\infty (1+g)^{(n-k-1)/2} \left[1+g(1-R^2)\right]^{-(n-1)/2} g^{-3/2} e^{-n/2g} dg \tag{3}$$

III. APPLICATION OF 2^3 FACTORIAL DESIGN

For example, the result of a 2^3 factorial design runs replicates four times. The purpose of the experiment is to determine the effect of different kinds of fertilizers Nitrogen (N), Potash (K) and Phosphate (P) each at two levels on potato crop yield. The ANOVA table for full factorial design is shown in Table 1, all main effects (N, P, K) and one first-order interaction effect (PK) are significant, other interaction effects are not significant. Now, our attention is on the factorial design which includes significant effects only. Thus, we go for a reduced factorial design with significant effects alone. By comparing the full and reduced factorial design we can identify the effect of non-significant factors in the full factorial design. In the 2^3 reduced factorial design, all effects are significant and the interaction effects PK is not in the ANOVA Table 3. If the design matrix $\mathbb{X}$, then $(X'X)$ and its inverse are no longer diagonal matrix, which means that the effect estimate is no longer orthogonal.

Table 1. ANOVA output for 2^3 factorial design

Source of Variation	Df	Sum Sq.	Mean Sq.	F value	P- value	
Block	3	742	247	0.727	0.547	
N	1	3828	3828	11.252	0.003	**
P	1	275653	275653	810.220	<2e-16	***
K	1	158766	158766	466.657	7.99e-16	***
NP	1	990	990	2.910	0.103	
NK	1	465	465	1.367	0.255	
PK	1	14706	14706	43.225	1.64e-6	***
NPK	1	66	66	0.194	0.664	
Residuals	21	7145	340			
Signif. Codes: 0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘’ 1						

Table 2. ANOVA output for 2^{3-1}_{III} fractional factorial design

Source of Variation	Df	Sum Sq.	Mean Sq.	F value	P- value	
Block	3	778	259	1.58	0.26107	
N	1	1764	1764	10.75	0.00956	**
P	1	149382	149382	910.10	2.36e-10	***
K	1	92416	92416	563.03	2.00e-09	***
Residuals	9	1477	164			
Signif. Codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '' 1						

Now, we opt usual fractional factorial design for the full factorial design. The half-fraction factorial design with resolution III design consists of the factors N, P, K, and second-order interactions NPK. The 2^{3-1}_{III} fractional factorial design ANOVA output in, Table 2, shows significant values for the factors as in the other two designs. By comparing the full, reduced, and fractional factorial design the level of significance are vary for the effects. Particularly, the main effect N is highly significant in reduced factorial design than full and fractional factorial design.

Table 3. ANOVA output for 2^3 reduced factorial design

Source of Variation	Df	Sum Sq.	Mean Sq.	F value	P- value	
Block	3	1481	497	3.481	0.0636	
N	1	160893	160893	1134.766	8.83e-11	***
P	1	20592	20592	145.235	7.42e-07	***
K	1	17391	17391	122.659	1.52e-06	***
Residuals	9	1276	142			
Signif. Codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '' 1						

Table 4. Bayes factor for full, reduced, and fractional factorial designs for actual data

Prior	Factorial Design		
	2^3 Full	2^{3-1} Fractional	2^3 Reduced
Jeffreys-Zellner-Siow (JZS)	18.1599	11.1800	8.8973

Now, the same comparisons would be made in a Bayesian framework also to check the factor effects for full, reduced, and fractional factorial designs. First, computed Bayes factor value for 2^3 full factorial, 2^3 reduced factorial, and 2^{3-1} fractional factorial models to the null model. The 2^3 reduced factorial and 2^{3-1} fractional factorial design is a nested model to the 2^3 full factorial design, so we may compare Bayes factor values to the two models. The Bayes Factor values for the actual data of 2^3 full, reduced and 2^{3-1} fractional factorial designs are in Table 4. The JZS prior provide almost the same results that data supports 18 times of full factorial design. But, in the fractionaland reduced factorial designs, the Bayes factor values are 11.18 and 8.8973 respectively. In the next section, we will discuss these designs for simulation datasets to check the behavior of the JZS prior for three types of designs.

A. Simulation for 2^3 full, Fractional and Reduced Factorial Designs

We have to simulate the datasets to the respective designs to obtain reliable conclusions. In this study, simulated 10,000 data with the error variance is 1 to compute 10,000 Bayes factor values for each of the Jeffreys-Zellner-Siow prior. In the same way, we computed Bayes factors for various datasets with the error variances 5, 25, and 50 respectively. The mean sum of squares due to error in the ANOVA result is huge, thus we took the large value of error variance to simulate datasets. This prior's Bayes factors for these simulated data to the full, reduced, and fractional factorial designs. The mean and standard deviation of Bayes factor values for full, reduced, and fractional factorial designs were presented in Table 5.

This depicts that the data supporting the full model is almost two times more than the fractional factorial model. Thus, we got more information in the full model than the reduced and half-fraction model. If the error variance is more, the Jeffreys-Zellner-Siow provided that the data support more or less double the times to the full model than the fractional model. But this prior is supporting data on full and fractional designs is slightly less (around one and half times) in original data and less error variance simulation data. The 2^3 reduced factorial design provides results close to the fractional factorial design. Furthermore, the reduced factorial design may produce the best result than fractional factorial design if more factors in reduced fractional design than fractional factorial design.

Table 5. Average (SD) of 10000 Bayes factor values to the simulation of a). 2^3 full factorial, b). 2^3reduced factorial and c). 2^{3-1} fractional factorial designs for JZS prior when the error variance σ_e^2 =1, 5, 25, and 50

Error Variance (σ_e^2)	2^3 Full factorial Design	2^{3-1}Fractional factorial Design	2^3 reduced Factorial Design
1	18.4674 (1.7211)	10.0869 (1.1334)	9.1568 (1.1461)
5	18.0553 (1.7117)	9.8160 (1.1165)	8.9099 (1.1259)
25	13.0440 (1.5516)	6.6238 (1.0389)	6.0521 (1.0415)
50	7.8885 (1.5542)	3.9790 (1.0368)	3.5020 (1.0397)

IV. CONCLUSION

For any factorial design we commence with full factorial design alone, but the number of factors is large then the size of the design is becoming very large. Alternative to the full factorial design is a fractional factorial design which can help reduce the number of runs for screening designs. Several fractional factorial designs exist, but the design resolution is very important. We considered an illustration for 2^3 factorial designs, to apply the Bayesian concept in the factorial designs to find the effects of the factors in the full, reduced, and fractional factorial designs. In the classical approach, we find the significance of factor from the ANOVA output and based on the results and aliased factors we formed a fractional factorial design. Furthermore, in this study, we introduced a new model known as reduced factorial design consisting of all significant factors. The Bayes Factor for Jeffreys-Zellner-Siow is computed for the 2^3 full, reduced and fractional factorial designs. It provided different results within the respective models. To generalize the Bayesian approach, we generated a large set of data by simulation with different error variances, it gives a wide range of ideas to compare the full, reduced and fractional factorial design with the existence of the factors. All the Bayes factor values are in the full model are almost two times as compared with the fractional factorial model. Thus, we lose half of the information in a fractional factorial design. In the simulation dataset with less error variance, the data support full, reduced, and fractional design, but if the error variance is high the simulation data support the null model. We conclude with, in this illustration, the mean square error value is huge then reduced and fractional factorial designs provides not much difference in results because both models has similar factors. Hence, the reduced factorial design has more significant factors it would express better results than fractional factorial design.

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and approved the final manuscript.

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