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*Original Article*

# An Extension of a Theorem of Herstein

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***Abstract*** *- Let R bearing with unity and satisfying certain conditions* $[x^m, y^m] = 0$ *and* $(xy)^n = (yx)^n$*, for all* $x, y \in R$*. In this paper, we extend a well known result.*

***Keywords*** *- Rings, Nil rings, Commutator, Hyper centre, Jacobson radical.*

## I. INTRODUCTION

Let R bean associative ring with unity with centre Z(R) and Jacobsonradical J(R). For any pair of x, y ring elements, a ring R is said to be commutative if and only if[x, y]=0, for all x, y ε R. In 1975, Herste in[10] introduced the concept of hyper centre. The hyper centreT(R) of aring R is the totality of all those elements of R which commute with some power of each element in R, the power may be localized in the sense that it may depend on the elements. Thus T(R) $=\{r \in R | rx^n = x^n r,$ wheren $=$ n(r, x) is a positive integer}. We see that Z(R)⊆T(R). There exist enough non commutative nil rings to show that in general T(R) need not coincide with Z(R). In 1976, Herste in [11] proved that a ring R in which given x, y ε R, there exist integers $m=m(x, y) \geq 1$,$n=n(x, y) \geq 1$ such that $[x^n, y^n]=0$. If in addition, R has nononzero nil ideals, then R must be commutative.

Motivated by these observations, one may conjecture that instead of torsion condition or absence of nil commutator ideal, some other constraints on the elements of R should also turn the ring commutative. Working on these lines, we extend this result imposing an additional condition on the underlying ring.

## II. MAIN RESULT

**Theorem**. Let R bearing with unity 1 in which the re exist positive integer s m and n satisfying
$(C_1)$ :$[x^m, y^m]=0$,for all $x, y \in R$.
$(C_2)$ : $[(xy)^n, (yx)^n]=0$,for all $x, y \in R$.
If in addition, integers m and n are relatively prime, then R must be commutative.

The following lemmas are required to prove our theorem:

**Lemma 2.1**[18], Lemma1]. If$[x, [x,y]]=0$, forall$x, y \in R$, then $[x^n,y]=nx^{n-1}[x,y]$ holds for every positive integer n.

**Lemma 2.2.** Let R be a ring with unity 1 and $f:R\to R$is a function such that $f(1 + x) = f(x)$.If there exists an integer $m=m(x) \geq 1$ such that $x^m f(x)=0$,thennecessarily$f(x)=0$.

***Proof.*** For the elements $x$ and $1+x$, there exist integers $m = m(x) \geq 1$ and $n = n(1+x) \geq 1$ such that

$$x^m f(x) = 0$$
$$(1+x)^n f(1+x) = 0 = (1+x)^n f(x)$$

If $N = \max(m,n)$, then we have

$$x^N f(x) = 0 \quad (2.1)$$

$$(1+x)^N f(1+x) = 0 = (1+x)^N f(x) \quad (2.2)$$

If $N=1$,then the result follows trivially. Suppose $N \geq 2$. We have

$$f(x) = [(1+x)-x]^{2N+1} f(x) = \{(1+x)^{2N+1} + C_1^{2N+1}(1+x)^{2N} + \cdots + (-1)^{2N+1} x^{2N+1}\} f(x)$$

= 0, by(2.1) and (2.2).

***Remark2.1.*** Notice that commutator function [$x$, $y$] satisfies the hypothesis of the lemma i.e.,[1+$x$, $y$]=[$x$, y] and so the above lemma can be restated as follows:

***Lemma2.3.*** In a ring with unity 1,$x^n\,[x, y] = 0$ implies that [$x$, $y$] = 0, for any positive integer $m$=$m$($x$) ≥1.

***Lemma 2.4.*** Let $R$ be a ring with unity 1 satisfying the identities ($C_1$) and ($C_2$). Then $U$($R$), the set of all invertible elements and $J$($R$), the Jacobson radical of $R$ are commutative.

***Proof.*** Since $m$ and $n$ are relatively prime, we may assume $rn$ -$sm$= 1, for some positive integers $r$ and $s$. If $k$ = $sm$, then $k$ + 1 = $rn$so that the identities ($C_1$) and($C_2$) of the hypothesi simply that

$$(xy)^k = (yx)^k, \text{ for all } x,\ y \in R. \tag{2.3}$$

and

$$x^{k+1}y^{k+1} = y^{k+1}x^{k+1}, \text{for all} x,\ y \in R. \tag{2.4}$$

Let $u, v \in U(R)$.Replacing $x$ by $u$ and $y$ by $u^{-1}v$ in (2.3), we get

$$uv^k = v^k u, \text{ for all u,v } \varepsilon \text{ R}. \tag{2.5}$$

Now replacement of $x$ by $u$ and $y$by$v$ in(2.4)yield s$u^{k+1}v^{k+1} = v^{k+1}\,ux^{k+1}$1and inview of (2.5), this implies that $uv = vu$, for all $u, v\ \varepsilon\ U(R)$. R is commutative.

Further, let $a$, $b$ε R. Then1+ $a$and1 +$b$ are invertible and commute with Hence each other. Thus $ab$=$ba$ and $J$($R$) is commutative.

***Lemma2.5.*** Let $R$ be a ring with unity 1 satisfying the identities ($C_1$) and ($C_2$). Then $R$/$J$($R$)is commutative.
***Proof.*** $R$/$J$($R$)is semi simple. We know that every semis imploring $R$ is isomorphic to a sub direct sum of primitive rings $R_\alpha$, each of which as ahomomorphic image of $R$ inherits the hypothesis placed on $R$ and so we assume that $R/J$($R$) is primitive satisfying the hypothesis of our theorem. Notice that no complete matrix ring satisfies the hypothesis as consideration of$x$ =$e_{1n}$and $y$ = $e_{n1}$ shows. Thus by the Jacobson Density Theorem [18, pp. 33], $R$ / $J$($R$) is a division ring. Hence $R/J$($R$) is commutative by Lemma 2.4.

Now we are ready to prove our theorem

***Proof of Theorem.***ByLemma2.5,

$$C(R) \subseteq J(R). \tag{2.6}$$

Replace $x$ by $u$ and $y$ by $u^{-1}y$ in (2.3), to get [$u$, $y^n$] = 0, for all $u$ε U(R) and $y$ ε R. Now, if $a \in J(R)$, then $1 + a \in U(R)$. Replacing $u$ by 1 + $a$, we obtain

$$[a, y^n] = 0, for\ all\ y \in R. \tag{2.7}$$

In view of (2.6), $[a, y^{n+1}] \in J(R)$and hence commute with $u$=1+$a$, for$a \in J(R)$by Lemma 2.4. Hence$0 = [u^{n+1}, y^{n+1}] = (n+1)u^n[u, y^{n+1}]$impliesthat$(n+1)u^n[u, y^{n+1}] = 0$. Replacing$u$by1+$a$,we find that

$$(n+1)[a, y^{n+1}] = 0, for\ all\ y \in R. \tag{2.8}$$

Using (2.7), we can assume that $n$[$a$, $y^n$]=0 and hence

$$n[a, y^n] = 0 = (n+1)[a, y^{n+1}], for\ all\ y \in R, a\ \in J(R) \tag{2.9}$$

Since $J^2$(R) ⊆ Z(R ), the only terms in the expansion of $(y+a)^{n+1}$ which do not commute with $y^{n+1}$are those involving $a$ exactly once. Hence

$$0 = [(y + a)^{n+1}, y^{n+1}]$$

$$0 = [y^n a + y^{n-1} ay + \cdots + yay^{n-1} + ay^n, y^{n+1}]$$

$$n(y^n a + y^{n-1} ay + \cdots + yay^{n-1} + ay^n)y^{n+1} = ny^{n+1}(y^n a + y^{n-1} ay + \cdots + yay^{n-1} + ay^n).$$

Using(2.8), we have

$$ny^{n+1}(y^n a + y^{n-1} ay + \cdots + yay^{n-1} + ay^n) = ny^{n+1}(y^n a + y^{n-1} ay + \cdots + yay^{n-1} + ay^n).$$

$$ny^{n+1}(y^n a + y^{n-1} ay + \cdots + yay^{n-1} + ay^n) = n(y^{2n} ay + \cdots + y^{n+1} ay) + nay^{2n+1}.$$

This gives that$(ay^{2n+1} - y^{2n+1}a) = 0$ and hence by the relation(2.9),$ny^n[a, y] = 0$.ByLemma2.3, this reduce to

$$n[a, y] = 0, for\ all\ y \in R, a \in J(R). \qquad (2.10)$$

Replace $y$ by $y^n$, to get $n\ [a\ ,y^{n+1}]=0$,which in view of(2.9) yields

$$n[a, y^{n+1}] = 0, for\ all\ y \in R, a \in J(R). \qquad (2.11)$$

By using (2.7)and (2.11), we have $[a, y]\,y^n = 0$.Application of Lemma2.3 yields that$[a, y\,] = 0$,for all $y \in R$ and$a \in J(R)$i.e. , J(R)⊆ Z(R ). Consequently,(2.6)gives

$$C(R) \subseteq J(R) \subseteq Z(R). \qquad (2.12)$$

The identity($C_1$) of the hypothesis implies that $x(xy)^n = x(yx)^n = (xy)^n x$i.e.,$[x, (xy)^n] = 0$. By Lemma 2.5 and (2.12), we have $(xy)^n - x^n y^n \in J(R) \subseteq Z(R)$and hence$[x, (xy)^n] - x^n[x, y^n] = 0$.Thus by Lemma 2.3,$[x, y^n] = 0$, for all $x, y \in R$.ApplicationofLemma2.1yieldsthat0= $[x, y^n] = ny^{n-1}[x, y]$. Lemma2.3implies that

$$n[x, y] = 0, for\ all\ x, y\ \in R. \qquad (2.13)$$

In view of (2.4),$[x^{n+1}, y^{n+1}] = 0$, which together with (2.13) and Lemma 2.1 imply that$(n + 1)x^n[x, y^{n+1}] = 0$,for all $x, y \in R$.ByLemma2.3,$(n + 1)[x, y^{n+1}] = 0$,for all $x, y \in R$. Arguing in the same fashion again, we have

$$(n + 1)^2[x, y] = 0, for\ all\ x, y \in R.$$

i.e.,

$$n^2[x, y] + 2n[x, y] + [x, y] = 0, for\ all\ x, y \in R.$$

Hence $[x, y]\ = 0$,for all$x, y \in R$by (2.13). This completes the proof of the theorem.

***Remark2.2.*** The following exampled emonstrates that if $m$ and $n$ are not relatively prime in the hypothesis of the above theorem, thering maybe badly noncommutative.

***Example 2.1.*** Consider the ring

$$R = \left\{ aI_3 + D_0 = \begin{pmatrix} 0 & bc & \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, I_3 is\ 3 \times 3 \text{, identity matrix and} a, b, c, d \in GF(2) \right\}$$

Here $R$ contains unity and satisfies the conditions$[x^2, y^2]$=0and$(xy)^2=(yx)^2$,for all $x$ , $y$ ε R but $(xy)^3 \neq (yx)^3$. However, R is a noncommutative ring.

***Remark2.3***. The following example justifies that the conditions imposed on the hypothesis of the theorem is not superfluous.

***Example 2.2.*** Let R = $\left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} ;\ a, b, c, d\ \in\ GF(3) \right\}$be a ring. It can be easily verified that $R$ is an on commutative ring satisfying the condition$[x^3, y^3]\ =\ 0$,for all $x, y \in R$. However, $R$ does not satisfies the condition$(xy)^2 = (yx)^2$,for all $x, y \in R$.

## CONCLUSION

In this paper, we extend the result of Herstein imposing an additional condition on the underlying ring and its proof is based on Jacobson structure theory of rings.

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