International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 / https://doi.org/10.14445/22315373/IJMTT-V68I3P512

Volume 68 Issue 3, 63-72, March 2022

Original Article

Boundary Value Problems for Cauchy-Riemann Systems in Some Low Dimensions

Dinh Thi Kim Nhung[1], Le Thi Hien[2], Doan Thi Linh[3]

[1,2,3]*Faculty of Management Information Systems, University of Finance and Business Administration, Hung Yen, Vietnam*

Received Date: 07 February 2022
Revised Date: 03 April 2022
Accepted Date: 09 April 2022

Abstract *- In this paper we introduce some notations in Clifford algebras and boundary value problems for Cauchy-Riemann systems in* $\mathbb{R}^d$ *with* $d = 3,4,5,6$.

Keywords *- Clifford analysis, Boundary value problems, Cauchy-Riemann system.*

I. INTRODUCTION

Let $\mathbb{R}^{n+1}$ be the Euclidean space which has an orthonomal basis $\{e_0, e_1, \dots, e_n\}$ and endswed with the standard Ecuclidean inner product $< x, y > = \sum_{j=0}^{n} x_j y_j$. The Clifford algebra $\mathcal{A}_n$ is defined as the2^n-dimensional real associated, noncommunitative algebra generated by $e_0, e_1, \dots, e_n$ and the multiplication rules

$$e_0^2 = 1,$$

$$e_j^2 = -1, j = 1, \dots, n,$$

$$e_i e_j + e_j e_i = 0, 1 \le i \ne j \le n.$$

An element $a \in \mathcal{A}_n$ has the following form

$$a = \sum_{A \in \mathcal{N}} a_A e_A \ , a_A \in \mathbb{R},$$

where$A = \{\alpha_1 \alpha_2 \dots \alpha_h\}, (\alpha_1 < \dots < \alpha_h\}$ is a subset of $\mathcal{N} = \{1,2, \dots, n\}$ and $e_A = e_{\alpha_1} e_{\alpha_2} \dots e_{\alpha_h}$. For $A = \emptyset$ we put $e_\emptyset = e_0 = 1$.

Vector in $\mathbb{R}^{n+1}$ are identified with 1-vector in $\mathcal{A}_n$ under the canonicalembeding

$$x \in \mathbb{R}^{n+1}, x = (x_1, \dots, x_n) \to \sum_{j=0}^{n} x_j e_j \coloneqq x \in \mathcal{A}_n.$$

The conjugation is defined by a mapping sending $a \mapsto \bar{a}$ with $\bar{e}_j = -e_j$ for $j = 1,2, \dots, n, \bar{e}_0 = e_0$and $\overline{ab} = \bar{b}\bar{a}$. The inner product in $\mathcal{A}_n$ is defined by

$$< a, b >_0 = 2^m . \sum_{A \in \mathcal{N}} a_A b_A \ , where\ b = \sum_{A \in \mathcal{N}} b_A e_A.$$

Hence a norm is defined by

$$\|a\|_0 = (< a, a >_0)^{1/2} = 2^{n/2} . \left(\sum_{A \in \mathcal{N}} a_A^2 \right)^{1/2}$$

which turns $\mathcal{A}_n$ into a Banach algebra of dimension 2^n. For other definition of Clifford algebra we refer reader to [F.Bracks, R.Delanghe and F.Somen].

Let Ω be a subset of the Euclidean space $\mathbb{R}^{n+1}$. We consider the function u defined in Ω and taking values in $\mathcal{A}_n$ as the mapping

$$u: \Omega \longrightarrow \mathcal{A}_n.$$

Then u can be presented by

$$u = \sum_{A \in \mathcal{N}} u_A(x) e_A,$$

where$u_A(x)$ are the (real valued) functions of $n+1$ variables $x_0, x_1, \dots, x_n$. We write $u \in \mathrm{C}(\Omega, \mathcal{A}_\mathrm{n})$, $\mathrm{C}^\mathrm{k}(\Omega, \mathcal{A}_\mathrm{n})$, $\mathrm{L}_\mathrm{p}(\Omega, \mathcal{A}_\mathrm{n})$, ... according $\mathrm{u}_\mathrm{A} \in C(\Omega), C^k(\Omega), L_p(\Omega), \dots$ respectively.

The Cauchy-Riemann operator and its adjoin defined by

$$D = \sum_{j=0}^{n} e_j \partial_j \ ; \ \partial_j = \frac{\partial}{\partial x_j} \ \ ; j = 1,2, \dots n,$$

$$\overline{D} = \partial_0 - \sum_{j=1}^{n} e_j \partial_j \ .$$

Definition 1

A function $u \in \mathrm{C}^1(\Omega, \mathcal{A}_\mathrm{n})$ is called monogenic if it sastisfies the Cauchy-Riemann system $Du = 0$.

Remark 1

From definition of Cauchy-Riemann operator, we have

$$\overline{D}D = D\overline{D} = \sum_{j=0}^{n} \partial_j^2 = \Delta_{n+1},$$

whereΔ_{n+1} is Laplace operator in $\mathbb{R}^{n+1}$.

Remark 2

From Remark 1 we see that, if $Du = 0$ then $\overline{D}Du = \Delta_{n+1} u_A = 0$.

Let Ω be a bounded domain in a Euclidean space $\mathbb{R}^{n+1}$ having a sufficiently smooth boundary $\partial\Omega$. Then we know from potential theory that to an arbitrarily chosen (continuous) function $g$on $\partial\Omega$, there exists a uniquely determined solution u of the Laplace equation such that $u = g$ on $\partial\Omega$.

In view of Remark 2 we know that all real-valued component u_A of a monogenic function in $\mathbb{R}^{n+1}$ are solutions to theLappace equation. This does not mean, however, that the boundary values of all real-valued components can be freely chosen because all components are connected by the Cauchy-Riemann system. This paper deals with the question how many components u_A can be chosen arbitrarily on thw whole boundary, and what can be prescribed for the remaning components.

II. BOUNDARY VALUE PROBLEMS FOR HOLOMORPHIC FUNCTIONS IN THE PLANE

The Cauchy-Riemann systhem for holomorphic function $w = u + iv$ in a (bounded) domain Ω in the complex plane can be prescribed by

$$\begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v. \end{cases} \tag{2.1}$$

We know that, the imaginary part v of w is uniquely determined by its boundary values. The system (2.1) leads for the real part u to the completely integrable first order system, and u is uniquely determined (in simply connected domains) up to a real constant. And so u is then uniquely determined by its values at one point of Ω (or $\overline{\Omega}$).

III. BOUNDARY VALUE PROBLEMS FOR CAUCHY-RIEMANN SYSTEM IN$\mathbb{R}^3$

In $\mathbb{R}^3$, The Clifford algebra $\mathcal{A}_2$ is defined as the 2^2-dimensional real associated, noncommunitative algebra generated by $e_0 = 1, e_1, e_2, e_1 e_2$ and the multiplication rules

$$e_0^2 = 1,$$

$$e_j^2 = -1, j = 1,2,$$

$$e_1 e_2 + e_2 e_1 = 0.$$

The function u taking values in $\mathcal{A}_2$ can be presented by

$$u = u_0 e_0 + u_1 e_1 + u_2 e_2 + u_{12} e_1 e_2,$$

and the Cauchy-Riemann operator is defined by

$$D = e_0 \partial_0 + e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3.$$

Let Ω be a cylindrical in $\mathbb{R}^3$ which is defined by

$$\{x = (x_0, x_1, x_2): \psi_1(x_1, x_2) < x_0 < \psi_1(x_1, x_2), (x_1, x_2) \in \Omega_0\}$$

whereΩ_0 is a domain in the x_1, x_2-plane.A similar situation occurs for monogenic function$u$in $\mathbb{R}^3$. TheDefinition 1 for monogenic function u leads to four real-vallued components u_0, u_1, u_2 and u_{12} satisfy the Cauchy-Riemann system

$$\partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 = 0 \quad (3.1)$$

$$\partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} = 0 \quad (3.2)$$

$$\partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 = 0 \quad (3.3)$$

$$\partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 = 0. \quad (3.4)$$

Suppose, further, that the monogenic function $u$is continuous in $\overline{\Omega}$. Since all components of a monogenic functions are solutions to the Laplace equation,u_1 and u_2 are uniquely determined already by their boundary values on the whole boundary $\partial\Omega$. Knowing $u_1$and u_2, then u_{12} can be determined by a simple integration in x_0-direction from the last eqution (3.4) provided one knows only the values g_{12} of u_{12} on the lower covering surface

$$S_0 = \{x = (x_0, x_1, x_2): x_0 = \psi_1(x_1, x_2), (x_1, x_2) \in \overline{\Omega}_0\}.$$

In order to be short, introduce the abbreviation

$$-\partial_1 u_2 + \partial_2 u_1 = F_{12}.$$

Then u_{12} can be represented in the form

$$u_{12}(x_0, x_1, x_2) = g_{12}(x_1, x_2) + \int_{\psi_1(x_1, x_2)}^{x_0} F_{12}(\xi_0, x_1, x_2) d\xi_0. \quad (3.5)$$

Notice that not for every choice of the initial functions g_{12} the function u_{12} turns out to be a solution to the Laplace eqaution. In order to simplify the calculations a little bit, we suppose that the lower covering surface of our cylindrical domain Ω is given by $\psi_1(x_1, x_2) \equiv 0$, and so we have

$$u_{12}(x_0, x_1, x_2) = g_{12}(x_1, x_2) + \int_0^{x_0} F_{12}(\xi_0, x_1, x_2) d\xi_0. \quad (3.6)$$

From (3.6) we obtain

$$\partial_0 u_{12}(x_0, x_1, x_2) = F_{12}(x_0, x_1, x_2),$$

$$\partial_0^2 u_{12}(x_0, x_1, x_2) = \partial_0 F_{12}(x_0, x_1, x_2). \quad (3.7)$$

Moreover, differentiating under the sign of integration, one gets

$$\partial_1^2 u_{12}(x_0, x_1, x_2) = \partial_1^2 g_{12}(x_1, x_2) + \int_0^{x_0} \partial_1^2 F_{12}(\xi_0, x_1, x_2) d\xi_0, \quad (3.8)$$

$$\partial_2^2 u_{12}(x_0, x_1, x_2) = \partial_2^2 g_{12}(x_1, x_2) + \int_0^{x_0} \partial_2^2 F_{12}(\xi_0, x_1, x_2) d\xi_0. \quad (3.9)$$

Now observe that the derivatives of a solution to the Laplace equation are also solutions to the Laplace equation. Thus

$$\partial_1^2 F_{12} + \partial_2^2 F_{12} = -\partial_0^2 F_{12}.$$

Taking into accout this relation, the addtion of the formulas (3.7), (3.8) and (3.9) leads to the relation

$$\Delta_3 u_{12}(x_0, x_1, x_2) = \Delta_2 g_{12}(x_1, x_2) + \partial_0 F_{12}(x_0, x_1, x_2) - \int_0^{x_0} \partial_0^2 F_{12}(\xi_0, x_1, x_2) d\xi_0.$$

Since $\partial_0 F_{12}(x_0, x_1, x_2)$ is a primitive of the integrand of the last integral, the last integral has the value

$$\partial_0 F_{12}(x_0, x_1, x_2) - \partial_0 F_{12}(0, x_1, x_2).$$

Thus $\Delta_3 u_{12}$ has everywhere in Ω the values

$$\Delta_3 u_{12}(x_0, x_1, x_2) = \Delta_2 g_{12}(x_1, x_2) + \partial_0 F_{12}(0, x_1, x_2).$$

And this, the differential equation $\Delta_3 u_{12} = 0$ is everywhere satisfied in Ω if the initial function g_{12} satisfies the following Poisson equation

$$\Delta_2 g_{12}(x_1, x_2) = -\partial_0 F_{12}(0, x_1, x_2)$$

Everywhere in the lower covering surface S_0 of the cylindrical domain Ω. And so the initial function g_{12} is uniquely determined by its boundary values on the one-dimensional boundary of the two-dimensional covering surface S_0.

Finally one can use the remaining three equation (3.1)-(3.3) in order to calculate the component u_0. The component u_0 can be constructed from the system

$$\partial_0 u_0 = \partial_1 u_1 + \partial_2 u_2 := p_0,$$

$$\partial_1 u_0 = -\partial_0 u_1 - \partial_2 u_{12} := p_1,$$

$$\partial_2 u_0 = -\partial_0 u_2 + \partial_1 u_{12} := p_2.$$

Since u_1, u_2 and u_{12} are solutions to the Laplace equation, the last system for u_0 turns out to be completely integrable, that is $\partial_k p_j = \partial_j p_k, k, j = 0,1,2$. For instance, to proof of $\partial_1 p_0 = \partial_0 p_1$, from the Laplace equation $\Delta_3 u_1 = 0$, we have

$$\begin{aligned}\partial_1 p_0 - \partial_0 p_1 &= \partial_1(\partial_1 u_1 + \partial_2 u_2) - \partial_0(-\partial_0 u_1 - \partial_2 u_{12}) \\ &= \partial_1^2 u_1 + \partial_1\partial_2 u_2 + \partial_0^2 u_1 + \partial_0\partial_2 u_{12} \\ &= -\partial_2^2 u_1 + \partial_2\partial_1 u_2 + \partial_2\partial_0 u_{12} \\ &= \partial_2(-\partial_2 u_1 + \partial_1 u_2 + \partial_2\partial_0 u_{12}) \\ &= \partial_2(0) = 0.\end{aligned}$$

Similarly, to proof of $\partial_2 p_1 = \partial_1 p_2$ and $\partial_2 p_0 = \partial_0 p_2$ one needs the Laplace equation$\Delta_3 u_{12} = 0$and $\Delta_3 u_2 = 0$.Provided Ω is homotopically simply connected, u_0 is already uniquely determined by its value at one point $P_0$of $\overline{\Omega}$.

To sum up, a monogenic funtion in $\mathbb{R}^3$ is completely determined by

- the values of two components u_1 and u_2 on the whole two-dimentional boundary $\partial\Omega$ of the three-dimentional domain Ω,
- the values of u_{12} on the one-dimensional boundary of the two-dimentional lower covering surface S_0 and
- the value of u_0 at one point P_0 in Ω.

IV. BOUNDARY VALUE PROBLEMS FOR CAUCHY-RIEMANN SYSTEM IN$\mathbb{R}^4$

In Euclidean space $\mathbb{R}^4$, Clifford algebra $\mathcal{A}_3$ has the basis elements

$$e_0 = 1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3.$$

And the multiplication rules

$$e_0^2 = 1,$$

$$e_j^2 = -1, j = 1,2,3,$$

$$e_i e_j + e_j e_i = 0; (1 \leq i \neq j \leq 3).$$

The function u taking values in $\mathcal{A}_3$ can be presented by

$$u = u_0 e_0 + u_1 e_1 + u_2 e_2 + u_3 e_3 + u_{12} e_1 e_2 + u_{13} e_1 e_3 + u_{23} e_2 e_3 + u_{123} e_1 e_2 e_3,$$

and the Cauchy-Riemann operator is defined by

$$D = e_0 \partial_0 + e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3.$$

The Cauchy-Riemann system $Du = 0$can be presented by the following system (here we denote $e_1 e_2 = e_{12}, e_1 e_2 e_3 = e_{123}, \dots$)

$$\partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 - \partial_3 u_3 = 0 \quad (4.1)$$

$$\partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} + \partial_3 u_{13} = 0 \quad (4.2)$$

$$\partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 + \partial_3 u_{23} = 0 \quad (4.3)$$

$$\partial_0 u_3 - \partial_1 u_{13} - \partial_2 u_{23} + \partial_3 u_0 = 0 \quad (4.4)$$

$$\partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 - \partial_3 u_{123} = 0 \quad (4.5)$$

$$\partial_0 u_{13} + \partial_1 u_3 + \partial_2 u_{123} - \partial_3 u_1 = 0 \quad (4.6)$$

$$\partial_0 u_{23} - \partial_1 u_{123} + \partial_2 u_3 - \partial_3 u_2 = 0 \quad (4.7)$$

$$\partial_0 u_{123} + \partial_1 u_{23} - \partial_2 u_{13} + \partial_3 u_{12} = 0. \quad (4.8)$$

Let Ω be the unit ball in $\mathbb{R}^4$, Ω_0 be the unit ball in (x_1, x_2, x_3)-space, and Ω_{01} be the unit ball in (x_2, x_3)-plane. Then we have the following theorem:

Theorem 1. The four components u_1, u_2, u_3, u_{123} can be found from their alues on the whole boundary. The two components u_{12} and u_{23} can be found from their values on the boundary of the three-dimensional distinguishing part Ω_0 of the boundary, while u_{23} can be calculated from the values on the boundaycurve Ω_{01}. The component u_0, finally, is completely determined by its value at the one point P_0 in $\overline{\Omega}$.

Proof:It is clearly that, if the value of four components: u_1, u_2, u_3, u_{123} on the whole boundary of $\partial\Omega$, then the corresponding components are uniquely determined in the whole domain Ω.

From equation (4.5), (4.6) we can calculate the components u_{12}, u_{13}, for instance, from (4.5), we have

$$\partial_0 u_{12} = -\partial_1 u_2 + \partial_2 u_1 + \partial_3 u_{123} = F_{12}(x_0, x_1, x_2, x_3). \quad (4.9)$$

If g_{12} are the values of u_{12} on the midle surface Ω_0, then we have

$$u_{12}(x_0, x_1, x_2, x_3) = g_{12}(x_1, x_2, x_3) + \int_0^{x_0} F_{12}(\xi_0, x_1, x_2, x_3) d\xi_0. \quad (4.10)$$

Similar situation for function u_{12}in $\mathbb{R}^3$, we can show that $\Delta_4 u_{12}(x) = 0$ in Ω if and only if the initial function g_{12} satisfies the Poisson equation in Ω_0

$$\Delta_3 g_{12} = -\partial_0 F_{12}(0, x_2, x_3, x_4).$$

Now we calculate for component u_{23}, from (4.7), (4.8), we get

$$u_{23}(x) = g_{23}(x_2, x_3) + \int_\gamma (p_{23}^0(\xi_0, \xi_1, x_2, x_3) d\xi_0 + p_{23}^1(\xi_0, \xi_1, x_2, x_3) d\xi_1), \quad (4.11)$$

where

$$\partial_0 u_{23} = p_{23}^0(x_0, x_1, x_2, x_3) = \partial_1 u_{123} - \partial_2 u_3 + \partial_3 u_2\ , \quad (4.12)$$

$$\partial_1 u_{23} = p_{23}^1(x_0, x_1, x_2, x_3) = -\partial_0 u_{123} + \partial_2 u_{13} - \partial_3 u_{12}\,, \tag{4.13}$$

andγ is any curve in Ω starting from $(0,0,x_2,x_3)$ to (x_0,x_1,x_2,x_3).

Using $\Delta_4 u_{123} = 0$, from the equation (4.5) and (4.6), we obtain

$$\partial_1 p_{23}^0 - \partial_0 p_{23}^1 = \partial_1^2 u_{123} - \partial_1\partial_2 u_3 + \partial_1\partial_3 u_2 + \partial_0^2 u_{123} - \partial_0\partial_2 u_{13} + \partial_0\partial_3 u_{12}$$

$$= \partial_0^2 u_{123} + \partial_0^2 u_{123} - \partial_2(\partial_1 u_3 + \partial_0 u_{13}) + \partial_3(\partial_1 u_2 + \partial_0 u_{12})$$

$$= \partial_0^2 u_{123} + \partial_0^2 u_{123} - \partial_2(-\partial_2 u_{123} + \partial_3 u_1) + \partial_3(\partial_2 u_1 + \partial_3 u_{123})$$

$$= \partial_0^2 u_{123} + \partial_1^2 u_{123} + \partial_2^2 u_{123} + \partial_3^2 u_{123} = \Delta_4 u_{123} = 0.$$

This implies that the intrgral in (4.11) does not depend on the special choice of γ, and we can prove that, the Laplace equation $\Delta_4 u_{23} = 0$ leads to the Poissoon equation

$$\Delta_3 g_{23}(x_2,x_3) = -\partial_0 p_{23}^0(0,0,x_2,x_3) - \partial_1 p_{23}^1(0,0,x_2,x_3) \ \ in\ \Omega_{01}.$$

Then g_{23} is uniquely determined by its values on the boundary curve $\partial\Omega_{01}$.

Finally, from (4.1)-(4.4), we have

$$\partial_0 u_0 = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 =: p^0$$

$$\partial_0 u_1 = -\partial_1 u_0 - \partial_2 u_{12} - \partial_3 u_{13} =: p^1$$

$$\partial_0 u_2 = \partial_1 u_{12} - \partial_2 u_0 - \partial_3 u_{23} =: p^2$$

$$\partial_0 u_3 = \partial_1 u_{13} + \partial_2 u_{23} - \partial_3 u_0 =: p^3$$

It is not difficult to prove that

$$\partial_j p^k = \partial_k p^j; 0 \le i \ne k \le 3.$$

Therefore, this systhem turns out to be completely integrable, so u_0 is uniquely determined by

$$u_0(P) = u_0(P_0) + \int_\gamma (p^0 d\xi_0 + p^1 d\xi_1 + p^2 d\xi_2 + p^3 d\xi_3),$$

where $P_0 \in \partial\Omega_{01}$, γ is arbitrary curve in Ω starting from P_0 to $P \in \Omega$.

V. BOUNDARY VALUE PROBLEMS FOR CAUCHY-RIEMANN SYSTEM IN$\mathbb{R}^5$

In the case $\mathbb{R}^5$, we have the Cauchy-Riemann operator $D = e_0\partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 + e_4\partial_4$, and a function u taking values in Clifford algebra $\mathcal{A}_4$ can be presented by

$$u = \sum_A u_A e_A,$$

where $A \in \{0,1,2,3,4,12,13,14,23,24,34,123,124,134,234,1234\}$, here the functions u has 16 components. The Cauchy-Riemann system $Du = 0$ is expressed by the following system

$$\partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 - \partial_3 u_3 - \partial_4 u_4 = 0 \tag{5.1}$$

$$\partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} + \partial_3 u_{13} + \partial_4 u_{14} = 0 \tag{5.2}$$

$$\partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 + \partial_3 u_{23} + \partial_4 u_{24} = 0 \tag{5.3}$$

$$\partial_0 u_3 - \partial_1 u_{13} - \partial_2 u_{23} + \partial_3 u_0 + \partial_4 u_0 = 0 \tag{5.4}$$

$$\partial_0 u_4 - \partial_1 u_{14} - \partial_2 u_{24} - \partial_3 u_{34} + \partial_4 u_0 = 0 \tag{5.5}$$

$$\partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 - \partial_3 u_{123} - \partial_4 u_{124} = 0 \tag{5.6}$$

$$\partial_0 u_{13} + \partial_1 u_3 + \partial_2 u_{123} - \partial_3 u_1 - \partial_4 u_{134} = 0 \tag{5.7}$$

$$\partial_0 u_{14} + \partial_1 u_4 + \partial_2 u_{124} + \partial_3 u_{134} - \partial_4 u_1 = 0 \quad (5.8)$$

$$\partial_0 u_{23} - \partial_1 u_{123} + \partial_2 u_3 - \partial_3 u_2 - \partial_4 u_{234} = 0 \quad (5.9)$$

$$\partial_0 u_{24} - \partial_1 u_{124} + \partial_2 u_4 + \partial_3 u_{234} - \partial_4 u_2 = 0 \quad (5.10)$$

$$\partial_0 u_{34} - \partial_1 u_{134} - \partial_2 u_{234} + \partial_3 u_4 - \partial_4 u_3 = 0 \quad (5.11)$$

$$\partial_0 u_{123} + \partial_1 u_{23} - \partial_2 u_{13} + \partial_3 u_{12} + \partial_4 u_{1234} = 0. \quad (5.12)$$

$$\partial_0 u_{124} + \partial_1 u_{24} - \partial_2 u_{14} - \partial_3 u_{1234} + \partial_4 u_{12} = 0. \quad (5.13)$$

$$\partial_0 u_{134} + \partial_1 u_{34} + \partial_2 u_{1234} - \partial_3 u_{14} + \partial_4 u_{13} = 0. \quad (5.14)$$

$$\partial_0 u_{234} - \partial_1 u_{1234} + \partial_2 u_{34} - \partial_3 u_{24} + \partial_4 u_{23} = 0. \quad (5.15)$$

$$\partial_0 u_{1234} + \partial_1 u_{234} - \partial_2 u_{134} + \partial_3 u_{124} - \partial_4 u_{123} = 0. \quad (5.16)$$

Let

$$\Omega = \{x = (x_0, x_1, \dots, x_4) \in \mathbb{R}^4 : \sum_{j=0}^{n} x_j^2 < 1\},$$

$$\Omega_0 = \{x = (x_0, x_1, \dots, x_4) \in \Omega : x_0 = 0\},$$

$$\Omega_{01} = \{x = (x_0, x_1, \dots, x_4) \in \Omega : x_0 = x_1 = 0\},$$

$$\Omega_{012} = \{x = (x_0, x_1, \dots, x_4) \in \Omega : x_0 = x_1 = x_2 = 0\},$$

and

$$\Lambda_0^5 = \{34\},$$

$$\Lambda_1^5 = \{23,24\},$$

$$\Lambda_2^5 = \{12,13,14,1234\},$$

$$\Lambda_3^5 = \{1,2,3,4,123,124,134,234\}.$$

Then we have following theorem.

Theorem 2. A monogenic function $u = \sum_A u_A e_A$ defined in Ω, taking values in $\mathcal{A}_4$ is completely determined by its values in distinguishing boundary of Ω, which as

- $u_A = \varphi_A, A \in \Lambda_3^5$in whole boundary of domain Ω, which is 4-dimentional,
- $u_A = \varphi_A, A \in \Lambda_2^5$in whole boundary of domain Ω_0, which is 3-dimentional,
- $u_A = \varphi_A, A \in \Lambda_1^5$in whole boundary of domain Ω_{01}, which is 2-dimentional,
- $u_A = \varphi_A, A \in \Lambda_0^5$in whole boundary of domain Ω_{012}, which is 1-dimentional,
- After all, the value of u_0 at the point $P_0 \in \partial\Omega$.

Proof: By assumption of the boundary values of eight components: u_1, u_2, u_3, u_4, u_{123}, u_{124}, u_{134}, u_{234} on the whole boundary of Ω, the corresponding components are uniquely determined in the whole domain Ω.

The equation (5.6), (5.7), (5.8), (5.16)allow to calcualte the components: u_{12}, u_{13}, u_{14}, u_{1234} by a simple integration in x_0-direction from the values on the distinguishing part Ω_0.

By the system (5.9, 5.12) and (5.10,5.13) we can calculate the components: u_{23}, u_{24} by an integrarion in x_0, x_1-direction from values on the distinguishing part Ω_{01}.

From the system (5.11, 5.14, 5.15), we can calculate the componentu_{34} by an integration in x_0, x_1, x_2-direction from the values on the distinguishing part Ω_{012}.

Finally, since the domain Ω is homotopically simply connected, from (5.1)-(5.5) we can show that, u_0 is uniquely determined by its value at one point $P_0$of $\partial\Omega_{012}$.

The theorem is proved.

VI. BOUNDARY VALUE PROBLEMS FOR CAUCHY-RIEMANN SYSTEM IN$\mathbb{R}^6$

In the case $\mathbb{R}^6$, we have the Cauchy-Riemann operator $D = e_0\partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 + e_4\partial_4 + e_5\partial_5$, and a function u taking values in Clifford algebra $\mathcal{A}_5$ can be presented by

$$u = \sum_A u_A e_A,$$

where $A \in \{0,1,2,\dots,5,12,\dots,15,23,\dots,123,\dots,12345\}$, here the functions u has 32 components. Therefore, Cauchy-Riemann system $Du = 0$ is expressed by the following system

$$\partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 - \partial_3 u_3 - \partial_4 u_4 - \partial_5 u_5 = 0 \quad (6.1)$$

$$\partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} + \partial_3 u_{13} + \partial_4 u_{14} + \partial_5 u_{15} = 0 \quad (6.2)$$

$$\partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 + \partial_3 u_{23} + \partial_4 u_{24} + \partial_5 u_{25} = 0 \quad (6.3)$$

$$\partial_0 u_3 - \partial_1 u_{13} - \partial_2 u_{23} + \partial_3 u_0 + \partial_4 u_{34} + \partial_5 u_{35} = 0 \quad (6.4)$$

$$\partial_0 u_4 - \partial_1 u_{14} - \partial_2 u_{24} - \partial_3 u_{34} + \partial_4 u_0 + \partial_5 u_{45} = 0 \quad (6.5)$$

$$\partial_0 u_5 - \partial_1 u_{15} - \partial_2 u_{25} - \partial_3 u_{35} - \partial_4 u_{45} + \partial_5 u_0 = 0 \quad (6.6)$$

$$\partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 - \partial_3 u_{123} - \partial_4 u_{124} - \partial_5 u_{125} = 0 \quad (6.7)$$

$$\partial_0 u_{13} + \partial_1 u_3 + \partial_2 u_{123} - \partial_3 u_1 - \partial_4 u_{134} - \partial_5 u_{135} = 0 \quad (6.8)$$

$$\partial_0 u_{14} + \partial_1 u_4 + \partial_2 u_{124} + \partial_3 u_{134} - \partial_4 u_1 - \partial_5 u_{145} = 0 \quad (6.9)$$

$$\partial_0 u_{15} + \partial_1 u_5 + \partial_2 u_{125} + \partial_3 u_{135} + \partial_4 u_{145} - \partial_5 u_1 = 0 \quad (6.10)$$

$$\partial_0 u_{23} - \partial_1 u_{123} + \partial_2 u_3 - \partial_3 u_2 - \partial_4 u_{234} - \partial_5 u_{235} = 0 \quad (6.11)$$

$$\partial_0 u_{24} - \partial_1 u_{124} + \partial_2 u_4 + \partial_3 u_{234} - \partial_4 u_2 - \partial_5 u_{245} = 0 \quad (6.12)$$

$$\partial_0 u_{25} - \partial_1 u_{125} + \partial_2 u_5 + \partial_3 u_{235} + \partial_4 u_{245} - \partial_5 u_2 = 0 \quad (6.13)$$

$$\partial_0 u_{34} - \partial_1 u_{134} - \partial_2 u_{234} + \partial_3 u_4 - \partial_4 u_3 - \partial_5 u_{345} = 0 \quad (6.14)$$

$$\partial_0 u_{35} - \partial_1 u_{135} - \partial_2 u_{235} - \partial_3 u_{345} + \partial_4 u_5 - \partial_5 u_4 = 0 \quad (6.15)$$

$$\partial_0 u_{45} - \partial_1 u_{145} - \partial_2 u_{245} - \partial_3 u_{345} + \partial_4 u_5 - \partial_5 u_4 = 0 \quad (6.16)$$

$$\partial_0 u_{123} + \partial_1 u_{23} - \partial_2 u_{13} + \partial_3 u_{12} + \partial_4 u_{1234} + \partial_5 u_{1235} = 0. \quad (6.17)$$

$$\partial_0 u_{124} + \partial_1 u_{24} - \partial_2 u_{14} - \partial_3 u_{1234} + \partial_4 u_{12} + \partial_5 u_{1245} = 0. \quad (6.18)$$

$$\partial_0 u_{125} + \partial_1 u_{25} - \partial_2 u_{15} - \partial_3 u_{1235} - \partial_4 u_{1245} + \partial_5 u_{12} = 0. \quad (6.19)$$

$$\partial_0 u_{134} + \partial_1 u_{34} + \partial_2 u_{1234} - \partial_3 u_{14} + \partial_4 u_{13} + \partial_5 u_{1235} = 0. \quad (6.20)$$

$$\partial_0 u_{135} + \partial_1 u_{35} + \partial_2 u_{1235} - \partial_3 u_{15} - \partial_4 u_{1345} + \partial_5 u_{13} = 0. \quad (6.21)$$

$$\partial_0 u_{145} + \partial_1 u_{45} + \partial_2 u_{1245} + \partial_3 u_{1345} - \partial_4 u_{15} + \partial_5 u_{14} = 0. \quad (6.22)$$

$$\partial_0 u_{234} - \partial_1 u_{1234} + \partial_2 u_{34} - \partial_3 u_{24} + \partial_4 u_{23} + \partial_5 u_{2345} = 0. \quad (6.23)$$

$$\partial_0 u_{235} - \partial_1 u_{1235} + \partial_2 u_{35} - \partial_3 u_{25} - \partial_4 u_{2345} + \partial_5 u_{23} = 0. \quad (6.24)$$

$$\partial_0 u_{245} - \partial_1 u_{1245} + \partial_2 u_{45} + \partial_3 u_{2345} - \partial_4 u_{25} + \partial_5 u_{24} = 0. \quad (6.25)$$

$$\partial_0 u_{345} - \partial_1 u_{1245} - \partial_2 u_{2345} + \partial_3 u_{45} - \partial_4 u_{35} + \partial_5 u_{34} = 0. \quad (6.26)$$

$$\partial_0 u_{1234} + \partial_1 u_{234} - \partial_2 u_{134} + \partial_3 u_{124} - \partial_4 u_{123} - \partial_5 u_{12345} = 0. \quad (6.27)$$

$$\partial_0 u_{1235} + \partial_1 u_{235} - \partial_2 u_{135} + \partial_3 u_{125} + \partial_4 u_{12345} - \partial_5 u_{123} = 0. \quad (6.28)$$

$$\partial_0 u_{1245} + \partial_1 u_{245} - \partial_2 u_{145} - \partial_3 u_{12345} + \partial_4 u_{125} - \partial_5 u_{124} = 0. \quad (6.29)$$

$$\partial_0 u_{1345} + \partial_1 u_{345} + \partial_2 u_{12345} - \partial_3 u_{145} + \partial_4 u_{135} - \partial_5 u_{134} = 0. \quad (6.30)$$

$$\partial_0 u_{2345} - \partial_1 u_{12345} + \partial_2 u_{345} - \partial_3 u_{245} + \partial_4 u_{235} - \partial_5 u_{234} = 0. \quad (6.31)$$

$$\partial_0 u_{12345} + \partial_1 u_{2345} - \partial_2 u_{1345} + \partial_3 u_{1245} - \partial_4 u_{1235} + \partial_5 u_{1234} = 0. \quad (6.32)$$

Let

$$\Omega = \{x = (x_0, x_1, \dots, x_5) \in \mathbb{R}^6 : \sum_{j=0}^{n} x_j^2 < 1\},$$

$$\Omega_0 = \{x = (x_0, x_1, \dots, x_5) \in \Omega : x_0 = 0\},$$

$$\Omega_{01} = \{x = (x_0, x_1, \dots, x_5) \in \Omega : x_0 = x_1 = 0\},$$

$$\Omega_{012} = \{x = (x_0, x_1, \dots, x_5) \in \Omega : x_0 = x_1 = x_2 = 0\},$$

$$\Omega_{0123} = \{x = (x_0, x_1, \dots, x_5) \in \Omega : x_0 = x_1 = x_2 = x_3 = 0\},$$

and

$$\Lambda_0^6 = \{45\},$$

$$\Lambda_1^6 = \{34, 35\},$$

$$\Lambda_2^6 = \{23,24,25,2345\},$$

$$\Lambda_3^6 = \{12,13,14,15,1234,1235,1245,1345\},$$

$$\Lambda_4^6 = \{1,2,3,4,5,123,124,125,134,135,145,234,235,245,345,12345\}.$$

Then similar in $\mathbb{R}^5$ we can proved the following theorem.

Theorem 3. A monogenic function $u = \sum_A u_A e_A$ defined in Ω, taking values in $\mathcal{A}_5$ is completely determined by its values in distinguishing boundary of Ω, which as

- $u_A = \varphi_A, A \in \Lambda_4^6$in whole boundary of domain Ω, which is 5-dimentional,
- $u_A = \varphi_A, A \in \Lambda_3^6$in whole boundary of domain Ω_0, which is 4-dimentional,
- $u_A = \varphi_A, A \in \Lambda_2^6$in whole boundary of domain Ω_{01}, which is 3-dimentional,
- $u_A = \varphi_A, A \in \Lambda_1^6$in whole boundary of domain Ω_{012}, which is 2-dimentional,
- $u_A = \varphi_A, A \in \Lambda_0^6$in whole boundary of domain Ω_{0123}, which is 1-dimentional,
- After all, the value of u_0 at the point $P_0 \in \partial\Omega$.

VII. CONCLUSION

The above results show that one can generaliza the concept of conjugate solution to higher dimensions: Given 2^{n-1} real-valued solutions to the Laplace equation in a (homotopically simply connected) domain in $\mathbb{R}^{n+1}$, one can find 2^{n-1}another real-valued functions which are also solutions to the Laplace equation, and the whole system of all 2^n real-valued functions are the real components of a monogenic function in $\mathbb{R}^{n+1}$. The 2^{n-1} real-valued conjugate solutions to the Laplace equation are uniquely determined by their initial values on some parts of the boundary.

ACKNOWLEDGMENT

We are grateful to Professor Le Hung Son, Hanoi University of Science and Technology, for his useful discussion to complete this paper.

REFERENCES

[1] O. Celebi and K. Koca, A Note on a Boundary Value Problem for Nonlinear Complex Differential Equations in Wiener-type Domains, InternationalConference on Applied Mathematics, (2004) 321-326.
[2] C. Miranda, Partial differential equations of elliptic type. Ergebnisse der Mathematik and ihrerGrenzgebiete, Band 2, Springer-Verlag, New York, Second revised edition.Translated from the Italian by Zane C. Motteler. (1970).
[3] D. Gilbarg and N.S.Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag Berlin Heidelbaerg New York Tokyo (1983).
[4] Doan Cong Dinh, Dirichlet boundary value problem for monogenic function in Clifford analysis, Complex Variables and Elliptic Equations, 2014. 59 (9) (2014) 1201-1213.
[5] Doan Cong Dinh, Generalized Clifford Analysis, The doctoral thesis, Graz University of Technology, (2012).
[6] D. Alayon-Solarz and C. J. Vanegas, Operators Associated to the CauchyRiemann Operators in Elliptic Complex Numbers, Adv. Appl. Clifford Algebras. 22 (2012) 257-270.
[7] F. Brackx, R. Delanghe and F. Sommen, Clifford analysis, Pitman, Research Notes, 76 (1982).
[8] Dao Viet Cuong, From distinguishing boundaries to bounday value problems for mononegic functions, Complex Analysis and Operator Theory (2021).

[9] A. Escassut, W. Tutschke and C. C. Yang (editors),Some topicson value distribution and differentiability in complex and p-adicanalysis. Science Press Beijing. (2008).
[10] Sha Huang, Yu Ying Qiao, and Guo Chun Wen, Real and ComplexClifford Analysis, Advances in Complex Analysis and Its Applications, Springer-Verlag. 5 (2006).
[11] J. Vanegas and F. Vargas, On weighted Dirac Operators and their Fundamental Solution for Anisotropic Media, Adv. Appl. Clifford Algebras 28 (2018).
[12] V. V. Kravchenko, Applied quaternionic analysis,Research and Exposition in Mathematics, Lemgo: HeldermannVerlag.28 (2003).
[13] Le Hung Son and W. Tutschke (editors), Algebraic Structures in Partial Differential Equations Related to Complex and Clifford Analysis, Ho Chi Minh CityUniversity of Education Press, Ho Chi Minh City, (2010).
[14] Le Hung Son and W. Tutschke, Complex Methods in Higher Dimensions |Recent Trends for Solving Boundary Value and Initial Value Problems, ComplexVariables, 50 (7-11) (2005) 673679.
[15] Le Hung Son and W. Tutschke, First order Differential Operators Associated to the Cauchy-Riemann Operator in the Plane, Complex Var. Theory Appl. 48(2003) 797-801.
[16] Le Hung Son and W Tutschke,Complex Methods In Higher Dimensions - Recent Trends for Solving Boundary Value and Initialvalue Problems. Complex Variables, 50 (. 7-11) (2005) 673-679.
[17] Le Hung Son and W. Tutschke (editors), Algebraic Structures Inpartial Differential Equations Related to Complex and Clifford analysis. Ho Chi Minh City University of Education Press.Ho Chi MinhCity, (2010).
[18] Muhammad SajidIqbal, Solutions of Boundary Value Problems for Nonlinear Partial Differential Equations by Fixed Point Methods, Disseration, Graz, (2011).
[19] A.S.A. Mshimba and W. Tutschke (editors), Functional AnalyticMethods in Complex Analysis and Applications to Partial Differential Equations, Proceedings of the Second Workshop held at theICTP in Trieste, January 25-29, (1993). World Scientific (1995).3940 Bibliography
[20] Sheldon Alex, Paul Bourdon and Wade Ramey, Harmonic Function Theory,Second Edition, Springer-Verlag New York, Inc, (2001).
[21] Richard Delanghe, Clifford Analysis,History and Perspective, Computational Methods and Function Theory.1(1) (2001) 107-153.
[22] W. Tutschke, An elementary approach to Clifford Analysis. Contained in the Collection of papers [8]. (1995) 402-408.
[23] W. Tutschke, Real and Complex Fundamental Solutions - a way for Unifyingmathematical Analysis. Bol. Asoc. Mat. Venez., 9(2) (2002) 141-179.
[24] W. Tutschke, The Distinguishing Surface for Monogenic Functions in Cliffordanalysis. Advances in Applied Clifford Analysis. Online:DOI 10.1007/s00006-014-0484-y.
[25] W. E. Hamilton, Elements of Quaternion, Publisher London, Longmans, Green,& Co. (1866).
[26] W.Tutschke, Generalized Analytic Functions in Higher Dimensions, GeorgianMath. J. (2007).