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Original Article

Higher Order Duality for Non Smooth Multiobjective Semi-infinite Programming Problems

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Abstract− *A nonsmooth multiobjective semi-infinite programming problems with square root term both in objective functions and constraints is considered. Higher order Mangasarian and Mond-Weir duals are formulated for this multi-objective semi-infinite programming problem. Various higher order duality results are established under the assumption that the functions involved are higher-order* (Φ, ρ, Ω)*-type I functions.*

Keywords− *Nonsmooth Multiobjective Semi-infinite Programming, Optimality Conditions, Mangasarian dual, Mond-Weir dual, Higher Order Duality*

I. INTRODUCTION

Multiobjective semi-infinite programming problem is an optimization problem in which more than one objective function is to be minimized over feasible set described by infinite number of inequality constraints. A special case of the nonsmooth multiobjective semi-infinite programming problem is a problem containing a square root of the quadratic form in each of the objective functions as well as in the constraints. It has many applications in various types of generalizations of convexity theory (see, [1], [3], [4], [5], [6], [16],[19], [22], [26]).

The concept of higher order duals was introduced by Mangasarian [15]. Higher order duals are significant as they provide tighter bounds for the value of objective functions and hence has computational advantage over the first order duals. Due to its importance, many researchers have worked in this field (see, [7], [8], [9], [10], [11], [13],[18], [20], [21], [24], [25]). Mond [2] defined second order convexity and used it to derive second order duality results. Hanson [12] generalized the concept of invexity and introduced the class of type 1 functions in nonlinear scalar optimization problems. Mishra and Rueda [17] considered a class of higher order type-1 functions for scalar optimization problems and proved higher order duality results for various number of higher order duals under the assumption that the functions involved are higher order type-1 functions. Recently, Antczak et al. [23] proved higher order duality results for a class of nonconvex nonsmooth multiobjective programming problems with square root of a quadratic form in each of the objective functions under the assumption that the functions considered are higher order (Φ, ρ)-type 1 functions.

In this paper, a multiobjective semi-infinite programming problem with square root term both in objective functions and constraints is considered. Problems of this type are encountered in stochastic programming and portfolio selection problems. A new class of higher order $(\Phi, \rho, \Omega)-$type I objective and constraint functions is defined. Mangasarian dual and Mond-Weir dual are formulated for the considered programming problem. Various higher order duality results for both the duals are derived under the assumption that the functions involved are higher order $(\Phi, \rho, \Omega)-$type I objective and constraint functions.

II. PRELIMINARIES AND DEFINITIONS

The following notations are used in this paper.
Let $\mathbb{R}^m$ be a finite dimensional Euclidean space. Let $u = (u_1, u_2, ..., u_m),\ v = (v_1, v_2, ..., v_m) \in \mathbb{R}^m$, then
$u \leqq v$ if and only if $u_n \leq v_n,\ \forall\ n = 1, 2, ..., m,$
$u < v$ if and only if $u_n < v_n,\ \forall\ n = 1, 2, ..., m,$
$u \leq v$ if and only if $u \neq v$ and $u_n \leq v_n,\ \forall\ n = 1, 2, ..., m.$

We consider the following nonsmooth semi-infinite multiobjective programming problem with square root terms in objective and constraints

$$\begin{aligned} \min \quad & f(x) = (f_1(x) + (x^T B_1 x)^{\frac{1}{2}}, f_2(x) + (x^T B_2 x)^{\frac{1}{2}}, ..., f_q(x) + \\ & \qquad (x^T B_q x)^{\frac{1}{2}}) \\ \text{s.t.} \quad & g_t(x) + (x^T C_t x)^{\frac{1}{2}} \leq 0,\ t \in T \\ & x \in \mathbb{R}^m \end{aligned} \tag{P}$$

where $f_n : X \to \mathbb{R},\ n \in I = \{1, 2, ..., q\}$ and $g_t : X \to \mathbb{R}, t \in T$ are differential functions on a non empty open convex set $X \subset \mathbb{R}^m$ and each B_n and C_t is a $m \times m$ positive semidefinite symmetric matrix. The index set T is arbitrary non empty set not necessarily finite.
The feasible set of (P) denoted by M is given as

$$M = \{x \in \mathbb{R}^m \mid g_t(x) \leq 0,\ \ \forall\ t \in T\}$$

and the set of active constraints is given as

$$T(x^*) = \{t \in T \mid g_t(x^*) = 0\}.$$

Remark 2.1. Note that if $B_n = 0,\ n \in I$ and $C_t = 0,\ t \in T$, then the considered multiobjective programming problem reduces to a usual semi-infinite multiobjective programming problem.

Let $k_n : X \times \mathbb{R}^m \to \mathbb{R},\ n \in \{1, 2, ..., q\}$ and $h_t : X \times \mathbb{R}^m \to \mathbb{R},\ t \in T$ be differentiable functions, and p be any vector in $\mathbb{R}^m$.

Definition 2.1. Suppose that the functions $\Phi : X \times X \times \mathbb{R}^{m+1} \to \mathbb{R}$ and $\rho = (\rho_{f_n}, \rho_{g_t})$ where $n \in I$ and $t \in \Omega$ are given. Let $\Omega \subset T$ be a non-empty set. Then $((f_n,\ n \in I), (g_t,\ t \in \Omega))$ is said to be higher-order $(\Phi, \rho, \Omega)-$type I objective and constraint functions at $u \in X$ on X if $\Phi(x, u, .)$ is convex on $\mathbb{R}^{m+1}$, $\Phi(x, u, (0, a)) \geq 0,\ \forall\ x \in X$ and any $a \in \mathbb{R}^+$ and

$$f_n(x) - f_n(u) - k_n(u, p) + p^T \nabla_p k_n(u, p) \geq \Phi(x, u, (\nabla_p k_n(u, p), \rho_{f_n})),\ \forall\ n \in I \tag{1}$$

$$-g_t(u) - h_t(u, p) + p^T \nabla_p h_t(u, p) \geq \Phi(x, u, (\nabla_p h_t(u, p), \rho_{g_t})),\ \forall\ t \in \Omega \tag{2}$$

If these inequalities (1) and (2) are satisfied at each $u \in X$, then $((f_n,\ n \in I), (g_t,\ t \in \Omega))$ is said to be higher-order $(\Phi, \rho, \Omega)-$type I objective and constraint functions on X.
If inequalities in (1) are strict for all $x \in X, (x \neq u)$ and $n \in I$, then $((f_n,\ n \in I), (g_t,\ t \in \Omega))$ is said to be higher-order strictly $(\Phi, \rho, \Omega)-$type I objective and constraint functions at $u \in X$ on X.

Example 2.1. *Consider the programming problem*

$$\begin{aligned} \min \quad & f(x) = (f_1, f_2) = \left((x_1^2 + x_2^2)^{\frac{1}{3}} + (x_1^2 + x_2^2)^{\frac{1}{2}}, (x_1^2 + x_2^2)^{\frac{1}{5}} + (x_1^2 + x_2^2)^{\frac{1}{2}}\right) \\ s.t. \quad & g_t(x) = -x_1 - x_2 - t \leq 0,\ t \in [0, 2] \\ & g_3(x) = (x_1)^2 + (x_2)^2 \leq 0 \\ & x \in \mathbb{R}^2 \end{aligned} \tag{3}$$

Let $u = (0,0)$. *Define* $\Phi(x,u,(v,\rho)) = v_1(u_1 - x_1^2) + v_2(u_2 - x_2^2) + \rho$, $k_i(u,p) = \frac{p_1+p_2}{u_1^2+u_2^2+1} - 1$, $h_t(u,p) = \frac{p_1+p_2}{u_1^2+u_2^2+1} - 1$, $h_3(u,p) = \frac{p_1+p_2}{u_1^2+u_2^2+1} - 1$, $\rho_{f_i} = 1$, $\rho_{g_t} = 2$, *and* $\rho_{g_3} = 2$
Then (f,g) *is higher-order* $(\phi,\rho,\Omega)-$*type I objective and constraint functions on the feasible region, where* $\Omega = [3,4] \cup \{0\}$

Definition 2.2. A feasible point x^* is said to be an efficient solution of the problem (P) if there does not exist a feasible point $x \in M$ such that

$$f_n(x) + (x^T B_n x)^{\frac{1}{2}} \leq f_n(x^*) + (x^{*^T} B_n x^*)^{\frac{1}{2}} \ \ \forall\, n \in \{1,2,...,q\}$$

and there exist atleast one $r \in \{1,2,...,q\}$ such that

$$f_r(x) + (x^T B_r x)^{\frac{1}{2}} < f_r(x^*) + (x^{*^T} B_r x^*)^{\frac{1}{2}}$$

Definition 2.3. An efficient solution x^* is said to be properly efficient solution if there exist a scalar $\mathcal{K} > 0$ such that for each $n \in \{1,2,...,q\}$ and $x \in M$, we have

$$[f_n(x^*) + (x^{*^T} B_n x^*)^{\frac{1}{2}}] - [f_n(x) + (x^T B_n x)^{\frac{1}{2}}]$$
$$\leq \mathcal{K}([f_l(x^*) + (x^{*^T} B_l x^*)^{\frac{1}{2}}] - [f_l(x) + (x^T B_l x)^{\frac{1}{2}}])$$

for atleast one $l \in \{1,2,...,q\}$ satisfying

$$f_l(x^*) + (x^{*^T} B_l x^*)^{\frac{1}{2}} < f_l(x) + (x^T B_l x)^{\frac{1}{2}}$$

Remark 2.2. Define the set

$$\Omega(x^*) = \bigcup_{n=1}^{q} \Omega_n(x^*)$$

where $\Omega_n(x^*) = \{w \in \mathbb{R}^m : w^T \nabla g_t(x^*) + \frac{w^T C_t x}{(x^T C_t x)^{\frac{1}{2}}} \leq 0,\ t \in T(x^*)$ and $w^T \nabla f_n(x^*) + \frac{w^T B_n x}{(x^T B_n x)^{\frac{1}{2}}} < 0,$ if $x^{*^T} B_n x^* > 0,\ w^T \nabla f_n(x^*) + (w^T B_n w)^{\frac{1}{2}} < 0$ if $x^{*^T} B_n x^* = 0\}$

Theorem 2.1. *(Generalized Schwartz inequality) Let* B *be a positive semidefinite symmetric matrix of order* m*. Then, for all* $x, z \in \mathbb{R}^m$*, we have*

$$x^T B z \leq (x^T B x)^{\frac{1}{2}} (z^T B z)^{\frac{1}{2}}$$

Note that the equality holds, if $Bx = \alpha B w$ *for some* $\alpha \in \mathbb{R}^m$ *with* $\alpha \geq 0$*. Moreover, if* $(z^T B z)^{\frac{1}{2}} \leq 1$*, then*

$$x^T B z \leq (x^T B x)^{\frac{1}{2}}$$

The following theorem is the generalization of Theorem 3.4 (ii) from Kanzi and Nobakhtian [14] for the multiobjective semi-infinite programming problem with square root terms in objective functions and constraints.

Theorem 2.2. *(Necessary Optimality Conditions) Let* x^* *be a properly efficient solution of the problem* (P')*, the set* $\Omega(x^*)$ *be empty and a suitable constraint qualification be satisfied at* x^**. Then, there exist* $\mu = (\mu_j) > 0,\ j \in J,\ \Omega \subseteq T(x^*)$ *such that* $|\,\Omega\,| < \infty$, $\beta = (\beta_t) \geq 0,\ t \in \Omega$ *and* $w \in \underbrace{(\mathbb{R}^n \times ... \times \mathbb{R}^n)}_{s\ times}$, $v \in \underbrace{(\mathbb{R}^n \times ... \times \mathbb{R}^n)}_{\Omega\ times}$ *such that*

$$\sum_{j=1}^{s} \mu_j [\nabla f_j(x^*) + B_j w_j] + \sum_{t \in \Omega} \beta_t [\nabla g_t(x^*) + C_t v_t] = 0, \tag{4}$$

$$\beta_t \left(g_t(x^*) + x^{*^T} C_t v_t \right) = 0,\ t \in \Omega, \tag{5}$$

$$\sum_{j=1}^{s} \mu_j = 1, \tag{6}$$

$$\left(x^{*^T}B_jx^*\right)^{\frac{1}{2}} = x^*B_jw_j,\ j \in J, \tag{7}$$

$$\left(x^{*^T}C_tx^*\right)^{\frac{1}{2}} = x^*C_tv_t,\ t \in \Omega, \tag{8}$$

$$w_j^TB_jw_j \leq 1,\ j \in J, \tag{9}$$

$$v_t^TC_tv_t \leq 1,\ t \in \Omega. \tag{10}$$

III. MAIN RESULTS

In this section, we derive weak duality and strong duality results using higher-order $(\Phi, \rho, \Omega)-$type I objective and constraint functions for Mangasarian dual and Mond-Weir dual .

I. Mangasarian Duality

For each $u \in X \subset \mathbb{R}^m,\ \Omega \subset T$ with $\mid \Omega \mid < \infty,\ p \in \mathbb{R}^m,\ w = (w_n),\ n \in I,\ w_n \in \mathbb{R}^m,\ v = (v_t),\ t \in \Omega,\ v_t \in \mathbb{R}^m$ and $\eta = (\eta_t),\ t \in \Omega$, we define

$$\mathcal{F}(u,w,v,\eta,\Omega,p) = \Bigg(f_1(u) + [u+p]^TB_1w_1 + k_1(u,p) + \sum_{t\in\Omega}\eta_t\left[g_t(u) + [u+p]^TC_tv_t + h_t(u,p)\right], ...,$$
$$f_q(u) + [u+p]^TB_qw_1 + k_q(u,p) + \sum_{t\in\Omega}\xi_t\left[g_t(u) + [u+p]^TC_tv_t + h_t(u,p)\right]\Bigg)$$

Then, the Mangasarian dual of the primal problem (P) is given by

$$Maximize \quad \mathcal{F}(u,w,v,\eta,\Omega,p) \tag{MD}$$

$$\text{s.t.} \quad \sum_{n=1}^{q}\mu_n\left[\nabla_pk_n(u,p) + B_nw_n\right] + \sum_{t\in\Omega}\eta_t\left[\nabla_ph_t(u,p) + C_tv_t\right] = 0 \tag{11}$$

$$w_n^TB_nw_n \leq 1 \quad n \in I \tag{12}$$

$$v_t^TB_nv_t \leq 1 \quad t \in \Omega \tag{13}$$

$$\sum_{n=1}^{p}\mu_n = 1 \tag{14}$$

$$u \in X,\ p \in \mathbb{R}^m,\ w_n \in \mathbb{R}^m,\ v_t \in \mathbb{R}^m,\ \mu_n > 0,\ n \in I,\ \eta_t \geq 0,\ t \in \Omega$$

Let $\mathcal{D}$ denotes the feasible set of the Mangasarian dual problem (MD).

Theorem 3.1. (Weak Duality Theorem) *Let x and $(u,\mu,\eta,w,v,\Omega,p)$ be any feasible solution for the problem (P) and (MD) respectively. Further, assume that $\left((f_n(.) + (.)^TB_nw_n,\ n \in I), (g_t(.) + (.)^TC_tv_t,\ t \in \Omega)\right)$ is higher-order $(\Phi, \rho, \Omega)-$ type I objective and constraint functions at u on $M \cup \mathcal{D}$. If $\sum_{n=1}^{p}\mu_n\rho_{f_n} + \sum_{t\in T}\eta_t\rho_{g_t} \geq 0$, then the following cannot hold:*

$$f(x) \leq \mathcal{F}(u,w,v,\eta,\Omega,p) \tag{15}$$

Proof Let x and $(u,\mu,\eta,w,v,\Omega,p)$ be any feasible solutions for the problem (P) and (MWD) respectively. Suppose that (15) holds, therefore, there exist atleast one $r \in I$ such that

$$f_n(x) + \left(x^TB_nx\right)^{\frac{1}{2}} \leq f_n(u) + [u+p]^TB_nw_n + k_n(u,p) + \sum_{t\in\Omega}\eta_t\left[g_t(u) + [u+p]^TC_tv_t + h_t(u,p)\right],$$
$$n \in I - \{r\} \tag{16}$$

$$f_r(x) + \left(x^TB_rx\right)^{\frac{1}{2}} < f_r(u) + [u+p]^TB_rw_r + k_r(u,p) + \sum_{t\in\Omega}\eta_t\left[g_t(u) + [u+p]^TC_tv_t + h_t(u,p)\right] \tag{17}$$

By generalized Schwarz inequality, (16) and (17), we obtain

$$f_n(x)+x^TB_nw_n \leq f_n(u)+[u+p]^TB_nw_n+k_n(u,p)+\sum_{t\in\Omega}\eta_t\left[g_t(u)+[u+p]^TC_tv_t+h_t(u,p)\right],\quad n\in I-\{r\} \tag{18}$$

$$f_r(x)+x^TB_rw_r < f_r(u)+[u+p]^TB_rw_r+k_r(u,p)+\sum_{t\in\Omega}\eta_t\left[g_t(u)+[u+p]^TC_tv_t+h_t(u,p)\right] \tag{19}$$

Since $(u,\mu,\eta,w,v,\Omega,p)$ is a feasible solution for the problem (MD), therefore, we have $\mu_n>0,\ n\in I$ and $\sum_{n=1}^q\mu_n=1$. Hence, from (18) and (19) we get

$$\sum_{n=1}^q\mu_n\left[f_n(x)+x^TB_nx\right]<\sum_{n=1}^q\mu_n\left[f_n(u)+[u+p]^TB_nw_n+k_n(u,p)\right]+\sum_{t\in\Omega}\eta_t\left[g_t(u)+[u+p]^TC_tv_t+h_t(u,p)\right] \tag{20}$$

The function $\left((f_n(.)+(.)^TB_nw_n,\ n\in I),(g_t(.)+(.)^TC_tv_t,\ t\in\Omega)\right)$ is higher-order $(\Phi,\rho,\Omega)-$ type I objective and constraint functions at u on $M\cup\mathcal{D}$, therefore, we have

$$f_n(x)+x^TB_nw_n-f_n(u)-u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)\geq\Phi\left(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n})\right),\quad\forall\, n\in I$$

$$-g_t(u)-u^TC_tv_t-h_t(u,p)+p^T\nabla_ph_t(u,p)\geq\Phi(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t})),\quad\forall\, t\in\Omega$$

We have $\mu_n>0,\ \forall\, n\in I$ and $\eta_t\geq 0,\ t\in\Omega$, therefore,

$$\mu_n\left[f_n(x)+x^TB_nw_n-f_n(u)+u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)\right]\geq\mu_n\Phi\left(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n})\right),\ \forall\, n\in I \tag{21}$$

$$-\eta_t\left[g_t(u)+u^TC_tv_t+h_t(u,p)-p^T\nabla_ph_t(u,p)\right]\geq\eta_t\Phi(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t})),\ \forall\, t\in\Omega \tag{22}$$

Adding (21) and (22), we get

$$\sum_{n=1}^q\mu_n\left[f_n(x)+x^TB_nw_n-f_n(u)-u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)\right]+\sum_{t\in\Omega}\eta_t\left[-g_t(u)-u^TC_tv_t-h_t(u,p)+p^T\nabla_ph_t(u,p)\right]\geq\sum_{n=1}^q\mu_n\Phi(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n}))]+\sum_{t\in\Omega}\eta_t[\Phi(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t})) \tag{23}$$

Put

$$\tilde{\mu}_n=\frac{\mu_n}{\sum_{n=1}^q\mu_n+\sum_{t\in\Omega}\eta_t},\ n\in I\text{ and }\tilde{\eta}_t=\frac{\eta_t}{\sum_{n=1}^q\mu_n+\sum_{t\in\Omega}\eta_t},\ t\in\Omega \tag{24}$$

Clearly $\tilde{\mu}_n,\ \tilde{\eta}_t\in[0,1],\ \forall\, n\in I,\ t\in\Omega$ and $\sum_{n=1}^q\tilde{\mu}_n+\sum_{t\in\Omega}\tilde{\eta}_t=1$
Using (23) and (24), we get

$$\sum_{n=1}^q\tilde{\mu}_n\left[f_n(x)+x^TB_nw_n-f_n(u)-u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)\right]+\sum_{t\in\Omega}\tilde{\eta}_t\left[-g_t(u)-u^TC_tv_t-h_j(u,p)+p^T\nabla_ph_t(u,p)\right]\geq\sum_{n=1}^q\tilde{\mu}_n\Phi\left(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n})\right)+\sum_{t\in\Omega}\tilde{\eta}_n\Phi\left(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t})\right) \tag{25}$$

Since, $\Phi(x,u,.)$ is convex on $\mathbb{R}^{m+1}$ and $\tilde{\mu}_n,\ \tilde{\eta}_t \in [0,1],\ \forall\ i \in I,\ t \in \Omega$ and $\sum_{n=1}^{q}\tilde{\mu}_n + \sum_{t\in\Omega}\tilde{\eta}_t = 1$, therefore, from (24) we have

$$\begin{aligned}
&\sum_{n=1}^{q}\tilde{\mu}_n\left[f_n(x)+x^TB_nw_n-f_n(u)-u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)\right]+\\
&\sum_{t\in\Omega}\tilde{\eta}_t\left[-g_t(u)-u^TC_tv_t-h_t(u,p)+p^T\nabla_ph_t(u,p)\right]\\
&\geq \Phi\left(x,u,\left(\sum_{n=1}^{q}\tilde{\mu}_n\left(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n}\right)+\sum_{t\in\Omega}\tilde{\eta}_t\left(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t}\right)\right)\right)
\end{aligned}\tag{26}$$

Using (11), (24) and (26), we get

$$\begin{aligned}
&\frac{1}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\xi_t}\sum_{n=1}^{q}\mu_n\left[f_n(x)+x^TB_nw_n-f_n(u)-u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)\right]+\\
&\frac{1}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\xi_t}\sum_{t\in\Omega}\eta_t\left[-g_t(u)-u^TC_tv_t-h_t(u,p)+p^T\nabla_ph_t(u,p)\right]\\
&\geq \Phi\left(x,u,\frac{1}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\xi_t}\left(0,\sum_{n=1}^{q}\mu_n\rho_{f_n}+\sum_{t\in\Omega}\eta_t\rho_{g_t}\right)\right)
\end{aligned}\tag{27}$$

Since $\Phi(x,u,(0,a)) \geqq 0$ for every $a \in \mathbb{R}^+$ and $\sum_{n=1}^{q}\mu_n\rho_{f_n}+\sum_{t\in\Omega}\xi_t\rho_{g_t} \geq 0$, therefore, we have

$$\Phi\left(x,u,\frac{1}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\xi_t}\left(0,\sum_{n=1}^{q}\rho_{f_n}+\sum_{t\in\Omega}\eta_t\rho_{g_t}\right)\right)\geq 0 \tag{28}$$

From (27) and (28), we get

$$\begin{aligned}
\sum_{n=1}^{q}\mu_n\left[f_n(x)+x^TB_nw_n\right] \geq &\sum_{n=1}^{q}\mu_n\left[f_n(u)+u^TB_nw_n+k_n(u,p)-p^T\nabla_pk_n(u,p)\right]\\
&+\sum_{t\in\Omega}\left[\eta_t(g_t(u)+u^TC_tv_t+h_t(u,p)-p^T\nabla_ph_t(u,p)\right]
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sum_{n=1}^{q}\mu_n\left[f_n(x)+x^TB_nw_n\right] \geq &\sum_{n=1}^{q}\mu_n\left[f_n(u)+[u+p]^TB_nw_n+k_n(u,p)\right]+\\
&\sum_{t\in\Omega}\left[\eta_t(g_t(u)+[u+p]^TC_tv_t+h_t(u,p)\right]-\\
&p^T\left(\sum_{n=1}^{q}\mu_n\left[\nabla_pk_n(u,p)+B_nw_n\right]+\sum_{t\in\Omega}\eta_t\left[\nabla_ph_t(u,p)+C_tv_t\right]\right)
\end{aligned}$$

By (11), we have

$$\begin{aligned}
\sum_{n=1}^{q}\mu_n[f_n(x)+x^TB_nw_n] \geqq &\sum_{n=1}^{q}\mu_n[f_n(u)+[u+p]^TB_nw_n+k_n(u,p)]+\\
&\sum_{t\in\Omega}\eta_t[g_t(u)+[u+p]^T+h_t(u,p)]
\end{aligned}$$

a contradiction to (20).
Hence (15) does not hold.

Theorem 3.2. (Strong Duality Theorem) *Let x^* be a properly efficient solution for the problem (P) such that the set $\Omega(x^*)$ is empty and and a suitable constraint qualification be satisfied at x^*. Further, assume that*

$$\begin{aligned} &k_n(x^*, 0) = 0 \text{ for all } n \in I;\ \nabla_p k(x^*, 0) = \nabla f(x^*) \\ &h_t(x^*, 0) = 0 \text{ for all } t \in T;\ \nabla_p h(x^*, 0) = \nabla g(x^*) \end{aligned} \tag{29}$$

Then, there exist $\mu^* = (\mu_n^*) > 0$, $\eta^* = (\eta_t^*) \geq 0$, $w^* = (w_n^*)$, $v^* = (v_t^*)$, $\Omega_1 \subseteq T(x^*)$ with $|\ \Omega_1 \ |< \infty$ such that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is feasible for (MD) and the corresponding objective values of (P) and (MD) are equal. Further, if weak duality holds, then $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is a properly efficient solution of the dual problem (MD).

Proof Let x^* be a properly efficient solution for the problem (P), the set $\Omega(x^*)$ is empty and the Linear Independence Constraint Qualification is satisfied at x^*. Then, by Necessary Optimality Theorem, it follows that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is feasible in (MD).
From (4) and (29), we obtain

$$f(x^*) = \mathcal{F}(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$$

Hence, the corresponding objective values of (P) and (MD) are equal.

Now, we first show that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is an efficient solution for problem (MD).
Suppose that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is not an efficient of (MD). Then, there exist $(\bar{x}, \bar{\mu}, \bar{\eta}, \bar{w}, \bar{v}, \Omega_2, \bar{p}) \in \mathcal{D}$ and $n^* \in I$ such that

$$\begin{aligned} &f_n(\bar{x}) + [\bar{x} + \bar{p}]^T B_n \bar{w}_n + k_n(\bar{x}, \bar{p}) + \sum_{t \in \Omega_2} \bar{\eta}_t \left[g_t(\bar{x}) + [\bar{x} + \bar{p}]^T C_t \bar{v}_t + h_t(\bar{x}, \bar{p})\right] \\ &\geq f_n(x^*) + {x^*}^T B_n w_n^* + k_n(x^*, 0) + \sum_{t \in \Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t^* + h_t(x^*, 0)\right],\ n \in I - \{n^*\} \\ &f_{n^*}(\bar{x}) + [\bar{x} + \bar{p}]^T B_{n^*} \bar{w}_{n^*} + k_{n^*}(\bar{x}, \bar{p}) + \sum_{t \in \Omega_2} \bar{\eta}_t \left[g_t(\bar{x}) + [\bar{x} + \bar{p}]^T C_t \bar{v}_t + h_t(\bar{x}, \bar{p})\right] \\ &> f_{n^*}(x^*) + {x^*}^T B_n w_n^* + k_{n^*}(x^*, 0) + \sum_{t \in \Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t^* + h_t(x^*, 0)\right] \end{aligned}$$

Using (29) in above equations, we get

$$\begin{aligned} &f_n(\bar{x}) + [\bar{x} + \bar{p}]^T B_n \bar{w}_n + k_n(\bar{x}, \bar{p}) + \sum_{t \in \Omega_2} \bar{\eta}_t \left[g_t(\bar{x}) + [\bar{x} + \bar{p}]^T C_t \bar{v}_t + h_t(\bar{x}, \bar{p})\right] \\ &\geq f_n(x^*) + {x^*}^T B_n w_n^* + \sum_{t \in \Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t^*\right],\ n \in I - \{n^*\} \end{aligned} \tag{30}$$

$$\begin{aligned} &f_{n^*}(\bar{x}) + [\bar{x} + \bar{p}]^T B_{n^*} \bar{w}_{n^*} + k_{n^*}(\bar{x}, \bar{p}) + \sum_{t \in \Omega_2} \bar{\eta}_t \left[g_t(\bar{x}) + [\bar{x} + \bar{p}]^T C_t \bar{v}_t + h_t(\bar{x}, \bar{p})\right] \\ &> f_{n^*}(x^*) + {x^*}^T B_n w_n^* + \sum_{t \in \Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t^*\right] \end{aligned} \tag{31}$$

From (4), (6), (7), (30) and (31), we get

$$\begin{aligned} &f_n(\bar{x}) + [\bar{x} + \bar{p}]^T B_n \bar{w}_n + k_n(\bar{x}, \bar{p}) + \sum_{t \in \Omega_2} \bar{\eta}_t \left[g_t(\bar{x}) + [\bar{x} + \bar{p}]^T C_t \bar{v}_t + h_t(\bar{x}, \bar{p})\right] \\ &\geq f_n(x^*) + \left({x^*}^T B_n x^*\right)^{\frac{1}{2}},\ n \in I - \{n^*\} \\ &f_{n^*}(\bar{x}) + [\bar{x} + \bar{p}]^T B_{n^*} \bar{w}_{n^*} + k_{n^*}(\bar{x}, \bar{p}) + \sum_{t \in \Omega_2} \bar{\eta}_t \left[g_t(\bar{x}) + [\bar{x} + \bar{p}]^T C_t \bar{v}_t + h_t(\bar{x}, \bar{p})\right] \\ &> f_{n^*}(x^*) + \left({x^*}^T B_{n^*} x^*\right)^{\frac{1}{2}} \end{aligned}$$

which implies,

$$f(x^*) \leq \mathcal{F}(\bar{x}, \bar{w}, \bar{v}, \bar{\eta}, \Omega_2, \bar{p})$$

a contradiction to Weak Duality Theorem.
Hence, $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is an efficient solution of (MD).
Now, we will prove that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is a properly efficient solution of the dual problem (MD).
Suppose that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is not a properly efficient solution for (MD). Then, there exists $(\bar{x}, \bar{\mu}, \bar{\eta}, \bar{w}, \bar{v}, \Omega_2, \bar{p}) \in \mathcal{D}$ and $n^* \in I$ such that

$$\begin{aligned} & f_{n^*}(\bar{x}) + [\bar{x}+\bar{p}]^T B_{n^*}\bar{w}_{n^*} + k_{n^*}(\bar{x},\bar{p}) + \sum_{t\in\Omega_2} \eta_t[g_t(\bar{x}) + [\bar{x}+\bar{p}]^T C_t v_t + h_t(\bar{x},\bar{p})] \\ & > f_{n^*}(x^*) + {x^*}^T B_{n^*} w^*_{n^*} + k_{n^*}(x^*,0) + \sum_{t\in\Omega_1} \eta_t[g_t(x^*) + {x^*}^T C_t v_t + h_t(x^*,0)] \end{aligned}$$

such that

$$\begin{aligned} & f_{n^*}(\bar{x}) + [\bar{x}+\bar{p}]^T B_{n^*}\bar{w}_{n^*} + k_{n^*}(\bar{x},\bar{p}) + \sum_{t\in\Omega_2} \eta_t \left[g_t(\bar{x}) + [\bar{x}+\bar{p}]^T C_t v_t + h_t(\bar{x},\bar{p})\right] - \\ & \left(f_{n^*}(x^*) + {x^*}^T B_{n^*} w^*_{n^*} + k_{n^*}(x^*,0) + \sum_{t\in\Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t + h_t(x^*,0)\right]\right) \\ & > M\left(f_r(x^*) + {x^*}^T B_r w^*_r + k_r(x^*,0) + \sum_{t\in\Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t + h_t(x^*,0)\right] - \right. \\ & \left. \left(f_r(\bar{x}) + [\bar{x}+\bar{p}]^T B_r \bar{w}_r + k_r(\bar{x},\bar{p}) + \sum_{t\in\Omega_2} \eta_t \left[g_t(\bar{x}) + [\bar{x}+\bar{p}]^T C_t v_t + h_t(\bar{x},\bar{p})\right]\right)\right) \end{aligned} \tag{32}$$

holds for each scalar $M > 0$ and all $r \in I$ satisfying

$$\begin{aligned} & f_r(x^*) + {x^*}^T B_r w^*_r + k_t(x^*,0) + \sum_{t\in\Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t + h_t(x^*,0)\right] \\ & > f_r(\bar{x}) + [\bar{x}+\bar{p}]^T B_r \bar{w}_r + k_t(\bar{x},\bar{p}) + \sum_{t\in\Omega_2} \eta_t \left[g_t(\bar{x}) + [\bar{x}+\bar{p}]^T C_t v_t + h_t(\bar{x},\bar{p})\right] \end{aligned} \tag{33}$$

Let I_1 denote the set of indices which satisfy (33) and $I_2 = I - (I_1 \cup n^*)$. Let $M > \frac{\mu_{n^*}}{\mu_n} \mid I_1 \mid$.
Therefore, from (33), we have

$$\begin{aligned} & \mu_{n^*}\left(f_{n^*}(\bar{x}) + [\bar{x}+\bar{p}]^T B_{n^*}\bar{w}_{n^*} + k_{n^*}(\bar{x},\bar{p}) + \sum_{t\in\Omega_2} \eta_t \left[g_t(\bar{x}) + [\bar{x}+\bar{p}]^T C_t v_t + h_t(\bar{x},\bar{p})\right] - \right. \\ & \left. \left(f_{n^*}(x^*) + {x^*}^T B_{n^*} w^*_{n^*} + k_{n^*}(x^*,0) + \sum_{t\in\Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t + h_t(x^*,0)\right]\right)\right) \\ & > \sum_{r\in I_1} \mu_r \left(f_r(x^*) + {x^*}^T B_r w^*_r + k_r(x^*,0) + \sum_{t\in\Omega_1} \eta_t \left[g_t(x^*) + {x^*}^T C_t v_t + h_t(x^*,0)\right] - \right. \\ & \left. \left(f_r(\bar{x}) + [\bar{x}+\bar{p}]^T B_r \bar{w}_r + k_r(\bar{x},\bar{p}) + \sum_{t\in\Omega_2} \eta_t \left[g_t(\bar{x}) + [\bar{x}+\bar{p}]^T C_t v_t + h_t(\bar{x},\bar{p})\right]\right)\right) \end{aligned} \tag{34}$$

Using the definition of I_2 and (34), we get

$$\sum_{n=1}^{q}\mu_n\left(f_n(x^*)+{x^*}^TB_nw_n+k_n(x^*,0)+\sum_{t\in\Omega_1}\eta_t\left[g_t(x^*)+{x^*}^TC_tv_t+h_t(x^*,0)\right]\right)$$

$$=\mu_{n^*}\left(f_{n^*}(x^*)+{x^*}^TB_{n^*}w_{n^*}+k_{n^*}(x^*,0)+\sum_{t\in\Omega_1}\eta_t\left[g_t(x^*)+{x^*}^TC_tv_t+h_t(x^*,0)\right]\right)+$$

$$\sum_{n\in I_1}\mu_n\left(f_n(x^*)+{x^*}^TB_nw_n+k_n(x^*,0)+\sum_{t\in\Omega_1}\eta_t\left[g_t(x^*)+{x^*}^TC_tv_t+h_t(x^*,0)\right]\right)+$$

$$\sum_{n\in I_2}\mu_n\left(f_n(x^*)+{x^*}^TB_nw_n+k_n(x^*,0)+\sum_{t\in\Omega_1}\eta_t\left[g_t(x^*)+{x^*}^TC_tv_t+h_t(x^*,0)\right]\right)$$

$$<\mu_{n^*}\left(f_{n^*}(\bar{x})+[\bar{x}+\bar{p}]^TB_{n^*}\bar{w}_{n^*}+k_{n^*}(\bar{x},\bar{p})+\sum_{t\in\Omega_2}\eta_t\left[g_t(\bar{x})+[\bar{x}+\bar{p}]^TC_tv_t+h_t(\bar{x},\bar{p})\right]\right)+$$

$$\sum_{n\in I_1}\mu_n\left(f_n(\bar{x})+[\bar{x}+\bar{p}]^TB_nw_n^*+k_n(\bar{x},\bar{p})+\sum_{t\in\Omega_2}\eta_t\left[g_t(\bar{x})+[\bar{x}+\bar{p}]^TC_tv_t+h_t(\bar{x},\bar{p})\right]\right)+$$

$$\sum_{n\in I_2}\mu_n\left(f_n(\bar{x})+[\bar{x}+\bar{p}]^TB_nw_n^*+k_n(\bar{x},\bar{p})+\sum_{t\in\Omega_2}\eta_t\left[g_t(\bar{x})+[\bar{x}+\bar{p}]^TC_tv_t+h_t(\bar{x},\bar{p})\right]\right)+$$

$$=\sum_{n=1}^{q}\mu_n\left(f_n(\bar{x})+[\bar{x}+\bar{p}]^TB_nw_n^*+k_n(\bar{x},\bar{p})+\sum_{t\in\Omega_2}\eta_t\left[g_t(\bar{x})+[\bar{x}+\bar{p}]^TC_tv_t+h_t(\bar{x},\bar{p})\right]\right)\tag{35}$$

From (4), (29) and (35) we have

$$\sum_{n=1}^{q}\mu_n\left(f_n(x^*)+(x^TB_nx^*)^{\frac{1}{2}}\right)<\sum_{n=1}^{q}\mu_n\left(f_n(\bar{x})+[\bar{x}+\bar{p}]^TB_nw_n^*+k_n(\bar{x},\bar{p})+\sum_{t\in\Omega_2}\eta_t\left[g_t(\bar{x})+[\bar{x}+\bar{p}]^T+h_t(\bar{x},\bar{p})\right]\right)\tag{36}$$

a contradiction to Weak Duality Theorem.
Hence, $(x^*,\mu^*,\eta^*,w^*,v^*,\Omega_1,p^*=0)$ is a properly efficient solution for the problem (MD).

Theorem 3.3. (Restricted Converse Duality) *Let x^* and $(\bar{u},\bar{\mu},\bar{\eta},\bar{w},\bar{v},\bar{\Omega},\bar{p})$ be feasible solutions for problem (P) and (MD) respectively, such that*

$$f_n(x^*)+{x^*}^TB_n\bar{w}_n\le f_n(\bar{u})+[\bar{u}+\bar{p}]^TB_n\bar{w}_n+k_n(\bar{u},\bar{p})+\sum_{t\in\bar{\Omega}}\bar{\eta}_t[g_t(\bar{u})+\bar{u}^TC_t\bar{v}_t+h_t(\bar{u},\bar{p})]\tag{37}$$

Further assume that $\left((f_n(.)+(.)^TB_n\bar{w}_n,\ n\in I),(g_t(.)+(.)^TC_t\bar{v}_t,\ t\in\bar{\Omega})\right)$ is higher-order strictly $(\Phi,\rho,\bar{\Omega})-$ type I objective and constraint functions at u on $M\cup\mathcal{D}$. If $\sum_{n=1}^{q}\mu_n\rho_{f_n}+\sum_{t\in\bar{\Omega}}\eta_t\rho_{g_t}\ge 0$, then $x^*=\bar{u}$.

Proof Suppose on the contrary that $x^*\neq\bar{u}$
Since $\left((f_n(.)+(.)^TB_n\bar{w}_n,\ n\in I),(g_t(.)+(.)^TC_t\bar{v}_t,\ t\in\bar{\Omega})\right)$ is higher-order strictly $(\Phi,\rho,\bar{\Omega})-$ type I objective and constraint functions at $\bar{u}$ on $M\cup\mathcal{D}$, therefore, we have

$$\begin{aligned}&f_n(x^*)+{x^*}^TB_n\bar{w}_n-f_n(\bar{u})-\bar{u}^TB_n\bar{w}_n-k_n(\bar{u},\bar{p})+\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})\\&>\Phi\left(x^*,\bar{u},\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n,\rho_{f_n}\right),\ n\in I\end{aligned}\tag{38}$$

$$-g_t(\bar{u})-\bar{u}^TC_t\bar{v}_t-h_t(\bar{u},\bar{p})+\bar{p}^T\nabla_ph_t(\bar{u},\bar{p})\ge\Phi\left(x^*,\bar{u},\nabla_ph_t(\bar{u},\bar{p})+C_t\bar{v}_t,\rho_{g_t}\right),\ t\in\bar{\Omega}\tag{39}$$

Since $\bar{\mu}_n>0,\ n\in I$ and $\bar{\eta}_t\ge 0,\ t\in\bar{\Omega}$, therefore, from (38) and (39), we have

$$\begin{aligned}&\bar{\mu}_n\left[f_n(x^*)+{x^*}^TB_n\bar{w}_n-f_n(\bar{u})-\bar{u}^TB_n\bar{w}_n-k_n(\bar{u},\bar{p})+\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})\right]\\&>\bar{\mu}_n\Phi\left(x^*,\bar{u},\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n,\rho_{f_n}\right),\ n\in I\end{aligned}\tag{40}$$

$$\bar{\eta}_t \left[-g_t(\bar{u}) - \bar{u}^T C_t \bar{v}_t - h_t(\bar{u},\bar{p}) + \bar{p}^T \nabla_p h_t(\bar{u},\bar{p})\right] \geq \bar{\eta}_t \Phi\left(x^*, \bar{u}, \nabla_p h_t(\bar{u},\bar{p}) + C_t\bar{v}_t, \rho_{g_t}\right),\ t \in \bar{\Omega} \tag{41}$$

Adding (40) and (41), we get

$$\sum_{n=1}^{p} \bar{\mu}_n \left[f_n(x^*) + {x^*}^T B_n \bar{w}_n - f_n(\bar{u}) - \bar{u}^T B_n \bar{w}_n - k_n(\bar{u},\bar{p}) + \bar{p}^T \nabla_p k_n(\bar{u},\bar{p})\right] +$$
$$\sum_{t\in\bar{\Omega}} \bar{\eta}_t \left[-g_t(\bar{u}) - \bar{u}^T C_t \bar{v}_t - h_t(\bar{u},\bar{p}) + \bar{p}^T \nabla_p h_t(\bar{u},\bar{p})\right]$$
$$> \sum_{n=1}^{p} \bar{\mu}_n \Phi\left(x^*, \bar{u}, \nabla_p k_n(\bar{u},\bar{p}) + B_n\bar{w}_n, \rho_{f_n}\right) + \sum_{t\in\bar{\Omega}} \bar{\eta}_t \Phi\left(x^*, \bar{u}, \nabla_p h_t(\bar{u},\bar{p}) + C_t\bar{v}_t, \rho_{g_t}\right) \tag{42}$$

Put

$$\mu_n^* = \frac{\bar{\mu}_n}{\sum_{n=1}^{p} \bar{\mu}_n + \sum_{t\in\bar{\Omega}} \bar{\eta}_t},\ n \in I \text{ and } \eta_t^* = \frac{\bar{\eta}_t}{\sum_{n=1}^{p} \bar{\mu}_n + \sum_{t\in\bar{\Omega}} \bar{\eta}_t},\ t \in \bar{\Omega} \tag{43}$$

Clearly $\mu_n^*,\ \eta_t^* \in [0,1]$ and $\sum_{n=1}^{p} \mu_n^* + \sum_{t\in\bar{\Omega}} \eta_t^* = 1$
Using (42) and (43), we get

$$\sum_{n=1}^{p} \mu_n^* [f_n(x^*) + {x^*}^T B_n \bar{w}_n - f_n(\bar{u}) - \bar{u}^T B_n \bar{w}_n - k_n(\bar{u},\bar{p}) + \bar{p}^T \nabla_p k_n(\bar{u},\bar{p})] +$$
$$\sum_{t\in\bar{\Omega}} \eta_t^* \left[-g_t(\bar{u}) - \bar{u}^T C_t \bar{v}_t - h_t(\bar{u},\bar{p}) + \bar{p}^T \nabla_p h_t(\bar{u},\bar{p})\right]$$
$$> \sum_{n=1}^{p} \mu_n^* \Phi(x^*, \bar{u}, \nabla_p k_n(\bar{u},\bar{p}) + B_n\bar{w}_n, \rho_{f_n}) + \sum_{t\in\bar{\Omega}} \eta_t^* \Phi\left(x^*, \bar{u}, \nabla_p h_t(\bar{u},\bar{p}) + C_t\bar{v}_t, \rho_{g_t}\right)$$

Since, $\Phi(x, u, .)$ is convex on $\mathbb{R}^{m+1}$, $\mu_n^*,\ \eta_t^* \in [0,1],\ n \in I,\ t \in T$ and $\sum_{n=1}^{p} \mu_n^* + \sum_{t\in\bar{\Omega}} \eta_t^* = 1$, therefore, we have

$$\sum_{n=1}^{p} \mu_n^* \left[f_n(x^*) + {x^*}^T B_n \bar{w}_n - f_n(\bar{u}) - \bar{u}^T B_n \bar{w}_n - k_n(\bar{u},\bar{p}) + \bar{p}^T \nabla_p k_n(\bar{u},\bar{p})\right] +$$
$$\sum_{t\in\bar{\Omega}} \eta_t^* \left[-g_t(\bar{u}) - \bar{u}^T C_t \bar{v}_t - h_t(\bar{u},\bar{p}) + \bar{p}^T \nabla_p h_t(\bar{u},\bar{p})\right]$$
$$> \Phi\left(x^*, \bar{u}, \left(\sum_{n=1}^{p} \mu_n^* \left(\nabla_p k_n\left(\bar{u},\bar{p}\right) + B_n\bar{w}_n\right), \rho_{f_n}\right) + \left(\sum_{t\in\bar{\Omega}} \tilde{\eta}_t \left(\nabla_p h_t\left(\bar{u},\bar{p}\right) + C_t\bar{v}_t, \rho_{g_t}\right)\right)\right) \tag{44}$$

From (43) and (44), we get

$$\frac{1}{\sum_{n=1}^{p} \bar{\mu}_n + \sum_{t\in\bar{\Omega}} \bar{\eta}_t} \sum_{n=1}^{p} \bar{\mu}_n \left[f_n(x^*) + {x^*}^T B_n \bar{w}_n - f_n(\bar{u}) - \bar{u}^T B_n \bar{w}_n - k_n(\bar{u},\bar{p}) + \bar{p}^T \nabla_p k_n(\bar{u},\bar{p}\right] +$$
$$\frac{1}{\sum_{n=1}^{p} \bar{\mu}_n + \sum_{t\in\bar{\Omega}} \bar{\eta}_t} \sum_{t\in\bar{\Omega}} \eta_t^* \left[-g_t(\bar{u}) - \bar{u}^T C_t \bar{v}_t - h_t(\bar{u},\bar{p}) + \bar{p}^T \nabla_p h_t(\bar{u},\bar{p})\right]$$
$$> \Phi\left(x^*, \bar{u}, \frac{1}{\sum_{n=1}^{p} \bar{\mu}_n + \sum_{t\in\bar{\Omega}} \bar{\eta}_t}\left(\sum_{n=1}^{p} \bar{\mu}_n (\nabla_p k_n(\bar{u},\bar{p}) + B_n\bar{w}_n) + \sum_{t\in\bar{\Omega}} \bar{\eta}_t \left(\nabla_p h_t(\bar{u},\bar{p}) + C_t\bar{v}_t\right),\right.\right.$$
$$\left.\left.\sum_{n=1}^{p} \bar{\mu}_n \rho_{f_n} \sum_{t\in\bar{\Omega}} \tilde{\eta}_t \rho_{g_t}\right)\right) \tag{45}$$

From (10) and (45), we get

$$\frac{1}{\sum_{n=1}^{p}\bar{\mu}_n+\sum_{t\in\bar{\Omega}}\bar{\eta}_t}\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(x^*)+{x^*}^TB_n\bar{w}_n-f_n(\bar{u})-\bar{u}^TB_n\bar{w}_n-k_n(\bar{u},\bar{p})+\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})\right]+$$
$$\frac{1}{\sum_{n=1}^{p}\bar{\mu}_n+\sum_{t\in\bar{\Omega}}\bar{\eta}_t}\sum_{t\in\bar{\Omega}}\tilde{\eta}_t\left[-g_t(\bar{u})-\bar{u}^TC_t\bar{v}_t-h_t(\bar{u},\bar{p})+\bar{p}^T\nabla_ph_t(\bar{u},\bar{p})\right]$$
$$>\Phi\left(x^*,\bar{u},0,\sum_{n=1}^{p}\bar{\mu}_n\rho_{f_n}\sum_{t\in\bar{\Omega}}\tilde{\eta}_t\rho_{g_t})\right) \tag{46}$$

Since $\Phi(x^*,\bar{u},(0,a))\geq 0$ for every $a\geq 0$ and $\sum_{n=1}^{p}\bar{\mu}_n\rho_{f_n}+\sum_{t\in\bar{\Omega}}\eta_t\rho_{g_t}\geq 0$, therefore, we have

$$\Phi\left(x^*,\bar{u},\left(0,\sum_{n=1}^{p}\bar{\mu}_n\rho_{f_n}\sum_{t\in\bar{\Omega}}\tilde{\eta}_t\rho_{g_t}\right)\right)\geq 0 \tag{47}$$

From (46) and (47), we have

$$\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(x^*)+{x^*}^TB_n\bar{w}_n\right]>\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(\bar{u})+\bar{u}^TB_n\bar{w}_n+k_n(\bar{u},\bar{p})-\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})\right]+$$
$$\sum_{t\in\bar{\Omega}}\bar{\eta}_t\left[g_t(\bar{u})+\bar{u}^TC_t\bar{v}_t+h_t(\bar{u},\bar{p})-\bar{p}^T\nabla_ph_t(\bar{u},\bar{p})\right]$$
$$=\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(\bar{u})+[\bar{u}+\bar{p}]^TB_n\bar{w}_n+k_n(\bar{u},\bar{p}\right]+$$
$$\sum_{t\in\bar{\Omega}}\bar{\eta}_t\left[g_t(\bar{u})+[\bar{u}+\bar{p}]^TC_t\bar{v}_t+h_t(\bar{u},\bar{p})\right]-$$
$$\bar{p}^T\left[\sum_{n=1}^{p}\bar{\mu}_n\left(\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n\right)+\sum_{t\in\bar{\Omega}}\bar{\eta}_t\nabla_p\left(h_t(\bar{u},\bar{p})+C_t\bar{v}_t\right)\right] \tag{48}$$

From (10) and (48), we have

$$\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(x^*)+{x^*}^TB_n\bar{w}_n\right]>\sum_{n=1}^{p}\bar{\mu}_n[f_n(\bar{u})+[\bar{u}+\bar{p}]^TB_n\bar{w}_n+k_n(\bar{u},\bar{p})]+$$
$$\sum_{t\in\bar{\Omega}}\bar{\eta}_t\left[g_t(\bar{u})+\bar{u}^TC_t\bar{v}_t+h_t(\bar{u},\bar{p})\right] \tag{49}$$

Since $\bar{\mu}_n>0,\ n\in I$ and $\sum_{n=1}^{p}\bar{\mu}_n=1$, therefore, from (37) we have

$$\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(x^*)+{x^*}^TB_n\bar{w}_n\right]\leqq\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(\bar{u})+[\bar{u}+\bar{p}]^TB_n\bar{w}_n+k_n(\bar{u},\bar{p})\right]+$$
$$\sum_{t\in\bar{\Omega}}\bar{\eta}_t\left[g_t(\bar{u})+\bar{u}^TC_t\bar{v}_t+h_t(\bar{u},\bar{p})\right]$$

a contradiction to (49).
Hence, $x^*=\bar{u}$.

II. Mond-Weir Duality

For each $u\in X,\ \Omega\subset T$ with $\mid\Omega\mid<\infty,\ p\in\mathbb{R}^m,\ w=(w_n),\ w_n\in\mathbb{R}^m, n\in I,\ v=v_t\in\mathbb{R}^m, t\in\Omega$ and $\eta=(\eta_t), t\in\Omega$
Define

$$\mathcal{G}(u,w)=(f_1(u)+u^TB_nw_n,f_2(u)+u^TB_2w_2,...,f_q(u)+u^TB_qw_q)$$

Then, the Mond-Weir dual of the primal problem (P) is given by

$$Maximize \quad \mathcal{G}(u, w, v, \eta, \Omega, p) \tag{MWD}$$

$$\text{s.t.} \quad \sum_{n=1}^{q} \mu_n [\nabla_p k_n(u, p) + B_n w_n] + \sum_{t \in \Omega} \eta_t [\nabla_p h_t(u, p) + C_t v_t] = 0 \tag{50}$$

$$\sum_{t \in \Omega} \eta_t [g_t(u) + u^T C_t v_t + h_t(u, p) - p^T \nabla_p h_t(u, p)] \geq 0 \tag{51}$$

$$\sum_{n=1}^{q} \mu_n [k_n(u, p) - p^T \nabla_p k_n(u, p)] \geq 0 \tag{52}$$

$$w_n^T B_n w_n \leq 1 \quad n \in I \tag{53}$$

$$v_t^T B_n v_t \leq 1 \quad t \in \Omega \tag{54}$$

$$u \in X, \ p \in \mathbb{R}^m, \ w_n \in \mathbb{R}^m, \ v_t \in \mathbb{R}^m, \ \mu_n > 0, \ n \in I, \ \xi_t \geq 0, \ t \in \Omega$$

Let $\mathcal{U}$ denotes the feasible set of the Mond-Weir dual problem (MWD).

Theorem 3.4. (Weak Duality Theorem) *Let x and (u, w) be any feasible solution for the problem (P) and (MWD) respectively. Further, assume that $\big((f_n(.) + (.)^T B_n w_n, \ n \in I), (g_t(.) + (.)^T C_t v_t, \ t \in \Omega)\big)$ is higher-order $(\Phi, \rho, \Omega)-$ type I objective and constraint functions at u on $M \cup \mathcal{U}$. If $\sum_{n=1}^{p} \mu_n \rho_{f_n} + \sum_{t \in T} \eta_t \rho_{g_t} \geq 0$, then the following cannot hold:*

$$f(x) \leq \mathcal{G}(u, w) \tag{55}$$

Proof Let x and (u, w) be any feasible solution for the problem (P) and (MWD) respectively.
Suppose that (55) holds, therefore, there exist atleast one $r \in I$ such that

$$f_n(x) + \left(x^T B_n x\right)^{\frac{1}{2}} \leq f_n(u) + u^T B_n w_n, \ n \in I - \{k\} \tag{56}$$

$$f_r(x) + \left(x^T B_r x\right)^{\frac{1}{2}} < f_r(u) + u^T B_r w_r \tag{57}$$

By generalized Schwarz inequality, (56) and (57), we have

$$f_n(x) + x^T B_n w_n \leq f_n(u) + u^T B_n w_n, \ n \in I - \{k\} \tag{58}$$

$$f_r(x) + x^T B_r w_r < f_r(u) + u^T B_r w_r \tag{59}$$

Since $(u, \mu, \eta, w, v, \Omega, p)$ is a feasible solution for the problem (MWD), therefore, we have $\mu_n > 0, \ n \in I$ and $\sum_{n=1}^{q} \mu_n = 1$.
Hence, from (58) and (59) we get

$$\sum_{n=1}^{q} \mu_n \left[f_n(x) + x^T B_n w_n\right] < \sum_{n=1}^{q} \mu_n \left[f_n(u) + u^T B_n w_n\right] \tag{60}$$

Since $\big((f_n(.) + (.)^T B_n w_n, \ n \in I), (g_t(.) + (.)^T C_t v_t, \ t \in \Omega)\big)$ is higher-order $(\Phi, \rho, \Omega)-$ type I objective and constraint functions at u on $M \cup \mathcal{U}$, therefore, we have

$$f_n(x) + x^T B_n w_n - f_n(u) - u^T B_n w_n - k_n(u, p) + p^T \nabla_p k_n(u, p) \geq \Phi\left(x, u, (\nabla_p k_n(u, p) + B_n w_n, \rho_{f_n})\right), \quad \forall\, n \in I$$

$$-g_t(u) - u^T C_t v_t - h_t(u, p) + p^T \nabla_p h_t(u, p) \geq \Phi(x, u, (\nabla_p h_t(u, p) + C_t v_t, \rho_{g_t})), \quad \forall\, t \in \Omega$$

Since $\mu_n > 0, \ \forall\, n \in I$ and $\eta_t \geq 0$, therefore, we have

$$\mu_n \left[f_n(x) + x^T B_n w_n - f_n(u) + u^T B_n w_n - k_n(u, p) + p^T \nabla_p k_n(u, p)\right] \geq \mu_n \Phi\left(x, u, (\nabla_p k_n(u, p) + B_n w_n, \rho_{f_n})\right), \ \forall\, n \in I \tag{61}$$

$$-\eta_t [g_t(u) + u^T C_t v_t + h_t(u, p) - p^T \nabla_p h_t(u, p)] \geq \eta_t \Phi(x, u, (\nabla_p h_t(u, p) + C_t v_t, \rho_{g_t})), \ \forall\, t \in \Omega \tag{62}$$

Adding (61) and (62), we get

$$\begin{aligned}&\sum_{n=1}^{q}\mu_n[f_n(x)+x^TB_nw_n-f_n(u)-u^TB_nw_n-k_n(u,p)+p^T\nabla_pk_n(u,p)]-\\&\sum_{t\in\Omega}\eta_t[g_t(u)+u^TC_tv_t+h_t(u,p)-p^T\nabla_ph_t(u,p)]\\&\geq\sum_{n=1}^{q}\mu_n[\Phi(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n}))]+\sum_{t\in\Omega}\eta_t[\Phi(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t}))]\end{aligned}\tag{63}$$

From (51), (52) and (63), we have

$$\sum_{n=1}^{q}\mu_n\Phi(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n}))+\sum_{t\in\Omega}\eta_t\Phi(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t}))<0\tag{64}$$

Put

$$\mu_n^*=\frac{\mu_n}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\eta_t},\ n\in I\text{ and }\eta_t^*=\frac{\eta_t}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\eta_t},\ t\in\Omega\tag{65}$$

Clearly $\mu_n^*,\ \eta_t^*\in[0,1],\ \forall\, n\in I,\ t\in\Omega$ and $\sum_{n=1}^{q}\mu_n^*+\sum_{t\in\Omega}\eta_t^*=1$
Using (64) and (65), we get

$$\sum_{n=1}^{q}\mu_n^*[\Phi\,(x,u,(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n}))]+\sum_{t\in\Omega}\eta_n^*[\Phi\,(x,u,(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t}))]<0\tag{66}$$

Since, $\Phi(x,u,.)$ is convex on $\mathbb{R}^{m+1}$ and $\mu_n^*,\ \eta_t^*\in[0,1],\ \forall\ i\in I,\ t\in\Omega$ and $\sum_{n=1}^{q}\mu_n^*+\sum_{t\in\Omega}\eta_t^*=1$, therefore, from (66) we have

$$\Phi\left(x,u,\left(\sum_{n=1}^{q}\mu_n^*\left(\nabla_pk_n(u,p)+B_nw_n,\rho_{f_n}\right)+\sum_{t\in\Omega}\eta_t^*\left(\nabla_ph_t(u,p)+C_tv_t,\rho_{g_t}\right)\right)\right)<0\tag{67}$$

Using (10), (66) and (67), we get

$$\Phi\left(x,u,\frac{1}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\xi_t}\left(0,\sum_{n=1}^{q}\mu_n\rho_{f_n}+\sum_{t\in\Omega}\eta_t\rho_{g_t})\right)\right)<0\tag{68}$$

Since $\Phi(x,u,(0,a))\geqq 0$ for every $a\in\mathbb{R}^+$ and $\sum_{n=1}^{q}\mu_n\rho_{f_n}+\sum_{t\in\Omega}\xi_t\rho_{g_t}\geq 0$, therefore, we have

$$\Phi\left(x,u,\frac{1}{\sum_{n=1}^{q}\mu_n+\sum_{t\in\Omega}\xi_t}\left(0,\sum_{n=1}^{q}\rho_{f_n}+\sum_{t\in\Omega}\eta_t\rho_{g_t}\right)\right)\geq 0$$

a contradiction to (68).
Hence, (55) does not hold.

Theorem 3.5. (Strong Duality Theorem) *Let x^* be a properly efficient solution for the problem (P) such that the set $\Omega(x^*)$ is empty and and a suitable constraint qualification be satisfied at x^*. Further, assume that*

$$\begin{aligned}&k_n(x^*,0)=0\ \textit{for all}\,n\in I;\ \nabla_pk(x^*,0)=\nabla f(x^*)\\&h_t(x^*,0)=0\ \textit{for all}\,t\in T;\ \nabla_ph(x^*,0)=\nabla g(x^*)\end{aligned}\tag{69}$$

Then, there exist $\mu^*=(\mu_n^*)>0,\ \eta^*=(\eta_t^*)\geq 0,\ w^*=(w_n^*),\ v^*=v_t^*),\ \Omega_1\subseteq T(x^*)$ with $\mid\Omega_1\mid<\infty$ such that $(x^*,\mu^*,\eta^*,w^*,v^*,\Omega_1,p^*=0)$ is feasible for (MWD) and the corresponding objective values of (P) and (MWD) are equal. Further, if weak duality holds, then $(x^*,\mu^*,\eta^*,w^*,v^*,\Omega_1,p^*=0)$ is a properly efficient solution of a maximum type in (MWD).

Proof Let x^* be a properly efficient solution for the problem (P), the set $\Omega(x)$ is empty and the Linear Independence Constraint Qualification is satisfied at x^*.
By Necessary Optimality Theorem and using (69), it follows that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is feasible in (MWD). From (6), we obtain

$$f(x^*) = \mathcal{G}(x^*, w^*)$$

First, we show that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is an efficient solution for the problem (MWD).
Suppose that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is not an efficient of the problem (MWD). Then, there exists $(\bar{x}, \bar{\mu}, \bar{\eta}, \bar{w}, \bar{v}, \Omega_2, \bar{p}) \in \mathcal{D}$ and $n^* \in I$ such that

$$f_n(\bar{x}) + \bar{x}^T B_n \bar{w}_n \geq f_n(x^*) + {x^*}^T B_n w_n^*, \; n \in I - \{n^*\} \tag{70}$$

$$f_{n^*}(\bar{x}) + \bar{x}^T B_{n^*} \bar{w}_{n^*} > f_{n^*}(x^*) + {x^*}^T B_n w_n^* \tag{71}$$

Hence by (6), (70) and (71), we get

$$f_n(\bar{x}) + \bar{x}^T B_n \bar{w}_n \geq f_n(x^*) + \left({x^*}^T B_n x^*\right)^{\frac{1}{2}}, \; n \in I - \{n^*\}$$

$$f_{n^*}(\bar{x}) + \bar{x}^T B_{n^*} \bar{w}_{n^*} > f_{n^*}(x^*) + \left({x^*}^T B_{n^*} x^*\right)^{\frac{1}{2}}$$

a contradiction to Weak Duality Theorem.
Hence, $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is an efficient solution for the problem (MWD).
Now, we will prove that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is a properly efficient solution of the dual problem.
Suppose that $(x^*, \mu^*, \eta^*, w^*, v^*, \Omega_1, p^* = 0)$ is not a properly efficient solution for (MWD). Then, there exists $(\bar{x}, \bar{\mu}, \bar{\eta}, \bar{w}, \bar{v}, \Omega_2, \bar{p}) \in \mathcal{D}$ and $n^* \in I$ such that

$$f_{n^*}(\bar{x}) + \bar{x}^T B_{n^*} \bar{w}_{n^*} > f_{n^*}(x^*) + {x^*}^T B_{n^*} w_{n^*}^*$$

such that

$$f_{n^*}(\bar{x}) + \bar{x}^T B_{n^*} \bar{w}_{n^*} - \left(f_{n^*}(x^*) + {x^*}^T B_{n^*} w_{n^*}^*\right) > M \left(f_r(x^*) + {x^*}^T B_r w_r^* - \left(f_r(\bar{x}) + \bar{x}^T B_r \bar{w}_r\right)\right) \tag{72}$$

holds for each scalar $M > 0$ and all $r \in I$ satisfying

$$f_r(x^*) + {x^*}^T B_r w_r^* > f_r(\bar{x}) + \bar{x}^T B_r \bar{w}_r$$

Let I_1 denote the set of indices which satisfy () and $I_2 = I - (I_1 \cup n^*)$.
Let $M > \frac{\mu_{n^*}}{\mu_n} \mid I_1 \mid$.
Therefore, from (72), we get

$$\mu_{n^*} \left(f_{n^*}(\bar{x}) + \bar{x}^T B_{n^*} \bar{w}_{n^*} - \left(f_{n^*}(x^*) + {x^*}^T B_{n^*} w_{n^*}^*\right)\right) > \sum_{r \in I_1} \mu_r \left(f_r(x^*) + {x^*}^T B_r w_r^* - \left(f_r(\bar{x}) + \bar{x}^T B_r \bar{w}_r\right)\right) \tag{73}$$

Using the definition of I_2 and (73), we get

$$\begin{aligned}
&\sum_{n=1}^{q} \mu_n (f_n(x^*) + {x^*}^T B_n w_n \\
&= \mu_{n^*} \left(f_{n^*}(x^*) + {x^*}^T B_{n^*} w_{n^*}\right) + \sum_{n \in I_1} \mu_n \left(f_n(x^*) + {x^*}^T B_n w_n\right) + \sum_{n \in I_2} \mu_n \left(f_n(x^*) + {x^*}^T B_n w_n\right) \\
&< \mu_{n^*} \left(f_{n^*}(\bar{x}) + \bar{x}^T B_{n^*} \bar{w}_{n^*}\right) + \sum_{n \in I_1} \mu_n \left(f_n(\bar{x}) + \bar{x}]^T B_n w_n^*\right) + \sum_{n \in I_2} \mu_n \left(f_n(\bar{x}) + \bar{x}^T B_n w_n^*\right) \\
&= \sum_{n=1}^{q} \mu_n \left(f_n(\bar{x}) + \bar{x}^T B_n w_n^*\right)
\end{aligned} \tag{74}$$

From (6)and (74), we have

$$\sum_{n=1}^{q}\mu_n(f_n(x^*)+(x^TB_nx^*)^{\frac{1}{2}})<\sum_{n=1}^{q}\mu_n\left(f_n(\bar{x})+\bar{x}^TB_nw_n^*\right) \tag{75}$$

a contradiction to Weak Duality Theorem.
Hence, $(x^*,\mu^*,\eta^*,w^*,v^*,\Omega_1,p^*=0)$ is a properly efficient solution of (MWD).

Theorem 3.6. (Restricted Converse Duality) *Let x^* and $(\bar{u},\bar{\mu},\bar{\eta},\bar{w},\bar{v},\bar{\Omega},\bar{p})$ be feasible solutions for problem (P) and (MWD) respectively, such that*

$$f_n(x^*)+{x^*}^TB_n\bar{w}_n\leq f_n(\bar{u})+\bar{u}^TB_n\bar{w}_n \tag{76}$$

Further assume that $\left((f_n(.)+(.)^TB_n\bar{w}_n,\ n\in I),(g_t(.)+(.)^TC_t\bar{v}_t,\ t\in\bar{\Omega})\right)$ is higher-order strictly $(\Phi,\rho,\bar{\Omega})-$ type I objective and constraint functions at u on $M\cup\mathcal{U}$. If $\sum_{n=1}^{q}\mu_n\rho_{f_n}+\sum_{t\in\bar{\Omega}}\eta_t\rho_{g_t}\geqq 0$, then $x^*=\bar{u}$.

Proof Suppose on the contrary that $x^*\neq\bar{u}$
Since $\bar{\mu}_n>0,\ n\in I$ and $\sum_{n=1}^{p}\bar{\mu}_n=1$, therefore from (76), we have

$$\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(x^*)+{x^*}^TB_n\bar{w}_n\right]\leq\sum_{n=1}^{p}\bar{\mu}_n\left[f_n(\bar{u})+\bar{u}^TB_n\bar{w}_n\right] \tag{77}$$

Since $\left((f_n(.)+(.)^TB_n\bar{w}_n,\ n\in I),(g_t(.)+(.)^TC_t\bar{v}_t,\ t\in\bar{\Omega})\right)$ is higher-order strictly $(\Phi,\rho,\bar{\Omega})-$ type I objective and constraint functions at $\bar{u}$ on $M\cup\mathcal{U}$, therefore, we have

$$\begin{aligned}&f_n(x^*)+{x^*}^TB_n\bar{w}_n-f_n(\bar{u})-\bar{u}^TB_n\bar{w}_n-k_n(\bar{u},\bar{p})+\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})\\&>\Phi(x^*,\bar{u},\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n,\rho_{f_n}),\ n\in I\end{aligned}$$

$$-g_t(\bar{u})-\bar{u}^TC_t\bar{v}_t-h_t(\bar{u},\bar{p})+\bar{p}^T\nabla_ph_t(\bar{u},\bar{p})\geq\Phi(x^*,\bar{u},\nabla_ph_t(\bar{u},\bar{p})+C_t\bar{v}_t,\rho_{g_t}),\ t\in\bar{\Omega}$$

Since $\bar{\mu}_n>0,\ n\in I$ and $\bar{\eta}_t\geq 0,\ t\in\bar{\Omega}$, therefore, we have

$$\begin{aligned}&\bar{\mu}_n[f_n(x^*)+{x^*}^TB_n\bar{w}_n-f_n(\bar{u})-\bar{u}^TB_n\bar{w}_n-k_n(\bar{u},\bar{p})+\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})]\\&>\bar{\mu}_n\Phi(x^*,\bar{u},\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n,\rho_{f_n}),\ n\in I\end{aligned} \tag{78}$$

$$\bar{\eta}_t[-g_t(\bar{u})-\bar{u}^TC_t\bar{v}_t-h_t(\bar{u},\bar{p})+\bar{p}^T\nabla_ph_t(\bar{u},\bar{p})]\geq\bar{\eta}_t\Phi(x^*,\bar{u},\nabla_ph_t(\bar{u},\bar{p})+C_t\bar{v}_t,\rho_{g_t}),\ t\in\bar{\Omega} \tag{79}$$

Adding (78) and (79), we get

$$\begin{aligned}&\sum_{n=1}^{q}\bar{\mu}_n[f_n(x^*)+{x^*}^TB_n\bar{w}_n-f_n(\bar{u})-\bar{u}^TB_n\bar{w}_n-k_n(\bar{u},\bar{p})+\bar{p}^T\nabla_pk_n(\bar{u},\bar{p})]-\\&\sum_{t\in\bar{\Omega}}\bar{\eta}_t[g_t(\bar{u})+\bar{u}^TC_t\bar{v}_t+h_t(\bar{u},\bar{p})-\bar{p}^T\nabla_ph_t(\bar{u},\bar{p})]\\&>\sum_{n=1}^{q}\bar{\mu}_n\Phi(x^*,\bar{u},(\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n,\rho_{f_n}))+\sum_{t\in\Omega}\bar{\eta}_t\Phi(x^*,\bar{u},(\nabla_ph_t(\bar{u},\bar{p})+C_t\bar{v}_t,\rho_{g_t}))\end{aligned} \tag{80}$$

From (51), (52) and (80), we have

$$\sum_{n=1}^{q}\bar{\mu}_n\Phi(x^*,\bar{u},(\nabla_pk_n(\bar{u},\bar{p})+B_n\bar{w}_n,\rho_{f_n}))+\sum_{t\in\bar{\Omega}}\bar{\eta}_t\Phi(x^*,\bar{u},(\nabla_ph_t(\bar{u},\bar{p})+C_t\bar{v}_t,\rho_{g_t}))<0 \tag{81}$$

Put

$$\mu_n^*=\frac{\bar{\mu}_n}{\sum_{n=1}^{p}\bar{\mu}_n+\sum_{t\in\bar{\Omega}}\bar{\eta}_t},\ n\in I\text{ and }\eta_t^*=\frac{\bar{\eta}_t}{\sum_{n=1}^{p}\bar{\mu}_n+\sum_{t\in\bar{\Omega}}\bar{\eta}_t},\ t\in\bar{\Omega} \tag{82}$$

Clearly $\mu_n^*,\ \eta_t^* \in [0,1]$ and $\sum_{n=1}^{p} \mu_n^* + \sum_{t\in\bar{\Omega}} \eta_t^* = 1$
Using (81) and (82), we get

$$\sum_{n=1}^{p} \mu_n^* \Phi(x^*, \bar{u}, \nabla_p k_n(\bar{u}, \bar{p}) + B_n \bar{w}_n, \rho_{f_n}) + \sum_{t\in\bar{\Omega}} \eta_t^* \Phi(x^*, \bar{u}, \nabla_p h_t(\bar{u}, \bar{p}) + C_t \bar{v}_t, \rho_{g_t}) < 0 \tag{83}$$

Since, $\Phi(x, u, .)$ is convex on $\mathbb{R}^{m+1}$, $\mu_n^*,\ \eta_t \in [0,1],\ n \in I,\ t \in T$ and $\sum_{n=1}^{p} \mu_n^* + \sum_{t\in\bar{\Omega}} \eta_t^* = 1$, therefore, we have

$$\Phi\left(x^*, \bar{u}, \left(\sum_{n=1}^{p} \mu_n^* \left(\nabla_p k_n\left(\bar{u}, \bar{p}\right) + B_n \bar{w}_n\right), \rho_{f_n}\right) + \left(\sum_{t\in\bar{\Omega}} \tilde{\eta}_t \left(\nabla_p h_t\left(\bar{u}, \bar{p}\right) + C_t \bar{v}_t, \rho_{g_t}\right)\right)\right) < 0 \tag{84}$$

From (50) and (84), we have

$$\Phi\left(x^*, \bar{u}, \frac{1}{\sum_{n=1}^{p} \bar{\mu}_n + \sum_{t\in\bar{\Omega}} \bar{\eta}_t}\left(0, \sum_{n=1}^{p} \bar{\mu}_n \rho_{f_n} \sum_{t\in\bar{\Omega}} \tilde{\eta}_t \rho_{g_t}\right)\right) < 0 \tag{85}$$

Since $\Phi(x^*, \bar{u}, (0, a)) \geq 0$ for every $a \geq 0$ and $\sum_{n=1}^{p} \bar{\mu}_n \rho_{f_n} + \sum_{t\in\bar{\Omega}} \eta_t \rho_{g_t} \geq 0$, therefore, we have

$$\Phi\left(x^*, \bar{u}, 0, \sum_{n=1}^{p} \bar{\mu}_n \rho_{f_n} \sum_{t\in\bar{\Omega}} \tilde{\eta}_t \rho_{g_t})\right) \geq 0$$

a contradiction to (85).
Hence, $x^* = \bar{u}$.

IV. CONCLUSION

In this paper, we proposed higher order Mangasarian and Mond-Weir duals for multiobjective semi-infinite programming problems with square root term both in objective functions and constraints. The results of this paper improve the results in [23].

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