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Original Article

Common Fixed Point Theorem on Partial Metric Space

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Abstract - *The aim of this paper to prove a common fixed point theorem for sequence of mappings on Partial Metric Spaces. Our result improve and generalize many existing results in the literature*.

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Keywords - *Partial metric space, Complete partial metric space, Coincidence point, Weakly compatible mappings, Fixed point , Bianchini – Grandolfi gauge function.*

I. INTRODUCTION

The study of fixed point theorems for satisfying contractive type conditions in partial metric spaces is a very active field of research activity recently. In 1994, Mathews [15] established the notion of a Partial metric spaces and proved common fixed point theorems for compatible maps in partial metric spaces. Also Matthews generated the Banach contraction principle is valid in partial metric space and it should be applied in program verification. Ciric [10] gave the Quasi contractions in metric spaces and established Banach contraction principle and many other fixed point theorems in metric spaces. In fact, it is clear that partial metric spaces play an important role for constructing models in the theory of computation.

Definition 1.1: A partial metric on a nonempty set X is a function $P : X \times X \to R^+$ such that

for all $x, y, z \in X$:

(p1): $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$

(p2) : $p(x,x) \leq p(x,y)$

(p3): $p(x,y) = p(y,x)$

(p4): $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then the pair (X,p) is called a partial metric space.

Remark 1.2: If $p(x,y) = 0$, then from (p1) and (p2) , we have $x = y$.. But if $x = y$,then $p(x,y)$ may not be 0.

If p is a partial metric on X , then the function $p^r : X \times X \to R^+$ given by

$P^r(x,y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

Definition1.3: Let (X, p) be a partial metric space and $\{x_n\}$ is a sequence in X ,then

(i) $\{x_n\}$ converges to a point $x \epsilon X$, iff $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) $\{x_n\}$ is called Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ is finite.

Definition1.4: A partial metric space $(X, p)$is said to be complete if every Cauchy sequence $\{x_n\}$ is converges to X , with respect to a point $x \in X$, such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Remark1.5: It is easy to see that every closed subset of a complete partial metric space is complete. **Lemma 1.6**: Let (X, P) be a partial metric space, then

(i) $\{x_n\}$ is a Cauchy sequence in (X, p) iff it is a Cauchy sequence in the metric space (X, p^r),

(ii)(X, p) is complete if and only if the metric space $(X, p^r)$is complete. Further more $\lim_{n\to\infty} p^r(x_n, x)$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Mathews[16] obtained the following Banach fixed point theorem for complete partial metric spaces.

Theorem 1.7: Let f be a mapping of a complete partial metric space (X, p) into itself such that there is a real number c with $0 \le c \le 1$, satisfying for all $x, y \epsilon X$, $p\left(f_x, f_y\right) \le cp(x, y)$. Then has a unique fixed point.

Definition 1.8 : A non decreasing function $\varphi : (0, \infty) \to (0, \infty)$ is said to be a Bianchini – Grandolfi gauge function on R^+ if $\sum_{n=0}^{\infty} \varphi^n(t)$ convergent for all $t \in R^+$, where φ^n is the nth iteration of φ .

and $\varphi^0(t) = t$ i.e. $\varphi^0(t) = t, \varphi^1(t) = \varphi(t), \varphi^2(t) = \varphi\big(\varphi(t)\big), \dots\dots\dots\dots$, $\varphi^n(t) = \varphi(\varphi^{n-1}(t))$.

Lemma 1.9 : If $\varphi \in \Psi$, then $\varphi(0) = 0$ and $\varphi(t) < t$, for all $t > 0$.

Definition 1.10 : Let X be a non-empty set and $A, B : X \to X$ are given self maps on X. If $u = Ax = Bx$ for some $x \in X$, then x is called a coincidence point of A, B and and u is called a point of coincidence of A and B .

Definition 1.11: Let X be a non-empty set and A $, B : X \to X$ are given self maps on X . Then the pair (A, B) is said to be weakly compatible if $ABt = BAt$, whenever $At = Bt$ for some $t \in X$.

II. MAIN RESULT

Our main result is the following:

Theorem 2.1: Let A, B, S and T are four self mappings of a complete partial metric space (X, p) such that

$A(X) \subseteq T(X), B(X) \subseteq S(X)$ and $p(Ax, By) \le \varphi\,(max\{p(Sx, Ty), [p(Sx, Ax) + p(Ty, By)], [p(Sx, By) + p(Ty, Ax)]\})$, $for\ all$ $x, y \epsilon X$, where $\varphi \in \boldsymbol{\Psi}$ (2.1)

If one the ranges $A(X), B(X), T(X) and\ S(X)$ is a closed subset of (X, p), then

(1) A and S have a coincidence point, (2) B and T have a coincidence point.

Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since $A(X) \subseteq T(X)$, there exists $x_1 \in X$ such that

$Tx_1 = Ax_0$. Since $B(X) \subseteq S(X)$, there exists $x_2 \in X$ such that S$Sx_2 = Bx_1$. Continuing this process we can construct sequences $\{x_n\}$ *and* $\{y_n\}$ in X defined by

$$y_{2n} = Tx_{2n+1} = Ax_{2n}\ ,\ y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \text{ for every } n \in N. \quad (2.2)$$

We have to prove that $\{y_n\}$ is a Cauchy sequence in the partial metric space (X , p).

Now $p(y_{2m}, y_{2m+1}) = p(A x_{2m}, Bx_{2m+1})$

$$\leq \varphi\ (\max\{p(Sx_{2m}, Tx_{2m+1}), [p(Sx_{2m}, Ax_{2m}) + p(Tx_{2m+1}, Bx_{2m+1})],$$

$$[p(Sx_{2m}, Bx_{2m+1}) + p(Tx_{2m+1}, Ax_{2m})]\})$$

$$\leq \varphi(\max\{p(y_{2m-1}, y_{2m}), [p(y_{2m-1}, y_{2m}) + p(y_{2m}, y_{2m+1})],$$

$$[p(y_{2m-1}, y_{2m+1}) + p(y_{2m}, y_{2m})]\})$$

$$\leq \varphi\ (\max\{p(y_{2m-1}, y_{2m}), [p(y_{2m-1}, y_{2m+1}) + p(y_{2m}, y_{2m})],$$

$$[p(y_{2m-1}, y_{2m+1}) + p(y_{2m}, y_{2m})]\}).$$

$$\leq \varphi\ (\max\{p\ (y_{2m-1}, y_{2m}), p(y_{2m-1}, y_{2m+1}), p(y_{2m-1}, y_{2m+1})\})$$

$$\leq \varphi\ (\max\{p\ (y_{2m-1}, y_{2m}), p(y_{2m-1}, y_{2m+1})\}) \quad (2.3)$$

Similarly we obtain,

$$p(y_{2m+1}, y_{2m+2}) \leq \varphi\ (\max\{p\ (y_{2m}, y_{2m+1}), p(y_{2m}, y_{2m+2})\}) \quad (2.4)$$

Therefore from (2.3) and (2.4) , we get

$$p(y_n, y_{n+1}) \leq \varphi(\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}) \quad (2.5)$$

Let there exists $m \in N$ such that

$p(y_{2m-1}, y_{2m}) = 0$, then we have $y_{2m-1} = y_{2m}$.

From (2.3) , we get

$p(y_{2m}, y_{2m+1}) \leq \varphi\{p\ (y_{2m}, y_{2m+1})\}$.

Since $\varphi(t) < t$ *for every* $t > 0$, we have $p\ (y_{2m}, y_{2m+1}) = 0$ and then $y_{2m} = y_{2m+1}$.

From (2.4) , we get

$p(y_{2m+1}, y_{2m+2}) \leq \varphi\ (\ p(y_{2m+1}, y_{2m+2})\})$,then $y_{2m+1} = y_{2m+2}$.

Hence, we have

$$y_{2m-1} = y_{2m} = y_{2m+1} = y_{2m+2} = \dots\dots\dots\dots \quad (2.6)$$

Hence $\{y_n\}$ is a Cauchy Sequence in(X, p) .

Now we assume that $p(y_n, y_{n+1}) > 0$, where n is large, then from (2.5) at $\varphi(t) < t$ for each

$t > 0$, we have $p(y_n, y_{n+1}) \leq \varphi(\max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}$

$= \varphi(p(y_{n-1}, y_n))$

$\Rightarrow p(y_n, y_{n+1}) \leq \varphi\{p(y_{n-1}, y_n)\}$, where n is large. (2.7)

Replacing this inequality n times, we get

$p(y_n, y_{n+1}) \leq \varphi^n\{p(y_0, y_1)\}$ (2.8)

By the properties (p1) and (p2) , we have

$\max\{p(y_n, y_n), p(y_{n+1}, y_{n+1})\} \leq p(y_n, y_{n+1})$ (2.9)

Therefore from(2.8) and (2.9) , we have

$\max\{p(y_n, y_n), p(y_{n+1}, y_{n+1})\} \leq \varphi^n\{p(y_0, y_1)\}$ (2.10)

Hence

$P^r(y_n, y_{n+1}) = 2p(y_n, y_{n+1}) - p(y_n, y_n) - p(y_{n+1}, y_{n+1})$

$\leq 2p(y_n, y_{n+1}) - p(y_n, y_n) - p(y_{n+1}, y_{n+1})$

$\leq 4\varphi^n\{p(y_0, y_1)\}$ (2.11)

Now for the metric p^r , by triangle inequality for any$l, m \in N^*$, we have

$P^r(y_n, y_{n+l}) \leq P^r(y_n, y_{n+1}) + P^r(y_{n+1}, y_{n+2}) + \cdots \ldots \ldots \ldots \ldots . + P^r(y_{n+l-1}, y_{n+l})$

$\leq 4\varphi^n\{p(y_0, y_1) + 4\varphi^{n+1}\{p(y_0, y_1)\} + \cdots \ldots \ldots \ldots . . + 4\varphi^{n+l-1}\{p(y_0, y_1)\}$

$\leq 4[\sum_{i=n}^{n+l-1} \varphi^i\{p(y_0, y_1)\}]$

Hence we conclude that for any arbitrary number $\varepsilon > 0$ there exists a positive number n_0

Such that

$P^r(y_n, y_{n+l}) \leq \varepsilon$ for every $n \geq n_0$ and all $l \in N^*$.

Thus $\{y_n\}$ is a Cauchy Sequence in the metric space(X, p^r).

Since (X, p) is a complete metric space, then from lemma (1.6), (X, p^r) is a complete metric space. Therefore the sequence $\{y_n\}$ converges to some $y \in X$.

i.e. $\lim_{n\to\infty} P^r(y_n, y) = 0$

From Lemma (1.6) , we have

$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n,m\to\infty} p(y_n, y_m)$ (2.12)

Moreover, since $\{y_n\}$ is a Cauchy Sequence in the metric space(X, p^r) , then

$\lim_{n,m\to\infty} p^r(y_n, y_m) = 0$. Therefore from Lemma (1.6), we have

$$\lim_{n\to\infty} p(y_n, y_m) = 0 \tag{2.13}$$

Thus from the definition of p^r and (2.13) , we have

$$\lim_{n\geq m\to\infty} p(y_n, y_m) = 0$$

Therefore from (2.12) , we have

$$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n\geq m\to\infty} p(y_n, y_m) = 0 \tag{2.14}$$

Thus

$$\lim_{n\to\infty} p(y_{2n}, y) = \lim_{n\to\infty} p(y_{2n-1}, y) = 0 \tag{2.15}$$

Hence from (2.15) , we have

$$\lim_{n\to\infty} p(Ax_{2n}, y) = \lim_{n\to\infty} p(Tx_{2n+1}, y) = 0 \tag{2.16}$$

and $\lim_{n\to\infty} p(Bx_{2n-1}, y) = \lim_{n\to\infty} p(Sx_{2n}, y) = 0$ (2.17)

Since $B(X) \subseteq S(X)$ and $y \in S(X)$, then from (2.17) there exists $u \in X$ such that $y = Su$.

We have to prove that $p(Au, y) = 0$. Now suppose that $p(Au, y) > 0$, then we have

$$\begin{aligned}
&\leq p(y, Bx_{2n+1}) + p(Bx_{2n+1}, Au) - p(Bx_{2n+1}, Bx_{2n+1}) \\
&\leq p(y, Bx_{2n+1}) + p(Bx_{2n+1}, Au) \\
&\leq p(y, Bx_{2n+1}) + p(Au, Bx_{2n+1}) \\
&\leq p(y, y_{2n+1}) + \varphi(\max[p(Su, Tx_{2n+1}), [p(Su, Au) + p(Tx_{2n+1}, Bx_{2n+1})], \\
&\quad [p(Su, Bx_{2n+1}) + p(Tx_{2n+1}, Au)]\}) \\
&\leq p(y, y_{2n+1}) + \varphi(\max[p(Su, y_{2n}), [p(Su, Au) + p(y_{2n}, y_{2n+1})], \\
&\quad [p(Su, y_{2n+1}) + p(y_{2n}, Au)]\}) \\
&\leq p(y, y_{2n+1}) + \varphi(\max\{p(y, y_{2n}), [p(y, Au) + p(y_{2n}, y_{2n+1})], \\
&\quad [p(y, y_{2n+1}) + p(y_{2n}, Au)]\})
\end{aligned}$$

Taking limit as $\to \infty$, we get

$$\begin{aligned}
p(y, Au) &\leq \lim_{n\to\infty} p(y, y_{2n+1}) + \varphi(\lim_{n\to\infty} \max\{p(y, y_{2n}), [p(y, Au) + p(y_{2n}, y_{2n+1})] \\
&\quad [p(y, y_{2n+1}) + p(y_{2n}, Au)]\}) \\
&= 0 + \varphi\left(p(y, Au)\right) = \varphi\left(p(y, Au)\right)
\end{aligned}$$

Hence for $t > 0$, $\varphi(t) < t$, *we have*

$p(y, Au) < p(y, Au)$, which is a contradiction.

Therefore $p(y, Au) = 0 \Rightarrow y = Au$. (2.18)

Since $y = Su$, then Au = Su. Therefore u is called coincidence point of A and S.

Hence the proof (1).

Since $A(X) \subseteq T(X)$, then from (2.18) ,we have $y \in T(X)$. Therefore there exists $v \in X$ such that

$y = Tv$.

We have to prove that $p(Bv, y) = 0$. Suppose that that $p(Bv, y) > 0$.

Now $y = Su = Au = Tv$ and from (2.1) , we have

$$
\begin{aligned}
p(y, Bv) &= p(Au, Bv) \\
&\leq \varphi(\max\{p(Su, Tv), [p(Su, Au) + p(Tv, Bv)], [p(Su, Bv) + p(Tv, Au)]\}) \\
&\leq \varphi(\max\{p(y, y), [p(y, y) + p(y, Bv)], [p(y, Bv) + p(y, y)]\}) \\
&\leq \varphi(\max\{p(y, Bv), p(y, Bv)\}) \\
&\leq \varphi(p(y, Bv)) \\
&< p(y, Bv).
\end{aligned}
$$

which is a contradiction. Therefore we have

$p(Bv, y) = 0$ *and* $y = Bv = Tv$. (2.19)

Therefore v is coincidence point of B and T. Hence (2) holds.

Since the pair $\{A, S\}$ is weakly compatible, then from (2.18) , we have

$Ay = ASu = SAu = Sy$.

We have to prove that $p(Ay, y) = 0$. Suppose that $(Ay, y) > 0$, then we have

$$
\begin{aligned}
p(Ay, y) &\leq p(Ay, y_{2n+1}) + p(y_{2n+1}, y) - p(y_{2n+1}, y_{2n+1}) \\
&\leq p(Ay, y_{2n+1}) + p(y_{2n+1}, y) \\
&\leq p(Ay, Bx_{2n+1}) + p(y_{2n+1}, y) \\
&\leq \varphi(\max\{p(Sy, Tx_{2n+1}), [p(Sy, Ay) + p(Tx_{2n+1}, Bx_{2n+1})], \\
&\qquad [p(Sy, Bx_{2n+1}) + p(Tx_{2n+1}, Ay)]\}) + p(y_{2n+1}, y) \\
&\leq \varphi(\max\{p(Ay, y_{2n}), [p(Ay, Ay) + p(y_{2n}, y_{2n+1})], \\
&\qquad [p(Ay, y_{2n+1}) + p(y_{2n}, Ay)]\}) + p(y_{2n+1}, y)
\end{aligned}
$$

$\leq \varphi(\max\{p(Ay, y), 0, [p(Ay, y) + p(y, Ay)]\})$

$\leq \varphi(p(y, Ay))$

$< p(y, Ay))$

$\Rightarrow p(Ay, y) < p(y, Ay))$, which is a contradiction.

Hence we have

$p(Ay, y) = 0 \; and \; Ay = Sy = y.$ (2.20)

Since the pair (B, T) is weakly compatible , therefore from (2.19), we have

$By = BTv = TBv = Ty$

We have to prove that

$p(By, y) = 0$.

Suppose that $p(By, y) > 0$, then from (2.1) and (2.20) , we have

$p(y, By) = P(Ay, By)$

$\leq \varphi$ (max$\{p(Sy, Ty), [p(Sy, Ay) + p(Ty, By)], [p(Sy, By) + p(Ty, Ay)]\}$

$\leq \varphi$ (max$\{p(y, By), [p(y, y) + p(By, By)], [p(y, By) + p(By, y)]\}$

$\leq \varphi(p(y, By))$

Which is a contradiction. Therefore we have

$p(By, y) = 0$ and $By = Ty = y$ (2.21)

Now from (2.20) and (2.21) , we have

$y = Ay = By = Sy = Ty$

Therefore y is a common fixed point of A, B, S and T .

Uniqueness: Let $w \in X$ is a common fixed point of A, B, S and T.

We have to prove that $w = y$. Suppose that $p(w, y) > 0$, then by using (2.1) , we get

$p(y, w) = P(Ay, Bw)$

$\leq \varphi$ (max$\{p(Sy, Tw), [p(Sy, Ay) + p(Tw, Bw)], [p(Sy, Bw) + p(Tw, Ay)]\}$

$\leq \varphi$ (max$\{p(y, w), [p(y, y) + p(w, w)], [p(y, w) + p(w, y)]\}$

$\leq \varphi$ (max$\{p(y, w), 0, [p(y, w) + p(w, y)]\}$

$\leq \varphi(2p(y, w))$

$< p(y, w)$, which is a contradiction .

Therefore we have , $p(y,\ w) = 0 \Rightarrow y = w$.

Hence uniqueness for the common fixed point theorem is proved.

Corollary 2.2 : Let A, B, S and T are self maps for a complete metric space (X, p) such that

$$A(X) \subseteq T(X), B(X) \subseteq S(X) \text{ and } p\,(Ax\,,\ By\,) \leq \varphi\,(max\,\{\,.\,p\,(Sx\,,Ty\,), p(Ax\,,Sx\,),$$

$$p(\,By\,,Ty), p(\,Ax\,,Ty\,), p(Sx,Ty\,)\}) \text{ , where } \varphi \in \boldsymbol{\Psi}\ .$$

If one of the ranges $A(X), B(X), T(X) and\ S(X)$ is a closed subset of (X, p), then

(1) A and S have a coincidence point, (2) B and T have a coincidence point.

Moreover, if the pairs $(A,\ S)$ and $(B,\ T)$ are weakly compatible, then $A\,, B, S\ and\ T$ have a unique common fixed point.

III. CONCLUSION

In this paper, we proved some fixed point results under Banach algebra on partial metric spaces. Our result extend and generalize several results from the existing literature in the context of partial metric spaces.

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